MULTIDIMENSIONAL TRANSONIC SHOCK WAVES AND FREE BOUNDARY PROBLEMS

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Abstract. We are concerned with free boundary problems arising from the analysis of multidimensional transonic shock waves for the Euler equations in compressible fluid dynamics. In this expository paper, we survey some recent developments in the analysis of multidimensional transonic shock waves and corresponding free boundary problems for the compressible Euler equations and related nonlinear partial differential equations (PDEs) of mixed type. The nonlinear PDEs under our analysis include the steady Euler equations for potential flow, the steady full Euler equations, the unsteady Euler equations for potential flow, and related nonlinear PDEs of mixed elliptic-hyperbolic type. The transonic shock problems include the problem of steady transonic flow past solid wedges, the von Neumann problem for shock reflection-diffraction, and the Prandtl-Meyer problem for unsteady supersonic flow onto solid wedges. We first show how these longstanding multidimensional transonic shock problems can be formulated as free boundary problems for the compressible Euler equations and related nonlinear PDEs of mixed type. Then we present an effective nonlinear method and related ideas and techniques to solve these free boundary problems. The method, ideas, and techniques should be useful to analyze other longstanding and newly emerging free boundary problems for nonlinear PDEs.

Contents

2010 *Mathematics Subject Classification.* 35Q31, 35M10, 35M12, 35R35, 35L65, 35B30, 35B40, 35D30, 35B65, 35J70, 76H05, 35L67, 35B45, 35B35, 35B36, 35L20, 35J67, 76N10, 76L05, 76N20, 76G25.

Key words and phrases. Nonlinear method, iteration scheme, free boundary problems, multidimensional transonic shocks, nonlinear PDEs, mixed type, mixed elliptic-hyperbolic type, Euler equations, potential flow, full Euler flow, sonic curves/surfaces, shock reflection/diffraction, von Neumann problem, von Neumann detachment conjecture, Prandtl-Meyer problem, degenerate ellipticity, *a priori* estimates, degree theory, geometric properties, convexity.

1. Introduction

We are concerned with free boundary problems arising from the analysis of multidimensional transonic shock waves for the Euler equations in compressible fluid dynamics. The purpose of this expository paper is to survey some recent developments in the analysis of multidimensional (M-D) transonic shock waves and corresponding free boundary problems for the Euler equations and related nonlinear partial differential equations (PDEs) of mixed type. We show how several M-D transonic shock problems can be formulated as free boundary problems for the compressible Euler equations and related nonlinear PDEs of mixed-type, and then present an efficient nonlinear method and related ideas and techniques to solve these free boundary problems.

Shock waves are steep wavefronts, which are fundamental in high-speed fluid flows (*e.g.*, [8, 9, 20, 35, 52, 53, 63, 90, 92, 103, 104, 109]). Such flows are governed by the compressible Euler equations in fluid dynamics. The time-dependent compressible Euler equations are a second-order nonlinear wave equation for potential flow, or a first-order nonlinear system of hyperbolic conservation laws for full Euler flow (*e.g.*, [21,35,52,53]). One of the main features of such nonlinear PDEs is that, no mater how smooth the given initial data start with, the solution develops singularity in a finite time to form shock waves (shocks, for short) generically, so that the classical notion of solutions has to be extended to the notion of entropy solutions in order to accommodate such discontinuity waves for physical variables, *that is*, the weak solutions satisfying the entropy condition that is consistent with the second law of thermodynamics (*cf.* [35, 52, 53, 73]).

General entropy solutions involving shocks for such PDEs have extremely complicated and rich structures. On the other hand, many fundamental problems in physics and engineering concern steady solutions (*i.e.*, time-independent solutions) or self-similar solutions (*i.e.*, the solutions depend only on the self-similar variables with form $\frac{\mathbf{x}}{t}$ for the space variables $\mathbf{x}$ and time-variable t); see [35, 52, 53, 63] and the references cited therein. Such solutions are governed by the steady or self-similar compressible Euler equations for potential flow, or full Euler flow. These governing PDEs in the new forms are time-independent and often are of mixed elliptic-hyperbolic type.

Mathematically, *M-D transonic shocks* are codimension-one discontinuity fronts in the solutions of the steady or self-similar Euler equations and related nonlinear PDEs of mixed elliptic-hyperbolic type, which separate two phases: one of them is supersonic phase (*i.e.*, the fluid speed is larger than the sonic speed) which is hyperbolic; the other is subsonic phase (*i.e.*, the fluid speed is smaller than the sonic speed) which is elliptic for potential flow, or elliptic-hyperbolic composite for full Euler flow (*i.e.*, elliptic equations coupled with some hyperbolic transport equations). They are formed in many physical situations, for example, by smooth supersonic flows or supersonic shock waves impinging onto solid wedges/cones or passing through de Laval nozzles, around supersonic or near-sonic flying bodies, or other physical processes. The mathematical analysis of shocks at least dates back to Stokes [101] and Riemann [95], starting from the one-dimensional (1-D) case. The mathematical understanding of M-D transonic shocks has been one of the most challenging and longstanding scientific research directions (*cf.* [35, 47, 51–53, 63, 65]). Such transonic shocks can be formulated as *free boundary problems* (FBPs) in the mathematical theory of nonlinear PDEs involving mixed elliptic-hyperbolic type.

Generally speaking, a *free boundary problem* is a boundary value problem for a PDE or system of PDEs which is defined in a domain, a part of whose boundary is *a priori unknown*; this part is accordingly named as a *free boundary*. The mathematical problem is then to determine both the location of the free boundary and the solution of the PDE/system in the resulting domain, which requires to combine analysis and geometry in sophisticated ways. The mathematical analysis of FBPs is one of the most important research directions in the analysis of PDEs, with wide applications across the sciences and real-world problems. On the other hand, it is widely regarded as a truly challenging field of mathematics. See [13, 14, 40, 54, 60, 70] and the references cited therein.

Transonic shock problems for steady or self-similar solutions are typically formulated as boundary value problems for a nonlinear PDE or system of mixed elliptic-hyperbolic type, whose type at a point is determined by the solution, as well as its gradient for some cases. For a system, the type is more complicated and may be either hyperbolic or mixed-composite elliptic-hyperbolic (also called *mixed*, for the sake of brevity when no confusion arises). General solutions of such nonlinear PDEs can be nonsmooth and of complicated structures (*e.g.* [17–19, 21, 35, 63, 72, 75, 78, 97, 98, 114, 116]), so that even the uniqueness issue has not been settled in many cases. However, in many problems, especially those motivated by physical phenomena, the expected structures of solutions are known from experimental/numerical results and underlying physics. The solutions are expected to be piecewise smooth, with some hyperbolic/elliptic regions separated by shocks, or sonic curves/surfaces of continuous type-transition (*i.e.*, the type of equations changes continuously in the physical variables such as the velocity, density, *etc.*). In this paper, we present the problems in which the hyperbolic part of the solution is *a priori* known, or can be determined separately from the elliptic part, in some larger regions. Then the problem is reduced to determining the region in which the underlying PDE is elliptic, with the transonic shock as a part of its boundary and the elliptic solution in that region. In other words, we need to solve a free boundary problem for the elliptic phase of the solution, with the transonic shock as a free boundary. Since the type of equations depends on the solution itself, the ellipticity in the region is a part of the results to be established. We remark that, in some other problems involving shocks, FBPs also need to be solved in order to find the hyperbolic part of the solution, which is beyond the scope of this paper.

