Plateau's problem in infinite-dimensional spaces

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Currents are functionals on differential forms

Fix integers $m \geq n \geq 0$. A classical *n*-current in $\mathbb{R}^m$ is a continuous linear functional

$$T\colon \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^m, \Lambda^n \mathbb{R}^m) \to \mathbb{R}$$

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Furthermore:

- its mass is $\mathbf{M}(T) := \sup\{|T(\omega)| \,:\, \|\omega\|_{\mathrm{L}^\infty} \leq 1\}$,
- its boundary is the $(n-1)$-current ∂T with $\partial T(\psi) := T(\mathrm{d}\psi)$.

Interpretation and motivation

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Integration over an n-dimensional submanifold (or a rectifiable set) Σ *in* $\mathbb{R}^m$ *with orientation* $\vec{\Sigma}$,

$$[\![\Sigma]\!](\omega) := \int_\Sigma \langle \omega(x), \vec{\Sigma}(x) \rangle \, \mathrm{d}\mathcal{H}^n(x)\,,$$

defines an n-current $[\![\Sigma]\!]$ *with* $\mathbf{M}\left([\![\Sigma]\!]\right) = \mathcal{H}^n(\Sigma)$, $\partial[\![\Sigma]\!] = [\![\partial\Sigma]\!]$.

Metric functionals

On a complete metric space E imitate classical currents with $(1+n)$-linear functionals

$$T:\ \underbrace{\mathrm{Lip_b}(E)}_{\text{bounded Lipschitz function } \varphi} \times \underbrace{\mathrm{Lip}(E)^n}_{\mathbb{R}^n\text{-valued Lipschitz function } \pi=(\pi_1,\pi_2,\ldots,\pi_n)} \to \mathbb{R}\,,$$

shortcut notation $\varphi\,\mathrm{d}\pi \equiv \varphi\,\mathrm{d}\pi_1 \wedge \mathrm{d}\pi_2 \wedge \ldots \wedge \mathrm{d}\pi_n$

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shortcut notation $\varphi\,\mathrm{d}\pi \equiv \varphi\,\mathrm{d}\pi_1 \wedge \mathrm{d}\pi_2 \wedge \ldots \wedge \mathrm{d}\pi_n$

and write $\|T\|$ for the least (tight) Borel measure on E with

$$|T(\varphi\,\mathrm{d}\pi)| \le \int_E |\varphi|\,\mathrm{d}\|T\| \prod_{i=1}^{n} \underbrace{\mathrm{Lip}(\pi_i)}_{\substack{\text{Lipschitz}\\ \text{constant}}} \qquad \forall \varphi, \pi\,.$$

The axioms of metric currents

Definition (Ambrosio & Kirchheim '00)

The space $\mathbf{M}_n(E)$ of metric n-currents in E with finite mass consists of all functionals T as before with

- finite mass: $\mathbf{M}(T) := \|T\|(E) < \infty$,
- continuity axiom: $\pi^l \to \pi$ *pointwise with* $\mathrm{Lip}(\pi^l)$ *bounded*
$$\implies T(\varphi\, \mathrm{d}\pi^l) \to T(\varphi\, \mathrm{d}\pi),$$
- locality axiom: π_{i_0} *constant on* $\{\varphi \neq 0\} \implies T(\varphi\, \mathrm{d}\pi) = 0.$

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- Remark: automatically alternating!
- Boundary operator: $\partial T(\varphi \, \mathrm{d}\pi) := T(1 \, \mathrm{d}(\varphi, \pi))$.

