PROBLEM SHEET 1

1.1 Find the radius and centre of the circle described by the equation

$$x^2 + y^2 - 2x - 4y + 1 = 0$$

by writing it in the form $(x-a)^2 + (y-b)^2 = c^2$ for suitable a, b and c.

1.2 Find the equation of the line perpendicular to $y = 3x$ passing through the point $(3, 9)$.

1.3 Given

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \quad \text{and} \quad \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

show that

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)] \quad \text{and} \quad \sin^2 A = \frac{1}{2}[1 - \cos 2A].$$

1.4 Show that

$$4\cos(\alpha t) + 3\sin(\alpha t) = 5\cos(\alpha t + \phi)$$

where $\phi = $arc $\tan(-3/4)$.

1.5 Show that, for $-1 \leq x \leq 1$,

$$\cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2}.$$

1.6 Given

$$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B \quad \text{and} \quad \cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B,$$

show that

$$\cosh A \cosh B = \frac{1}{2}[\cosh(A+B) + \cosh(A-B)] \quad \text{and} \quad \sinh^2 A = \frac{1}{2}[\cosh 2A - 1].$$

1.7 Given that

$$\sinh x = \frac{1}{2}[e^x - e^{-x}],$$

show that

$$\sinh^{-1} x = \ln\left[x + \sqrt{1 + x^2}\right].$$

1.8 Express

$$\frac{x}{(x-1)(x-2)}$$

in partial fractions.

1.9 If $a_n = \frac{1}{n}$, find $\sum_{i=1}^{5} a_n$ as a fraction.

1.10 If $S = \sum_{i=0}^{N} x^i$, show that $xS = \sum_{i=1}^{N+1} x^i$. Hence show that $S - xS = 1 - x^{N+1}$ and therefore that

$$S = \frac{1 - x^{N+1}}{1 - x}.$$

PROBLEM SHEET 2

2.1 Given that

$$\sinh x = \frac{1}{2}[e^x - e^{-x}]$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cosh x.$$

2.2 Given that

$$\cosh x = \frac{1}{2}[e^x + e^{-x}],$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh x.$$

2.3 Let n be a positive integer. Show that

$$\frac{\mathrm{d}^n(x^n)}{\mathrm{d}x^n} = n!$$

2.4 If $y = \ln x$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}; \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-1}{x^2}; \qquad \frac{\mathrm{d}^{100}y}{\mathrm{d}x^{100}} = \frac{-99!}{x^{100}}.$$

2.5 Find the equation of the tangent to the curve $y = x^2$ at $(1, 1)$.

2.6 Find the slope of the curve $y = 4x + e^x$ at $(0, 1)$.

2.7 Find the angle of inclination of the tangent to the curve $y = x^2 + x + 1$ at the point $(0, 1)$.

2.8 The displacement $y(t)$ metres of a body at time t seconds ($t \geq 0$) is given by $y(t) = t - \sin t$. At what times is the body at rest?

2.9 A particle has displacement $y(t)$ metres at time t seconds given by $y(t) = 3t^3 + 4t + 1$. Find its acceleration at time $t = 4$ seconds.

2.10 If

$$y = \sum_{n=0}^{N} a_n x^n$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{n=1}^{N} n a_n x^{n-1}.$$

PROBLEM SHEET 3

3.1 If $y = \ln(1 + x^2)$, find $\mathrm{d}y/\mathrm{d}x$.

3.2 If

$$y = \frac{x}{1 + x^2}$$

find $\mathrm{d}y/\mathrm{d}x$.

