ALGEBRA EXERCISES 1

1. (a) Find the remainder when $n^2 + 4$ is divided by 7 for $0 \le n < 7$.

Deduce that $n^2 + 4$ is not divisible by 7, for every positive integer n. [Hint: write $n = 7k + r$ where $0 \le r < 7$.]

(b) Now k is an integer such that $n^3 + k$ is not divisible by 4 for all integers n. What are the possible values of k?

2. (i) Prove that if a, b are positive real numbers then

$$\sqrt{ab} \le \frac{1}{2}(a+b).$$

(ii) Now let $a_1, a_2, \ldots, a_n$ be positive real numbers. Let $S = a_1 + a_2 + \cdots + a_n$ and $P = a_1 a_2 \cdots a_n$.

Suppose that a_i and a_j are distinct. Show that replacing a_i and a_j with $(a_i + a_j)/2$ and $(a_i + a_j)/2$ increases P without changing S.

Deduce that

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

3. (i) Let n be a positive integer. Show that

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right).$$

(ii) Let a also be a positive integer. Show that if $a^n - 1$ is prime then $a = 2$ and n is prime.

Is it true that if n is prime then $2^n - 1$ is also prime?

4. Let a, b, r, s be rational numbers with $s \ne 0$. Suppose that the number $r + s\sqrt{2}$ is a root of the quadratic equation

$$x^2 + ax + b = 0.$$

Show that $r - s\sqrt{2}$ is also a root.

5. (i) The cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots α, β, γ, and so factorises as

$$a\,(x-\alpha)\,(x-\beta)\,(x-\gamma)\,.$$

Determine

$$\alpha + \beta + \gamma, \quad \alpha\beta + \beta\gamma + \gamma\alpha, \quad \alpha\beta\gamma,$$

in terms of a, b, c, d. What does $\alpha^2 + \beta^2 + \gamma^2$ equal?

(ii) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(iii) By considering the roots of the equation $4x^3 - 3x - \cos 3\theta = 0$ deduce that

$$\cos\theta\cos(\theta + 2\pi/3)\cos(\theta + 4\pi/3) = \frac{\cos(3\theta)}{4}.$$

What do

$$\cos\theta + \cos(\theta + 2\pi/3) + \cos(\theta + 4\pi/3) \quad \text{and} \quad \cos^2\theta + \cos^2(\theta + 2\pi/3) + \cos^2(\theta + 4\pi/3)$$

equal?