Normal and integral currents

- Normal currents:

$$\mathbf{N}_n(E) := \{T \in \mathbf{M}_n(E) \,:\, \partial T \in \mathbf{M}_{n-1}(E)\}\,.$$

- Integral currents:

$$\mathbf{I}_n(E) := \left\{ \theta[\![\Sigma]\!] \in \mathbf{N}_n(E) \,:\, \begin{array}{c} \Sigma \text{ oriented } \mathcal{H}^n\text{-rectifiable set} \\ \theta\colon \Sigma \to \mathbb{Z} \text{ measurable} \end{array} \right\}.$$

Classical and generalized Plateau problem

- Classical Plateau problem: Given a closed curve S in $\mathbb{R}^3$ look among surfaces T with boundary S for one of minimal area.
- Generalized Plateau problem: For a given $(n-1)$-current $S \in \mathbf{I}_{n-1}(E)$ in a complete metric space E study optimal n-currents T in

$$\mathrm{Fillvol}_E(S) := \inf\{\mathbf{M}(T) \,:\, T \in \mathbf{I}_n(E), \partial T = S\}.$$

Existence for compact spaces and boundaries

- Basic result: E (locally) compact
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- From now on focus on infinite-dimensional Banach spaces (linear, but never locally compact):

Theorem 1 (Ambrosio & Kirchheim '00)

Consider a separable normed space X such that $Y := X^*$ has isoperimetric inequalities and $S \in \mathbf{I}_{n-1}(Y)$ with $\partial S \equiv 0$ such that $\mathrm{spt}\,\|S\|$ is compact. Then $\mathrm{Fillvol}_Y(S)$ is finite and is attained.

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- applies for reflexive Y, in particular in all Hilbert spaces.
- proved via Gromov's isometric embedding of an equi-compact sequence in a compact metric space, projection back to Y, and w^*-compactness.

Existence for general boundaries

Now consider also

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Theorem 2 (Ambrosio & S. '12)

Consider a separable dual Y and $S \in \mathbf{M}_{n-1}(Y)$. Whenever $\mathrm{Fillmass}_Y(S)$ or $\mathrm{Fillvol}_Y(S)$ is finite, then it is also attained.

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- proved via w^*-compactness, but intrinsically in Y.
- criteria for finiteness:
  - $\partial S \equiv 0$, $\mathrm{spt}\,\|S\|$ bounded $\implies \mathrm{Fillmass}_Y(S) < \infty$,
  - $S \in \mathbf{I}_{n-1}(Y)$, $\partial S \equiv 0 \iff \mathrm{Fillvol}_Y(S) < \infty$.

Existence with a free boundary

Theorem 3 (Ambrosio & S. '12)

Consider a separable dual Y and $S \in \mathbf{M}_{n-1}(Y)$. Then the infima

$$\mathbf{F}^{\mathbf{N}}_{Y}(S) := \inf\{\mathbf{M}(T) + \mathbf{M}(S - \partial T) \ : \ T \in \mathbf{N}_n(Y)\},$$
$$\mathbf{F}^{\mathbf{I}}_{Y}(S) := \inf\{\mathbf{M}(T) + \mathbf{M}(S - \partial T) \ : \ T \in \mathbf{I}_n(Y)\}$$

are attained.

Existence in non-separable duals

Theorem 4 (Wenger '05/'11/'12)

The isoperimetric inequality for currents holds in every Banach space. Moreover, Theorem 1 and the $\mathbf{I}_n$*-parts of Theorems 2, 3 remain valid in every dual* Y.

- existence results proved via ultralimit completion and projection arguments plus refined isometric embeddings.

w^*-convergence

Definition (Ambrosio & Kirchheim '00)

For currents $T, T_1, T_2, \ldots \in \mathbf{M}_n(Y)$ *in a dual space* Y *one says that* T_k w^**-converges to* T *if* $T_k(\varphi \, \mathrm{d}\pi)$ *converges to* $T(\varphi \, \mathrm{d}\pi)$ *for all* w^**-continuous* $(\varphi, \pi) \in \mathrm{Lip}_\mathrm{b}(Y) \times \mathrm{Lip}(Y)^n$.

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- in fact a "weak-weak*-convergence": distributional *and* with w^*-topology on Y.
- example: $\delta_{y_k} \stackrel{\mathrm{w}^*}{\to} \delta_y$ in $\mathbf{M}_0(Y) \iff y_k \stackrel{\mathrm{w}^*}{\to} y$ in Y.
- necessarily requires φ and π with unbounded support.

w^*-compactness

The main ingredient in our existence proof is ...