For several problems under our discussion below, the PDEs involved are single second-order quasilinear PDEs, whose coefficients and types (elliptic, hyperbolic, or mixed) depend on the gradient of the solution. In the other problems, the PDEs are first-order nonlinear systems, whose types are hyperbolic or composite-mixed elliptic-hyperbolic, and are determined by the solution only. In all the problems, the PDEs (or parts of the systems) are expected to be elliptic for our solutions in the regions determined by the free boundary problems. That is, we solve an expected elliptic free boundary problem. However, the available methods and approaches of elliptic FBPs do not directly apply to our problems, such as the variational methods of Alt-Caffarelli [1] and Alt-Caffarelli-Friedman [2–4], the Harnack inequality approach of Caffarelli [10–12], and other methods and approaches in many further works. The main reason is that the type of equations needs to be first controlled in order to apply these methods, which requires some strong estimates *a priori*. To overcome the difficulties, we exploit the global structure of the problems, which allows us to derive certain properties of the solution (such as the monotonicity, *etc.*) so that the type of equations and the geometry of the problem can be controlled. With this, we solve the free boundary problem by the iteration procedure.

Notice that the existence of multiple wild solutions for the Cauchy problem of the compressible Euler equations has been shown; see [50, 71] and the references cited therein for both the isentropic and full Euler cases. In this paper, we focus on the solutions of specific structures motivated by underlying physics; for these solutions, the uniqueness can be shown for all the cases as we discuss below. Since we are interested in the solutions of specific structures, we construct the solution in a carefully chosen class of solutions, called admissible solutions. This class of solutions needs to be defined with two somewhat opposite features: the conditions need not only to be flexible enough so that this class contains all possible solutions of the problem which are of the desired structure, but also to be rigid enough to force the desired structure of the solutions with the sufficient analytic and geometric control such that the expected estimates for these solutions can be derived, so that eventually a solution can be constructed in this class by the iteration procedure. In order to define such a class, we start with the solutions near some background solutions:

(i) to make sure that the solutions obtained are still in the same desired structure via careful estimates, which is the structure of transonic shock solutions in our application;
(ii) to gain the insight and motivation for the structure and properties of the solutions that are not near the background solution but have the required configuration to form the conditions on which the *a priori* estimates and fixed point argument are based.

In several problems, we consider only the solutions near the background solution, as in §2–§3 below. In the other problems, say in §4–§5, we carry out both steps described above and construct admissible solutions which are not close to any known background solution.

Furthermore, we emphasize that the elliptic and hyperbolic regions may be separated not only by shocks, which are discontinuity fronts of physical variables such as the velocity and the density, but also by sonic curves/surfaces where the type of equations changes continuously in the physical variables, as pointed out earlier. This means that the ellipticity and hyperbolicity degenerate near the sonic curves/surfaces. This presents additional difficulties in the analysis of such solutions. Moreover, the sonic curves/surfaces may intersect the transonic shocks (see *e.g.* Fig. 4.1, point P_1) so that, near such points, the analysis of solutions is even more involved.

The organization of this paper is as follows: In §2, we start with our presentation of M-D transonic shocks and free boundary problems for the compressible Euler equations for potential flow in a setup as simple as possible, and show how a transonic shock problem can be formulated as a free boundary problem for the corresponding nonlinear PDEs of mixed elliptic-hyperbolic type. Then we describe an efficient nonlinear method and related ideas and techniques, first developed in Chen-Feldman [29], with focus on the key points in solving such free boundary problems through this simplest setup. In §3, we describe how they can be applied to establishing the existence, stability, and asymptotic behavior of 2-D steady transonic flows with transonic shocks past curved wedges for the full Euler equations, by reformulating the problems as free boundary problems via two different approaches. In §4, we describe how the nonlinear method and related ideas and techniques presented in §2–3 can be extended to the case of self-similar shock reflection/diffraction for the compressible Euler equations for potential flow, including the von Neumann problem for shock reflection-diffraction and the Prandtl-Meyer problem for unsteady supersonic flow onto solid wedges, where the solutions have the sonic arcs in addition to the transonic shocks. In §5, we discuss some recent developments in the analysis of geometric properties of transonic shocks as free boundaries in the 2-D self-similar coordinates for compressible fluid flows with focus on the convexity properties of the self-similar transonic shocks obtained in §4.

2. Multidimensional Transonic Shocks and Free Boundary Problems for the Steady Euler Equations for Potential Flow

For clarity, we start with our presentation of M-D transonic shocks and free boundary problems for the compressible Euler equations in a setup as simple as possible, and show how a transonic shock problem can be formulated as a free boundary problem for the corresponding nonlinear PDEs of mixed elliptic-hyperbolic type. Then we describe a method first developed in Chen-Feldman [29], with focus on the key points to solve such free boundary problems through this simplest setup.

The steady Euler equations for potential flow, consisting of the conservation law of mass and the Bernoulli law for the velocity, can be written as the following second-order nonlinear PDE of mixed elliptic-hyperbolic type for the velocity potential $\varphi : \mathbb{R}^d \to \mathbb{R}$ (*i.e.*, $\mathbf{u} = D\varphi$ is the velocity):

$$\operatorname{div}\left(\rho(|D\varphi|^2)D\varphi\right) = 0, \tag{2.1}$$

by scaling so that the density function $\rho(q^2)$ has the form:

$$\rho(q^2) = \big(1 - \frac{\gamma-1}{2}q^2\big)^{\frac{1}{\gamma-1}}, \tag{2.2}$$

where $\gamma > 1$ is the adiabatic exponent and $D := (\partial_{x_1}, \dots, \partial_{x_d})$ is the gradient with respect to $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$.

Equation (2.1) can be written in the non-divergence form:

$$\sum_{i,j=1}^{d} \big(\rho(|D\varphi|^2)\delta_{ij} + 2\rho'(|D\varphi|^2)\varphi_{x_i}\varphi_{x_j}\big)\varphi_{x_i x_j} = 0, \tag{2.3}$$

where the coefficients of the second-order nonlinear PDE (2.3) depend on $D\varphi$, the gradient of the unknown function φ.