3.3 If $y = \cosh(x^4)$, find $\mathrm{d}y/\mathrm{d}x$.

3.4 If $y = x^2 \ln x$, find $\mathrm{d}^2y/\mathrm{d}x^2$.

3.5 Find $\mathrm{d}y/\mathrm{d}x$ for $y = (1 + x^2)^{-1/2}$.

3.6 Show that for $y = \sinh^{-1} x$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1 + x^2}}.$$

3.7 Show that for $y = \ln[x + \sqrt{1 + x^2}]$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1 + x^2}}.$$

3.8 Find $\mathrm{d}y/\mathrm{d}x$ for $y = \cos^{-1}(\sin x)$.

3.9 A curve is given in polar coordinates by $r = 1 + \sin^2\theta$. Find $\mathrm{d}y/\mathrm{d}x$ at $\theta = \pi/4$.

3.10 Show that if

$$y = \frac{1}{2a} \ln\left|\frac{x - a}{x + a}\right|, \quad \text{then} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2 - a^2}.$$

PROBLEM SHEET 4

4.1 Given $f(x - ct)$, where x and c are constant, show that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(x - ct) = c^2 f''(x - ct),$$

and calculate this expression when $f(u) = \sin u$.

4.2 Classify the stationary point of $y = x^{-2}\ln x$, where $x > 0$.

4.3 Classify the stationary points of $y(x) = x^2 - 3x + 2$.

4.4 The numbers x and y are subject to the constraint $x + y = \pi$. Find the values of x and y for which $\cos(x)\sin(y)$ takes its minimum value.

4.5 Sketch the graph of

$$y = \frac{x}{1 + x^2}.$$

4.6 Sketch the graph of

$$y(x) = \tan(2x) \quad \text{for} \quad -\frac{3\pi}{4} \leq x \leq \frac{3\pi}{4}.$$

4.7 Sketch the graph of $y = x\ln x$ for $x > 0$.

4.8 Sketch the graph of

$$y = \frac{x^3}{2x - 1}$$

showing clearly on your sketch any asymptotes.

4.9 Sketch the graph of

$$y = x\cos(3x) \quad \text{for} \quad 0 \leq x \leq 2\pi.$$

PROBLEM SHEET 5

5.1 Verify the following Taylor expansions (taking the ranges of validity for granted).

(a)

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ... + \frac{1}{n!}x^n + ...\text{valid for any } x.$$

(b)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + ...\text{valid for any } x.$$

(c)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + \frac{(-1)^n x^{2n}}{(2n)!} + ...\text{valid for any } x.$$

(d) Let α be a constant.

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + ...\text{valid for } -1 < x < 1$$

(e)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + \frac{(-1)^{n-1}x^n}{n} + ...\text{valid for } -1 < x \leq 1.$$

5.2 Obtain a four-term Taylor polynomial approximation valid near $x = 0$ for each of the following.

(a) $(1+x)^{1/2}$, (b) $\sin(2x)$, (c) $\ln(1+3x)$.

PROBLEM SHEET 6

6.1 Reduce to standard form

$$(a)\ \ \frac{3+i}{4-i}, \quad \text{and} \quad (b)\ \ (1+i)^5.$$

6.2 Prove

$$(a)\ \ |z_1 z_2| = |z_1|\,|z_2|\,, \quad \text{and} \quad (b)\ \ \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \ \text{ when } z_2 \neq 0.$$

6.3 Given that $e^{i\theta} = \cos\,\theta + i\,\sin\,\theta$, prove that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

6.4 Let $z = 1+i$. Find the following complex numbers in standard form and plot their corresponding points in the Argand diagram:-

$$(a)\ \ \bar{z}^2, \quad \text{and} \quad (b)\ \ \frac{z}{\bar{z}}.$$

6.5 Find the modulus and principal arguments of (a) $-2+2i$, (b) $3+4i$.

6.6 Find all the complex roots of
(a) $\cosh z = 1$;
(b) $\sinh\, z = 1$;
(c) $e^z = -1$;
(d) $\cos z = \sqrt{2}$.

6.7 Show that the mapping

$$w = z + \frac{c}{z},$$

where $z = x+iy$, $w = u+iv$ and c is a real number, maps the circle $|z| = 1$ in the z plane into an ellipse in the w plane and find its equation.