Theorem 5 (Ambrosio & S. '12)

Consider $T_1, T_2, \ldots \in \mathbf{N}_n(Y)$ in a separable dual $Y = X^*$ with

$$\sup_{k\in\mathbb{N}} \big[\mathbf{M}(T_k) + \mathbf{M}(\partial T_k)\big] < \infty \qquad \text{and} \qquad \bigcup_{k=1}^{\infty} \operatorname{spt} \|T_k\| \text{ bounded}\,.$$

Then T_k w^*-converges to some $T \in \mathbf{N}_n(Y)$.

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- equi-bound for $\operatorname{spt}\|T_k\|$ can be weakened to w^*-tightness.
- $T_k \in \mathbf{I}_n(Y) \implies T \in \mathbf{I}_n(Y)$ (by closure theorem of AK '00).

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- still true in $\mathbf{I}_n(Y)$ if merely X is separable (Wenger '12).

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- still true in $\mathbf{I}_n(Y)$ if merely X is separable (Wenger '12).
- both separability assumptions are sharp!

Construction of a limit object

The intrinsic proof of Theorem 5:

For $R \gg 1$ consider the compact metric space $K := (\mathrm{B}_R^Y, \mathrm{d}_{\mathrm{w}^*})$.

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We can assume[‡]

$$\left.\begin{aligned} \|T_k\| &\overset{\mathrm{w}^*}{\to} \mu \\ \|\partial T_k\| &\overset{\mathrm{w}^*}{\to} \nu \\ T_k(\cdot\,\mathrm{d}\pi) &\overset{\mathrm{w}^*}{\to} F^\pi \end{aligned}\right\} \quad \text{in } C^0(K)^*$$

for all π in a countable set A^n.

[‡] μ, ν are Borel measures, as $\mathcal{B}_{\mathrm{w}^*}(Y) = \mathcal{B}(Y)$ holds by separability.

w^*-separability

A is chosen according to the following . . .

Lemma

There exists a countable subset A of $\mathrm{Lip}(\mathrm{B}_R^Y)$*, dense with respect to pointwise convergence, such that A contains only* w^**-continuous functions.*

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There exists a countable subset A of $\mathrm{Lip}(\mathrm{B}_R^Y)$, dense with respect to pointwise convergence, such that A contains only w^*-continuous functions.

We can now define the seed of the limit current T by

$$T(\varphi\,\mathrm{d}\pi) := \langle F^\pi, \varphi \rangle \qquad \text{for } (\varphi, \pi) \in A \times A^n\,.$$

Equi-continuity and conclusions

By equi-continuity $\forall(\varphi,\pi),(\widetilde{\varphi},\widetilde{\pi}) \in A \times A^n$:

$$\begin{aligned} &|T_k(\widetilde{\varphi}\,\mathrm{d}\widetilde{\pi}) - T_k(\varphi\,\mathrm{d}\pi)| \\ &\qquad \leq C\left[\int_Y |\widetilde{\varphi}-\varphi|\,\mathrm{d}\underbrace{\|T_k\|}_{\overset{\mathrm{w}^*}{\to}\mu} + \int_Y |\widetilde{\pi}-\pi|\,\mathrm{d}\underbrace{(\|T_k\|+\|\partial T_k\|)}_{\overset{\mathrm{w}^*}{\to}\mu+\nu}\right]. \end{aligned}$$

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- ⇝ T is suitably uniformly continuous on $A \times A^n$,
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- ⇝ T is suitably uniformly continuous on $A\times A^n$,
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  (locality axiom requires additional w^*-separation lemma),
- ⇝ conclusion: $T_k\stackrel{\mathrm{w}^*}{\to}T$ with $\|T\|\le\mu$, $\|\partial T\|\le\nu$. □

The existence proof

Recall:

Theorem 2

In a separable dual Y: If $\begin{matrix}\mathrm{Fillmass}_Y(S)\\ \mathrm{Fillvol}_Y(S)\end{matrix}$ *is finite, it is attained.*

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- $\rightsquigarrow$ by Theorem 5*: T_k w^*-converges to some T,
- $\rightsquigarrow$ by w^*-semicontinuity: $\mathbf{M}(T) \le \lim_k \mathbf{M}(T_k) = \mathrm{Fillvol}_Y(S)$. □