The nonlinear PDE (2.1), or equivalently (2.3) for smooth solutions, is strictly elliptic at $D\varphi$ with $|D\varphi| = q$ if

$$\rho(q^2) + 2q^2\rho'(q^2) > 0, \tag{2.4}$$

and is strictly hyperbolic if

$$\rho(q^2) + 2q^2\rho'(q^2) < 0. \tag{2.5}$$

In fluid dynamics, the elliptic regions of equation (2.1) correspond to the *subsonic flow*, the hyperbolic regions of (2.1) to the *supersonic flow*, and the regions with $\rho(q^2) + 2q^2\rho'(q^2) = 0$ for $q = |D\varphi|$ to the *sonic flow*.

2.1. **Steady Transonic Shocks and Free Boundary Problems.** Let $\Omega \subset \mathbb{R}^d$ be a domain (*i.e.*, simply connected open subset). A function $\varphi \in W^{1,\infty}(\Omega)$ is a *weak solution* of (2.1) in Ω if

(i) $|D\varphi(\mathbf{x})| \leqslant \sqrt{2/(\gamma-1)}$ *a.e.* $\mathbf{x} \in \Omega$, that is, the physical region so that $\rho(|D\varphi(\mathbf{x})|^2)$ is well defined via (2.2) for *a.e.* $\mathbf{x} \in \Omega$;

(ii) for any test function $\zeta \in C_0^\infty(\Omega)$,

$$\int_\Omega \rho(|D\varphi|^2) D\varphi \cdot D\zeta \,\mathrm{d}\mathbf{x} = 0. \tag{2.6}$$

We are interested in the weak solutions with shocks (*i.e.*, the surfaces of jump discontinuity of $D\varphi$ of the solution φ with codimension one) satisfying the physical entropy condition that is consistent with the Second Law of Thermodynamics in Continuum Physics. More precisely, let Ω^+ and Ω^- be open nonempty subsets of Ω such that

$$\Omega^+ \cap \Omega^- = \varnothing, \qquad \overline{\Omega^+} \cup \overline{\Omega^-} = \overline{\Omega},$$

and $\mathcal{S} := \partial\Omega^+ \backslash \partial\Omega$. Let $\varphi \in W^{1,\infty}(\Omega)$ be a weak solution of (2.1) so that $\varphi \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ and $D\varphi$ has a jump across $\mathcal{S}$.

We now derive the necessary conditions on $\mathcal{S}$ that is a C^1–surface of codimension one. First, the requirement that φ is in $W^{1,\infty}(\Omega)$ yields $\mathrm{curl}(D\varphi) = 0$ in the sense of distributions, which implies

$$\varphi^+_{\boldsymbol{\tau}} = \varphi^-_{\boldsymbol{\tau}} \qquad \text{on } \mathcal{S}, \tag{2.7}$$

where

$$\varphi^\pm_{\boldsymbol{\tau}} := D\varphi^\pm - (D\varphi^\pm \cdot \boldsymbol{\nu})\boldsymbol{\nu}$$

are the trace values of the tangential gradients of φ on $\mathcal{S}$ in the tangential space with $(d-1)$-dimension on the $\Omega^\pm$ sides, respectively, and $\boldsymbol{\nu}$ is the unit normal to $\mathcal{S}$ from Ω^- to Ω^+. Then we simply write $\varphi_{\boldsymbol{\tau}} := \varphi^\pm_{\boldsymbol{\tau}}$ on $\mathcal{S}$ and choose

$$\varphi^+ = \varphi^- \qquad \text{on } \mathcal{S} \tag{2.8}$$

to be consistent with the $W^{1,\infty}$–requirement of φ.

Now, for $\zeta \in C_0^\infty(\Omega)$, we use (2.6) to compute

$$\begin{aligned}
0 &= \Big(\int_{\Omega^+} + \int_{\Omega^-}\Big) \rho(|D\varphi|^2) D\varphi \cdot D\zeta \,\mathrm{d}\mathbf{x} \\
&= -\int_{\partial\Omega^+} \rho(|D\varphi|^2) D\varphi \cdot \boldsymbol{\nu}\,\zeta \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\Omega^-} \rho(|D\varphi|^2) D\varphi \cdot \boldsymbol{\nu}\,\zeta \,\mathrm{d}\mathcal{H}^{d-1} \\
&= \int_{\mathcal{S}} \big(-\rho(|D\varphi^+|^2) D\varphi^+ \cdot \boldsymbol{\nu} + \rho(|D\varphi^-|^2) D\varphi^- \cdot \boldsymbol{\nu}\big)\,\zeta \,\mathrm{d}\mathcal{H}^{d-1},
\end{aligned}$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-D Hausdorff measure, *i.e.*, the surface area measure. Thus, the other condition on $\mathcal{S}$, which measures the trace jump of the normal derivative of φ across $\mathcal{S}$, is

$$\rho(|D\varphi^+|^2)\varphi^+_{\boldsymbol{\nu}} = \rho(|D\varphi^-|^2)\varphi^-_{\boldsymbol{\nu}} \qquad \text{on } \mathcal{S}, \tag{2.9}$$

where $\varphi^\pm_{\boldsymbol{\nu}} = D\varphi^\pm \cdot \boldsymbol{\nu}$ are the trace values of the normal derivative of φ along $\mathcal{S}$ on the $\Omega^\pm$ sides, and

$$\rho(|D\varphi^\pm|^2) = \Big(1 - \frac{\gamma-1}{2}\big(|\varphi^\pm_{\boldsymbol{\tau}}|^2 + |\varphi^\pm_{\boldsymbol{\nu}}|^2\big)\Big)^{\frac{1}{\gamma-1}},$$

respectively.

Conditions (2.8)–(2.9) are called the Rankine-Hugoniot conditions for potential flow in fluid dynamics. On the other hand, it can also be shown that any $\varphi \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ that is a C^2–solution of (2.1) in $\Omega^\pm$ respectively, such that $D\varphi$ has a jump across $\mathcal{S}$ satisfying the Rankine-Hugoniot conditions (2.8)–(2.9), must be a weak solution of (2.1) in the whole domain Ω. Therefore, the necessary and sufficient conditions for $\varphi \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ that is a solution of (2.1) in $\Omega^\pm$ respectively to be a weak solution of (2.1) in the whole domain Ω are the Rankine-Hugoniot conditions (2.8)–(2.9).

