# Topology Advanced Class

## Modular Functors from Conformal Blocks of Rational VOAs

Hilary Term 2026

**Talk 1. Overview (Adrià Marín Salvador)**

Overview of the topic [DW25].

**Talk 2. Introduction to Vertex Operator Algebras (Josep Fontana McNally)**

Introduction and intuition to the theory of VOAs that we will need during the term [FBZ04, Kac98]. We define vertex algebras [FBZ04, Sec. 1.3] and vertex operator algebras [FBZ04, Sec. 2.5] and provide examples. We introduce (admissible) modules over VOAs [FBZ04, Sec. 5.1] and the notion of rational vertex algebras [FBZ04, Sec. 5.5]. We get to the notion of strongly rational VOA, which is the hypothesis used in the Damiolini-Woike paper. A possible reference is [McR21, Sec. 2].

**Talk 3. $\mathcal{D}$-modules. (Simon Felten)**

Introduction to $\mathcal{D}$-modules. We introduce $\mathcal{D}$-modules [Gin] and $\mathcal{D}$-modules on stacks [Cai]. We go over $\mathcal{D}$-modules twisted by a line bundle raised to possibly non-integer power. We finalize with logarithmic $\mathcal{D}$-modules on stratified spaces [Del70, WZ21, Ogu18].

**Talk 4. The moduli spaces of marked curves. (Sam Moore)**

Introduction the different versions of the moduli space of $n$-marked genus $g$ (possibly nodal) curves that we will need later. For the classical story ($\overline{\mathcal{M}}_{g,n}$), we may follow the original paper [DM69]. We then introduce the versions with local coordinates ($\widehat{\overline{\mathcal{M}}}_{g,n}$) and local tangent vectors ($\overline{\mathcal{J}}^{1,\times}_{g,n}$) following [DGT21, Sec. 2].

**Talk 5. Modular functors.**

Definition of topological modular functor [DW25, Sec. 3] and [BK01, Sec. 5.1]. Definition of complex modular functor [BK01, Sec. 6.4]. The Surf modular functor and its relationship to the tower of groupoids $\Pi_1(\tilde{M})$ [BK01, 5.6]. Equivalence between the notions of topological and complex modular functors [BK01, Thm. 6.4.2]. The goal should be to understand [DW25, Def. 3.4.2] and the perspective outlined in [DW25, Rk. 3.3.2 c], cf [DW25, Sec. 4.1].

**Talk 6. The construction of conformal blocks (I)**

Definition of the sheaf of conformal blocks away from the nodal curves and its $\mathcal{D}$-module structure. In the first part, we define the vector bundle associated to a VOA and a smooth complex curve [FBZ04, Sec. 6.5] and the space conformal blocks associated to this data and a smooth projective curve [FBZ04, Sec. 9.2 and 10.1, 10.2]. We discuss the propagation of vacua, see for example [Cod20, Thm. 3.6] or [DGT21, Thm. 5.1]. In the second part, we discuss the sheaf of conformal blocks on the whole

moduli space of $n$-marked genus $g$ smooth projective curves [FBZ04, Ch. 17]. Crucially, we describe the twisted $\mathcal{D}$-module structure of this sheaf [FBZ04, Thm. 17.3.2, 17.3.15, 17.3.20].

**Talk 7. The construction of conformal blocks (II)**

Generalization of the previous talk to nodal curves [DGT21]. We describe the sheaf associated to a VOA and a (possibly nodal) curve [DGT21, Sec. 3.2.3 and what is needed earlier from 3.1, 3.2] (or [DGT24, Sec. 2]), the Lie algebra associated to a VOA and a marked curve [DGT21, Sec. 3.2.3 and what is needed earlier from 4.1, 4.2] (or [DGT24, Sec. 3]) and the spaces and sheaves of coinvariants on (possibly nodal) curves, first on a single curve [DGT21, Sec. 4.3] and then on the whole moduli space [DGT21, 5.1, 5.2, 5.3.1]. We finish with the crucial fact that the sheaf of coinvariants is a twisted logarithmic $\mathcal{D}$-module [DGT21, Sec. 6].

**Talk 8. Proof of the main Theorem**

Proof that the sheaf of conformal blocks on the moduli space of $n$-marked genus $g$ (possibly nodal) curves with tangent vectors is a modular functor. We first show that the sheaf of coinvariants is a vector bundle [DGT24, Sec. 8.8, VB Corollary]. For that, we introduce the Factorization Theorem [DGT24, Thm. 7.0.1] and the Sewing Theorem [DGT24, Sec. 8]. We then show that the sheaf of coinvariants of a strongly rational VOA defines a modular functor, [DW25, Thm. 4.2.2], by checking the necessary properties [DW25, Sec. 4].

# References


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