6.8 Show that

$$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$$

PROBLEM SHEET 7

7.1 The matrix $A = (a_{ij})$ is given by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & -2 & 4 \\ 1 & 5 & -3 \end{pmatrix}$$

Identify the elements a_{13} and $a_{31.}$

7.2 Given that

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ -1 & 1 \\ -0 & 1 \end{pmatrix},$$

verify the distributive law $A(B + C) = AB + AC$ for the three matrices.

7.3 Let

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}.$$

Show that $AB = 0$, but that $BA \neq 0$.

7.4 A general $n \times n$ matrix is given by $A = (a_{ij})$. Show that $A + A^T$ is a symmetric matrix, and that $A - A^T$ is skew-symmetric.

Express the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}.$$

as the sum of a symmetric matrix and a skew-symmetric matrix.

7.5 Let the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ b & c & 1 \end{pmatrix}.$$

Find A^2. For what relation between a, b, and c is $A^2 = I$ (the unit matrix)? In this case, what is the inverse matrix of A? What is the inverse matrix of A^{2n-1} (n a positive integer)?

7.6 Using the rule for inverses of 2×2 matrices, write down the inverse of

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

7.7 If A and B are both $n \times n$ matrices with A non-singular, show that

$$(A^{-1}BA)^2 = A^{-1}B^2A.$$

PROBLEM SHEET 8

8.1 Obtain the components of the vectors below where L is the magnitude and θ the angle made with the positive direction of the x axis ($-180° < \theta \leq 180°$).

(a) $L = 3, \theta = 60°$;

(b) $L = 3, \theta = -150°$.

8.2 Two ships, S_1 and S_2 set off from the same point Q. Each follows a route given by successive displacement vectors. In axes pointing east and north, S_1 follows the path to B via $\overrightarrow{QA} = (2, 4)$, and $\overrightarrow{AB} = (4, 1)$. S_2 goes to E via $\overrightarrow{QC} = (3, 3)$ $\quad \overrightarrow{CD} = (1, 1)$ and $\overrightarrow{DE} = (2, -3)$. Find the displacement vector $\overrightarrow{BE}$ in component form.

8.3 Sketch a diagram to show that if A, B, C are any three points, then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$. Formulate a similar result for any number of points.

8.4 Sketch a diagram to show that if A, B, C, D are any four points, then $\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AD}$. Formulate a similar result for any number of points.

8.5 Two points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. In terms of $\mathbf{a}$ and $\mathbf{b}$ find the position vectors of the following points on the straight line passing through A and B.

(a) The mid-point C of AB;

(b) a point U between A and B for which $AU/UB = 1/3$.

8.6 Suppose that C has position vector $\mathbf{r}$ and $\mathbf{r} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$ where λ is a parameter, and A, B are points with $\mathbf{a}, \mathbf{b}$ as position vectors. Show that C describes a straight line. Indicate on a diagram the relative positions of A, B, C, when $\lambda < 0, 0 < \lambda < 1$, and $\lambda > 1$.

8.7 Find the shortest distance from the origin of the line given in vector parametric form by $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where

$$\mathbf{a} = (1, 2, 3) \quad \text{and} \quad \mathbf{b} = (1, 1, 1)\,,$$

and t is the parameter (Hint: use a calculus method, with t as the independent variable.)

8.8 $ABCD$ is any quadrilateral in three dimensions. Prove that if P, Q, R, S are the mid-points of AB, BC, CD, DA respectively, then $PQRS$ is a parallelogram.

8.9 ABC is a triangle, and P, Q, R are the mid-points of the respective sides BC, CA, AB. Prove that the medians AP, BQ, CR meet at a single point G (called the centroid of ABC; it is the centre of mass of a uniform triangular plate.)