For given $K > 0$, consider the function:

$$\Phi_K(p) := \big(K - \frac{\gamma-1}{2}p^2\big)^{\frac{1}{\gamma-1}} p \qquad \text{for } p \in [0, \sqrt{2K/(\gamma-1)}]. \tag{2.10}$$

Then $\Phi_K \in C([0, \sqrt{2K/(\gamma-1)}])$ and

$$\Phi_K(p) > 0 \text{ for } p \in (0, \sqrt{2K/(\gamma-1)}), \qquad \Phi_K(0) = \Phi_K(\sqrt{2K/(\gamma-1)}) = 0, \tag{2.11}$$

$$0 < \Phi_K'(p) \leqslant K^{\frac{1}{\gamma-1}} \text{ for } p \in (0, p^K_{\text{sonic}}), \tag{2.12}$$

$$\Phi_K'(p) < 0 \text{ for } p \in (p^K_{\text{sonic}}, \sqrt{2K/(\gamma-1)}), \tag{2.13}$$

$$\Phi_K''(p) < 0 \text{ for } p \in (0, p^K_{\text{sonic}}], \tag{2.14}$$

where

$$p^K_{\text{sonic}} := \sqrt{2K/(\gamma+1)}. \tag{2.15}$$

By direct calculation, condition (2.4) is equivalent to $\Phi_1'(q) > 0$, and condition (2.5) is equivalent to $\Phi_1'(q) < 0$. Thus, using (2.12), we obtain that PDE (2.1) is strictly elliptic at $D\varphi$ if $|D\varphi| < p^1_{\text{sonic}}$ and is strictly hyperbolic if $|D\varphi| > p^1_{\text{sonic}}$, where we have used notation (2.15).

Suppose that $\varphi(x)$ is a solution satisfying

$$|D\varphi| < p^1_{\text{sonic}} = \sqrt{2/(\gamma+1)} \text{ in } \Omega^+, \qquad |D\varphi| > p^1_{\text{sonic}} \text{ in } \Omega^-, \tag{2.16}$$

and

$$D\varphi^\pm \cdot \boldsymbol{\nu} > 0 \qquad \text{on } \mathcal{S}, \tag{2.17}$$

besides (2.8) and (2.9). Then $\varphi(x)$ is a *transonic shock solution* with *transonic shock* $\mathcal{S}$ that divides the *subsonic region* Ω^+ from the *supersonic region* Ω^-. In addition, $\varphi(\mathbf{x})$ satisfies the *physical entropy condition* (see Courant-Friedrichs [52]; also see [53, 73]):

$$\rho(|D\varphi^-|^2) < \rho(|D\varphi^+|^2), \tag{2.18}$$

which implies, by (2.17), that *the density ρ increases in the flow direction*; that is, the transonic shock solution is an entropy solution. Note that equation (2.1) is elliptic in the subsonic region Ω^+ and hyperbolic in the supersonic region Ω^-.

For clarity of presentation of the nonlinear method, first developed in Chen-Feldman [29], we focus first on the free boundary problem in the simplest setup, while the method and related ideas and techniques have been applied to more general free boundary problems involving transonic shocks for the nozzle problems and other important problems, some of which will be discussed in §3–§5.

Let $(\mathbf{x}', x_d)$ be the coordinates of $\mathbb{R}^d$ with $\mathbf{x}' = (x_1, \cdots, x_{d-1}) \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. From now on, in this section, we focus on $\Omega := (0,1)^{d-1} \times (-1, 1)$ for simplicity, without loss of our main objectives.

Let $q^- \in (p^1_{\text{sonic}}, \sqrt{2/(\gamma-1)})$ and $\varphi_0^-(x) := q^- x_d$. Then φ_0^- is a supersonic solution in Ω. From (2.11)–(2.14), there exists a unique $q^+ \in (0, p^1_{\text{sonic}})$ such that

$$\big(1 - \frac{\gamma-1}{2}(q^+)^2\big)^{\frac{1}{\gamma-1}} q^+ = \big(1 - \frac{\gamma-1}{2}(q^-)^2\big)^{\frac{1}{\gamma-1}} q^-. \tag{2.19}$$

In particular, $q^+ < q^-$. Define $\varphi_0^+(\mathbf{x}) := q^+ x_d$ in Ω. Then the function:

$$\varphi_0(\mathbf{x}) = \min(\varphi_0^+(\mathbf{x}), \varphi_0^-(\mathbf{x})) \tag{2.20}$$

is a transonic shock solution in Ω, in which $\Omega_0^- = \{x_d \leqslant 0\} \cap \Omega$ and $\Omega_0^+ = \{x_d \geqslant 0\} \cap \Omega$ are the supersonic and subsonic regions of $\varphi_0(\mathbf{x})$, respectively. Also, the boundary condition: $(\varphi_0)_{\boldsymbol{\nu}} = 0$ holds on $\partial(0,1)^{d-1} \times [-1,1]$.

We start with perturbations of the background solution $\varphi_0(\mathbf{x})$ defined in (2.20). We use the following Hölder norms: For $\alpha \in (0,1)$ and any non-negative integer k,

$$[u]_{k,0,\Omega} = \sum_{|\boldsymbol{\beta}|=k} \sup_{x\in\Omega} |D^{\boldsymbol{\beta}} u(\mathbf{x})|, \qquad [u]_{k,\alpha,\Omega} = \sum_{|\boldsymbol{\beta}|=k} \sup_{\mathbf{x},\mathbf{y}\in\Omega, \mathbf{x}\neq\mathbf{y}} \frac{|D^{\boldsymbol{\beta}} u(\mathbf{x}) - D^{\boldsymbol{\beta}} u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}, \tag{2.21}$$

$$\|u\|_{k,0,\Omega} = \sum_{j=0}^{k} [u]_{j,0,\Omega}, \qquad \|u\|_{k,\alpha,\Omega} = \|u\|_{k,0,\Omega} + [u]_{k,\alpha,\Omega},$$

where $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_d)$, $\beta_l \geqslant 0$ integers, $D^{\boldsymbol{\beta}} = \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d}$, and $|\boldsymbol{\beta}| = \beta_1 + \cdots + \beta_d$.

Then the transonic shock problem can be formulated as the following problem:

Problem 2.1. *Given a supersonic solution φ^- of (2.1) in Ω, which is a $C^{2,\alpha}$–perturbation of φ_0^-:*

$$\|\varphi^- - \varphi_0^-\|_{2,\alpha,\Omega} \leqslant \sigma \tag{2.22}$$

for some $\alpha \in (0,1)$ with small $\sigma > 0$ and satisfies

$$\varphi^-_{\boldsymbol{\nu}} = 0 \qquad \text{on } \ \partial(0,1)^{d-1} \times [-1,1], \tag{2.23}$$

find a transonic shock solution φ in Ω such that

$$\varphi = \varphi^- \qquad \text{in } \ \Omega^- := \Omega \backslash \overline{\Omega^+},$$

where $\Omega^+ := \{\mathbf{x} \in \Omega \ : \ |D\varphi(\mathbf{x})| < p^1_{\text{sonic}}\}$ is the subsonic region of φ, which is the complementary set of the supersonic region of φ in Ω, and

$$\begin{cases} (\varphi, \varphi_{x_n}) = (\varphi^-, \varphi^-_{x_n}) & \text{on } \ (0,1)^{d-1} \times \{-1\}, \\ \varphi = \varphi_0^+ & \text{on } \ (0,1)^{d-1} \times \{1\}, \\ \varphi_{\boldsymbol{\nu}} = 0 & \text{on } \ \partial(0,1)^{d-1} \times [-1,1]. \end{cases} \tag{2.24}$$

Since $\varphi = \varphi^-$ in Ω^-, $|D\varphi| < p^1_{\text{sonic}} < |D\varphi^-|$ in Ω^+, $|D\varphi^-| \sim \partial_{x_d}\varphi^- > p^1_{\text{sonic}}$ in Ω, and it is expected that $\Omega^+ = \{x_d > f(\mathbf{x}')\} \cap \Omega$ and $|D\varphi| \sim \partial_{x_d}\varphi < p^1_{\text{sonic}}$ in Ω^+ with (2.8) across the transonic shock $\mathcal{S} = \{x_d = f(\mathbf{x}')\} \cap \Omega$, then φ should satisfy

$$\varphi(\mathbf{x}) \leqslant \varphi^-(\mathbf{x}) \qquad \text{for } \ \mathbf{x} \in \Omega. \tag{2.25}$$

This motivates the following reformulation of Problem 2.1 as a free boundary problem for the subsonic (elliptic) part of the solution:

Problem 2.2 (Free Boundary Problem)**.** *Find $\varphi \in C(\overline{\Omega})$ such that*

(i) *φ satisfies (2.25) in Ω and (2.24) on $\partial\Omega$;*
(ii) *$\varphi \in C^{2,\alpha}(\overline{\Omega^+})$ is a solution of (2.1) in $\Omega^+ = \{\mathbf{x} \in \Omega \ : \ \varphi(\mathbf{x}) < \varphi^-(\mathbf{x})\}$, the non-coincidence set;*
(iii) *the free boundary $\mathcal{S} = \partial\Omega^+ \cap \Omega$ is given by $x_d = f(\mathbf{x}')$ for $\mathbf{x}' \in (0,1)^{d-1}$ so that $\Omega^+ = \{x_d > f(\mathbf{x}') \ : \ x' \in (0,1)^{d-1}\}$ with $f \in C^{2,\alpha}([0,a]^{d-1})$;*

(iv) *the free boundary condition* (2.9) *holds on* $\mathcal{S}$.

In the free boundary problem (Problem 2.2) above, phase φ^- is not required to be a solution of (2.1) and φ is not necessary to be subsonic in Ω^+, although we require the subsonicity in Problem 2.1 so that the free boundary is a transonic shock.

It is proved in Chen-Feldman [29] that, if perturbation $\varphi^- - \varphi_0^-$ is small enough in $C^{2,\alpha}$, then the free boundary problem (Problem 2.2) has a solution that is subsonic on Ω^+, so that Problem 2.1 has a transonic shock solution. Furthermore, the transonic shock is stable under any small $C^{2,\alpha}$–perturbation of φ^-.

Theorem 2.1 (Chen-Feldman [29])**.** *Let* $q^+ \in (0, p^1_{\text{sonic}})$ *and* $q^- \in (p^1_{\text{sonic}}, \sqrt{2/(\gamma-1)})$ *satisfy* (2.19). *Then there exist positive constants* σ_0, C_1, *and* C_2 *depending only on* (q^+, d, γ) *and* Ω *such that, for every* $\sigma \leqslant \sigma_0$ *and any function* φ^- *satisfying* (2.22)–(2.23), *there exists a unique solution* φ *of the free boundary problem,* Problem 2.2, *satisfying*

$$\|\varphi - \varphi_0^+\|_{2,\alpha,\Omega^+} \leqslant C_1\sigma$$

and $|D\varphi| < p^1_{\text{sonic}}$ *in* Ω^+. *Moreover,* $\Omega^+ = \{x_d > f(\mathbf{x}')\} \cap \Omega$ *with* $f : \mathbb{R}^{d-1} \to \mathbb{R}$ *satisfying*

$$\|f\|_{2,\alpha,\mathbb{R}^{d-1}} \leqslant C_2\sigma, \qquad D_{\mathbf{x}'}f(\mathbf{x}') = 0 \ \text{ on } \partial(0,1)^{d-1},$$

that is, the free boundary $\mathcal{S} = \{(\mathbf{x}', x_d) \,:\, x_d = f(\mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{d-1}\} \cap \Omega$ *is in* $C^{2,\alpha}$ *and orthogonal to* $\partial\Omega$ *at their intersection points.*

In particular, we obtain

Corollary 2.1. *Let* $q^\pm$ *be as in* Theorem 2.1, *and let* σ_0 *be the constant defined in* Theorem 2.1. *If* $\varphi^-(\mathbf{x})$ *is a supersonic solution of* (2.1) *satisfying* (2.22)–(2.23) *with* $\sigma \leqslant \sigma_0$, *then there exists a transonic shock solution* φ *of* Problem 2.1 *with shock* $\mathcal{S} = \{(\mathbf{x}', x_d) \,:\, x_d = f(\mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{d-1}\} \cap \Omega$ *such that* φ *and* f *satisfy the properties stated in* Theorem 2.1.

Indeed, under the conditions of Corollary 2.1, solution φ of Problem 2.2 obtained in Theorem 2.1, along with the free boundary $\mathcal{S} = \{(\mathbf{x}', x_d) \,:\, x_d = f(\mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{d-1}\} \cap \Omega$, forms a transonic shock solution of Problem 2.1.

The following features of equation (2.1) and the free boundary condition (2.9) are employed in the proof of Theorem 2.1.

(i) The nonlinear equation (2.1) is uniformly elliptic only if $|D\varphi| < p^1_{\text{sonic}} - \varepsilon$ in Ω^+ for some $\varepsilon > 0$;
(ii) $|D\varphi^+| = \left(|\varphi^+_{\boldsymbol{\nu}}|^2 + |\varphi_{\boldsymbol{\tau}}|^2\right)^{1/2}$ on $\mathcal{S}$ is subsonic only if $\varphi_{\boldsymbol{\tau}}$ is sufficiently small;
(iii) The free boundary condition (2.9) is uniformly non-degenerate (*i.e.*, $\varphi^-_{\boldsymbol{\nu}} - \varphi^+_{\boldsymbol{\nu}}$ is bounded from below by a positive constant on $\mathcal{S}$) only if $\varphi^-_{\boldsymbol{\nu}} > p^K_{\text{sonic}} + \varepsilon$ on $\mathcal{S}$ for some $\epsilon > 0$ with $K = 1 - \frac{\gamma-1}{2}|\varphi_{\boldsymbol{\tau}}|^2$.

By (2.22), these conditions hold if, for any $\mathbf{x} \in \mathcal{S}$, the unit normal $\boldsymbol{\nu}(\mathbf{x})$ to $\mathcal{S}$ is sufficiently close to being orthogonal to $\{x_d = 0\}$.