8.10 Show that the vectors $\overline{0A} = (1, 1, 2), \overline{0B} = (1, 1, 1)$, and $\overline{0C} = (5, 5, 7)$ all lie in one plane.

PROBLEM SHEET 9

9.1 The figure $ABCD$ has vertices at $(0,0),(2,0),(3,1)$ and $(1,1)$.

Find the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$. Find $\overrightarrow{AC}\cdot\overrightarrow{BD}$.

Hence show that the angles between the diagonals of $ABCD$ have cosine $-1/\sqrt{5}$.

9.2 Show that the vectors $\mathbf{a}=\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $\mathbf{b}=-2\mathbf{i}+6\mathbf{j}-4\mathbf{k}$ are perpendicular.

Obtain any vector $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$ which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

9.3 Find the value of λ such that the vectors $(\lambda,2,-1)$ and $(1,1,-3\lambda)$ are perpendicular.

9.4 Find a constant vector parallel to the line given parametrically by

$$x=1-\lambda, y=2+3\lambda, z=1+\lambda.$$

9.5 A circular cone has its vertex at the origin and its axis in the direction of the unit vector $\hat{\mathbf{a}}$. The half-angle at the vertex is α. Show that the position vector $\mathbf{r}$ of a general point on its surface satisfies the equation

$$\hat{\mathbf{a}}\cdot\mathbf{r}=|\mathbf{r}|\cos\alpha.$$

Obtain the cartesian equation when $\hat{\mathbf{a}}=(2/7,-3/7,-6/7)$ and $\alpha=60^\circ$.

PROBLEM SHEET 10

10.1 For vectors **a** and **b**, show

$$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}| = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2) \quad \text{and} \quad \mathbf{a}\cdot\mathbf{b} = \frac{1}{4}(|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2)$$

10.2 In component form, let $\mathbf{a} = (1, -2, 2)$, $\mathbf{b} = (3, -1, -1)$, and $\mathbf{c} = (-1, 0, -1)$. Evaluate

$$\mathbf{a}\times\mathbf{b}, \qquad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}), \qquad \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}).$$

10.3 What is the geometrical significance of $\mathbf{a}\times\mathbf{b} = \mathbf{0}$?

10.4 Show that the vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ are perpendicular. Find a vector which is perpendicular to **a** and **b**.

10.5 Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-coplanar vectors, and **v** be any vector. Show that **v** can be expressed as

$$\mathbf{v} = X\mathbf{a} + Y\mathbf{b} + Z\mathbf{c}$$

where X, Y, Z, are constants given by

$$X = \frac{\mathbf{v}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}, \qquad Y = \frac{\mathbf{v}\cdot(\mathbf{c}\times\mathbf{a})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}, \qquad Z = \frac{\mathbf{v}\cdot(\mathbf{a}\times\mathbf{b})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}.$$

(Hint: start by forming, say, $\mathbf{v}\cdot(\mathbf{b}\times\mathbf{c})$).

PROBLEM SHEET 11

11.1 Integrate $\cos(3x+4)$.

11.2 Integrate $(1-2x)^{10}$.

11.3 Integrate e^{4x-1}.

11.4 Integrate $(4x+3)^{-1}$.

11.5 Find the equation of the curve passing through the point $(1,2)$ satisfying $\mathrm{d}y/\mathrm{d}x = 2x$.

11.6 A particle has acceleration $(3t^2+4)\ ms^{-2}$ at time t seconds. If its initial speed is $5ms^{-1}$, what is its speed at time $t=2$ seconds?

11.7 Find the area between the graph of $y = \sin x$ and the $x-$axis from $x=0$ to $x=\pi/2$.

11.8 Find the area between the graph

$$y = \frac{1}{x-1}$$

and the $x-$axis between $x=2$ and $x=3$.

11.9 Find the signed area between the graph $y=2x+1$ and the x-axis between $x=-1$ and $x=3$.

11.10 Find y, given that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sin x - \frac{4}{x^3}.$$