2.2. **A Nonlinear Method for Solving the Free Boundary Problems for Nonlinear PDEs of Mixed Elliptic-Hyperbolic Type.** We now describe a nonlinear method and related ideas and techniques, developed first in Chen-Feldman [29], for the construction of solutions of the free boundary problems for nonlinear PDEs of mixed elliptic-hyperbolic type, through Problem 2.2 as the simplest setup. We present the version of the method that is restricted to this setup. The key ingredient is an iteration scheme, based on the non-degeneracy of the free boundary condition: the jump of the normal derivative of solutions across the free boundary has a strict lower bound. Since the PDE is of mixed type, we make a cutoff (truncation) of the nonlinearity near the value related to the background solution in order to fix the type of equation (to make it elliptic everywhere) and, at the fixed point of the iteration, we remove the cutoff eventually by a required estimate. The iteration set consists of the functions close to the background solution – in the $C^{2,\alpha}$–norm in the present case. Then, for each function from the iteration set, the nondegeneracy allows of using one of the Rankine-Hugoniot conditions, equality (2.8), to define the iteration free boundary, which is a smooth graph. In domain Ω^+ determined by the iteration free boundary, we solve a boundary value problem with the truncated PDE, the condition on the shock derived from the other Rankine-Hugoniot condition (2.9) by a truncation (similar to the truncation of the PDE) and other appropriate modifications to achieve the uniform obliqueness, and the same boundary conditions as in the original problem for the iteration problem on the other parts of the boundary of the iteration domain. The solution of this iteration problem defines the iteration map. We exploit the estimates for the iteration problem to prove the existence of a fixed point of the iteration map, and then we show that a fixed point is a solution of the original problem.

In some further problems, we look for the solutions that are not close to a known background solution. Some of these problems, as well as the corresponding versions of the nonlinear method described above, are discussed in §4. A related method for the construction of perturbations of transonic shocks for the steady transonic small disturbance model was proposed in [16], in which the type of equation depends on the solution only (but not on its gradient) so that the ellipticity can be controlled by the maximum principle; also see [15].

2.2.1. Subsonic Truncations – Shiffmanization. In order to solve the free boundary problem, we first reformulate Problem 2.2 as a truncated one-phase free boundary problem, motivated by the argument introduced originally in Shiffman [100], now so called the *shiffmanization* (*cf.* Lax [74]); also see [4, pp. 87–90]. This is achieved by modifying both the nonlinear equation (2.1) and the free boundary condition (2.9) to make the equation uniformly elliptic and the free boundary condition non-degenerate. Then we solve the truncated one-phase free boundary problem with the modified equation in the downstream region, the modified free boundary condition, and the given hyperbolic phase in the upstream region. By a careful gradient estimate later on, we prove that the solution in fact solves the original problem. We note that, for the steady potential flow equation (2.1), the coefficients of its non-divergent form (2.3) depend on $D\varphi$, so the type of equation depends on $D\varphi$.

We first recall that the ellipticity condition for (2.1) at $|D\varphi| = q$ is (2.4), which is equivalent to

$$\Phi_1'(q) > 0, \tag{2.26}$$

where $\Phi_K(p)$ is the function defined in (2.10). By (2.12), inequality (2.26) holds for $q \in (0, p^1_{\rm sonic})$.

The truncation is done by modifying $\Phi_1(q)$ so that the new function $\tilde\Phi_1(q)$ satisfies (2.26) uniformly for all $q > 0$ and, around q^+, $\tilde\Phi_1(q) = \Phi_1(q)$. More precisely, the procedure consists of the following steps:

1. Denote $\varepsilon := \frac{p^1_{\rm sonic} - q^+}{2}$. Let $y = c_0 q + c_1$ be the tangent line of the graph of $y = \Phi_1(q)$ at $q = p^1_{\rm sonic} - \varepsilon$. Then, using (2.12), we obtain $c_0 = \Phi_1'(p^1_{\rm sonic} - \varepsilon) > 0$. Define $\tilde\Phi_1 : [0,\infty) \to \mathbb{R}$ as

$$\tilde\Phi_1(q) = \begin{cases} \Phi_1(q) & \text{if } 0 \leqslant q < p^1_{\rm sonic} - \varepsilon, \\ c_0 q + c_1 & \text{if } q > p^1_{\rm sonic} - \varepsilon, \end{cases} \tag{2.27}$$

which satisfies $\tilde\Phi_1 \in C^{1,1}([0,\infty))$.

2. Define

$$\tilde\rho(s) = \frac{\tilde\Phi_1(\sqrt{s})}{\sqrt{s}} \qquad \text{for } s \in [0,\infty). \tag{2.28}$$

Then $\tilde\rho \in C^{1,1}([0,\infty))$ and

$$\tilde\rho(q^2) = \rho(q^2) \qquad \text{if } \ 0 \leqslant q < p^1_{\rm sonic} - \varepsilon. \tag{2.29}$$

By (2.12)–(2.14) and the definition of $\tilde\Phi_1$ in (2.27),

$$0 < c_0 = \Phi_1'(p^1_{\rm sonic} - \varepsilon) \leqslant \tilde\Phi_1'(q) = \tilde\rho(q^2) + 2q^2\tilde\rho'(q^2) \leqslant C \qquad \text{for } \ q \in (0,\infty)$$

for some constant $C > 0$. Then the equation:

$$\tilde{\mathcal{L}}\varphi := \operatorname{div}\left(\tilde\rho(|D\varphi|^2) D\varphi\right) = 0 \tag{2.30}$$

is uniformly elliptic, with ellipticity constants depending only on q^+ and γ.

3. We also do the corresponding truncation of the free boundary condition (2.9):

$$\tilde{\rho}(|D\varphi|^2)\varphi_{\boldsymbol{\nu}} = \rho(|D\varphi^-|^2)D\varphi^- \cdot \boldsymbol{\nu} \qquad \text{on } \mathcal{S}. \tag{2.31}$$

On the right-hand side of (2.31), we use the non-truncated function ρ since $\rho \neq \tilde{\rho}$ on the range of $|D\varphi^-|^2$. Note that (2.31), with the right-hand side considered as a known function, is the conormal boundary condition for the uniformly elliptic equation (2.30).

4. Introduce the function:

$$u := \varphi^- - \varphi.$$

Then, by (2.25), the problem is to find $u \in C(\overline{\Omega})$ with $u \geqslant 0$ such that

(i) $u \in C^{2,\alpha}(\overline{\Omega^+})$ is a solution of

$$\operatorname{div} A(Du, \mathbf{x}) = F(\mathbf{x}) \qquad \text{in } \Omega^+ := \{u > 0\} \cap \Omega \text{ (the non-coincidence set)}, \tag{2.32}$$

$$A(Du, \mathbf{x}) \cdot \boldsymbol{\nu} = G(\boldsymbol{\nu}, \mathbf{x}) \qquad \text{on } \mathcal{S} := \partial\Omega^+ \backslash \partial\Omega, \tag{2.33}$$

and the boundary condition on $\partial\Omega$ determined by (2.24) and $\varphi^-(\mathbf{x})$:

$$\begin{cases} u = 0 & \text{on } (0,1)^{d-1} \times \{-1\}, \\ u = \varphi^- - \varphi_0^+ & \text{on } (0,1)^{d-1} \times \{1\}, \\ u_{\boldsymbol{\nu}} = 0 & \text{on } \partial(0,1)^{d-1} \times [-1,1], \end{cases} \tag{2.34}$$

where $\boldsymbol{\nu}$ is the unit normal to $\mathcal{S}$ towards the unknown phase and

$$A(P, \mathbf{x}) = \tilde{\rho}(|D\varphi^-(\mathbf{x}) - P|^2)(D\varphi^-(\mathbf{x}) - P) - \tilde{\rho}(|D\varphi^-(\mathbf{x})|^2)D\varphi^-(\mathbf{x}) \qquad \text{for } P \in \mathbb{R}^d,$$
$$F(\mathbf{x}) = -\operatorname{div}\left(\tilde{\rho}(|D\varphi^-(\mathbf{x})|^2)D\varphi^-(\mathbf{x})\right),$$
$$G(\boldsymbol{\nu}, \mathbf{x}) = \big(\rho(|D\varphi^-(\mathbf{x})|^2) - \tilde{\rho}(|D\varphi^-(\mathbf{x})|^2)\big)D\varphi^-(\mathbf{x}) \cdot \boldsymbol{\nu}.$$

Note that condition (2.23) has been used to determine the third condition in (2.34).

(ii) the free boundary $\mathcal{S} := \partial\Omega^+ \cap \Omega = \{x_d = f(\mathbf{x}') : \mathbf{x}' \in (0,1)^{d-1}\}$ so that $\Omega^+ = \{x_d > f(\mathbf{x}')\} \cap \Omega$ with $f \in C^{2,\alpha}([0,a]^{d-1})$ and $D_{\mathbf{x}'} f = 0$ on $\partial((0,1)^{d-1} \times [-1,1])$.

2.2.2. Domain Extension. We then extend domain Ω of the truncated free boundary problem in §2.2.1 above to domain $\Omega_{\rm e}$, so that the whole free boundary lies in the interior of the extended domain. This is possible owing to the simple geometry of the domain, as considered in this section.

Notice that, for a function $\phi \in C^{2,\alpha}(\overline{\Omega})$ with $\Omega := (0,1)^{d-1} \times (-1,1)$ satisfying

$$\phi_{\boldsymbol{\nu}} = 0 \qquad \text{on } \partial(0,1)^{d-1} \times [-1,1], \tag{2.35}$$

we can extend ϕ to $\mathbb{R}^{d-1} \times [-1,1]$ so that the extension (still denoted) ϕ satisfies

$$\phi \in C^{2,\alpha}(\mathbb{R}^{d-1} \times [-1,1]),$$

and, for every $m = 1, \cdots, n-1$, and $k = 0, \pm1, \pm2, \cdots$,

$$\phi(x_1, \cdots, x_{m-1}, k - z, x_{m+1}, \cdots, x_d) = \phi(x_1, \cdots, x_{m-1}, k + z, x_{m+1}, \cdots, x_d), \tag{2.36}$$

that is, ϕ is symmetric with respect to every hyperplane $\{x_m = k\}$. Indeed, for $\mathbf{k} = (k_1, \cdots, k_{d-1}, 0)$ with integers $k_j, j = 1, \cdots d-1$, we define

$$\phi(\mathbf{x} + \mathbf{k}) = \phi(\eta(x_1, k_1), \cdots, \eta(x_{d-1}, k_{d-1}), x_d) \qquad \text{for } \mathbf{x} \in (0,1)^{d-1} \times [-1,1]$$

with

$$\eta(t,k) = \begin{cases} t & \text{if } k \text{ is even}, \\ 1-t & \text{if } k \text{ is odd}. \end{cases}$$

It follows from (2.36) that $\phi(\mathbf{x}', x_d)$ is 2-periodic in each variable of $(x_1, \cdots, x_{d-1})$:

$$\phi(\mathbf{x} + 2\mathbf{e}_m) = \phi(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \mathbb{R}^{d-1} \times [-1,1],\ m = 1, \cdots, d-1,$$

where $\mathbf{e}_m$ is the unit vector in the direction of x_m.

Thus, with respect to the 2-periodicity, we can consider ϕ as a function on $\Omega_{\mathrm{e}} := \mathbb{T}^{d-1} \times [-1,1]$, where $\mathbb{T}^{d-1}$ is a flat torus in $d-1$ dimensions with its coordinates given by cube $(0,2)^{d-1}$. Note that (2.36) represents an extra symmetry condition, in addition to $\phi \in C^{2,\alpha}(\mathbb{T}^{d-1} \times [-1,1])$, and (2.36) implies (2.35).

Then we can extend φ^- in the same way by (2.23), that is, $\varphi^- \in C^{2,\alpha}(\Omega_{\mathrm{e}})$ satisfies (2.36). Notice that $\varphi_0^\pm$ can also be considered as the functions in Ω_{e} satisfying (2.36), since $\varphi_0^\pm(\mathbf{x}) = q^\pm x_d$ in $\mathbb{R}^{d-1} \times [-1,1]$ which are independent of $\mathbf{x}'$.

Therefore, we have reduced the transonic shock problem, Problem 2.2, into the following free boundary problem:

Problem 2.3. *Find $u \in C(\overline{\Omega_{\mathrm{e}}})$ with $u \geqslant 0$ such that*

(i) $u \in C^{2,\alpha}(\overline{\Omega_{\mathrm{e}}^+})$ *is a solution of* (2.32) *in* $\Omega_{\mathrm{e}}^+ := \{u(\mathbf{x}) > 0\} \cap \Omega_{\mathrm{e}}$, *the non-coincidence set*;

(ii) *the first two conditions in* (2.34) *hold on* $\partial\Omega_{\mathrm{e}}$, *i.e.,* $u = 0$ *on* $\partial\Omega_{\mathrm{e}} \cap \{x_n = -1\}$ *and*
$$u = \varphi^- - \varphi_0^+ \qquad \text{on } \partial\Omega_{\mathrm{e}} \cap \{x_n = 1\}; \tag{2.37}$$

(iii) *the free boundary* $\mathcal{S} = \partial\Omega^+ \cap \Omega_{\mathrm{e}}$ *is given by* $x_d = f(\mathbf{x}')$ *for* $\mathbf{x}' \in \mathbb{T}^{d-1}$ *so that* $\Omega^+ = \{x_d > f(\mathbf{x}') \;:\; \mathbf{x}' \in \mathbb{T}^{d-1}\}$ *with* $f \in C^{2,\alpha}(\mathbb{T}^{d-1})$;

(iv) *the free boundary condition* (2.33) *holds on* $\mathcal{S}$.

As indicated in §1, one of the main difficulties for solving the modified free boundary problem, Problem 2.3, is that the methods presented in the previous works for elliptic free boundary problems do not directly apply. Indeed, equation (2.32) is quasilinear, uniformly elliptic, but does not have a clear variational structure, while $G(\boldsymbol{\nu}, \mathbf{x})$ in the free boundary condition (2.33) depends on $\boldsymbol{\nu}$. Because of these features, the variational methods in [1,3] do not directly apply to Problem 2.3. Moreover, the nonlinearity in our problem makes it difficult to apply the Harnack inequality approach of Caffarelli in [10–12]. In particular, a boundary comparison principle for positive solutions of elliptic equations in Lipschitz domains is unavailable in our case that the nonlinear PDEs are not homogeneous with respect to (D^2u, Du, u) here. Therefore, a different method is required to overcome these difficulties for solving Problem 2.3.

2.2.3. Iteration Scheme for Solving Free Boundary Problems. The iteration scheme, developed in Chen-Feldman [29], is based on the non-degeneracy of the free boundary condition: the jump of the normal derivative of a solution across the free boundary has a strictly positive lower bound.

Denote $u_0 := \varphi^- - \varphi_0^+$. Note that u_0 satisfies the nondegeneracy condition: $\partial_{x_d} u_0 = q^- - q^+ > 0$ in Ω_{e}. Assume that (2.22) holds with $\sigma \leqslant \frac{q^- - q^+}{10}$. Let a function w on Ω_{e} be given such that $\|w - (\varphi^- - \varphi_0^+)\|_{C^{2,\alpha}(\overline{\Omega_{\mathrm{e}}})} \leqslant \frac{q^- - q^+}{10}$, which implies that w satisfies the nondegeneracy condition: $\partial_{x_d} w \geqslant \frac{q^- - q^+}{2} > 0$ in Ω_{e}. Define domain $\Omega^+(w) := \{w > 0\} \subset \Omega_{\mathrm{e}}$. Then
$$\Omega^+(w) = \{x_d > f(\mathbf{x}') \;:\; \mathbf{x}' \in \mathbb{T}^{d-1}\}, \qquad \mathcal{S}(w) := \partial\Omega^+(w) \backslash \partial\Omega_{\mathrm{e}} = \{x_d = f(\mathbf{x}') \;:\; \mathbf{x}' \in \mathbb{T}^{d-1}\}$$
with $f \in C^{2,\alpha}(\mathbb{T}^{d-1})$. We solve the oblique derivative problem (2.32)–(2.33) and (2.37) in $\Omega^+(w)$ to obtain a solution $u \in C^{2,\alpha}(\overline{\Omega^+(w)})$. However, u is not identically zero on $\mathcal{S}(w)$ in general, so that u is not a solution of the free boundary problem. Next, the estimates for problem (2.32)–(2.33) and (2.37) in $\Omega^+(w)$ show that $\|u - (\varphi^- - \varphi_0^+)\|_{C^{2,\alpha}(\overline{\Omega^+(w)})}$ is small. Then we extend u to the whole domain Ω_{e} so that $\|u - (\varphi^- - \varphi_0^+)\|_{C^{2,\alpha}(\overline{\Omega_{\mathrm{e}}})}$ is small. This defines the iteration map: $w \mapsto u$. The fixed point $u = w$ of this process determines a solution of the free boundary problem, since u is a solution of (2.32)–(2.33) and (2.37) in $\Omega^+(u)$, and u satisfies $u = w > 0$ on $\Omega^+(u) = \Omega^+(w) := \{w > 0\}$ and $u = w = 0$ on $\mathcal{S} := \partial\Omega^+(w) \backslash \partial\Omega_{\mathrm{e}}$. Then it remains to show the existence of a fixed point. Since the right-hand side of the free boundary condition (2.33) depends on $\boldsymbol{\nu}$, we need to exploit the structure of our problem, in

addition to the elliptic estimates, to obtain the better estimates for the iteration and prove the existence of a fixed point. More precisely, the nonlinear method can be described in the following five steps:

1. **Iteration set**. Let $M \geqslant 1$. Set

$$\mathcal{K}_M := \big\{w \in C^{2,\alpha}(\overline{\Omega_{\rm e}}) \;\;:\;\; w \text{ satisfies (2.36) and } \|w - (\varphi^- - \varphi_0^+)\|_{2,\alpha,\Omega_{\rm e}} \leqslant M\sigma\big\}, \tag{2.38}$$

where $\varphi_0^+(\mathbf{x}) = q^+ x_d$. Then $\mathcal{K}_M$ is convex and compact in $C^{2,\beta}(\Omega_{\rm e})$ for $0 < \beta < \alpha$.

Let $w \in \mathcal{K}_M$. Since $q^- > q^+$, it follows that, if

$$\sigma \leqslant \frac{q^- - q^+}{10(M+1)}, \tag{2.39}$$

then combining (2.22) and (2.38) with (2.39) implies

$$w_{x_d}(\mathbf{x}) \geqslant \frac{q^- - q^+}{2} > 0. \tag{2.40}$$

By the implicit function theorem, $\Omega^+(w) := \{w(\mathbf{x}) > 0\} \cap \Omega_{\rm e}$ has the form:

$$\Omega^+(w) = \{x_d > f(\mathbf{x}') \;:\; \mathbf{x}' \in \mathbb{T}^{d-1}\}, \qquad \|f\|_{2,\alpha,\mathbb{T}^{d-1}} \leqslant CM\sigma < 1, \tag{2.41}$$

where C depends on $q^- - q^+$, and the last inequality is obtained by choosing small σ. The corresponding unit normal on $\mathcal{S}(w) := \{x_d = f(\mathbf{x}')\}$ is

$$\boldsymbol{\nu}(\mathbf{x}') = \frac{(-D_{\mathbf{x}'} f(\mathbf{x}'), 1)}{\sqrt{1 + |D_{\mathbf{x}'} f(\mathbf{x}')|^2}} \in C^{1,\alpha}(\mathbb{T}^{d-1}; \mathbb{S}^{d-1})$$

with

$$\|\boldsymbol{\nu} - \boldsymbol{\nu}_0\|_{1,\alpha,\mathbb{R}^{d-1}} \leqslant CM\sigma, \tag{2.42}$$

where $\boldsymbol{\nu}_0$ is defined by

$$\boldsymbol{\nu}_0 := \frac{D(\varphi_0^- - \varphi_0^+)}{|D(\varphi_0^- - \varphi_0^+)|} = (0, \cdots, 0, 1)^\top. \tag{2.43}$$

Also, $\boldsymbol{\nu}(\cdot)$ can be considered as a function on $\mathcal{S}(w)$. Since $\Omega^+(w) = \{w(x) > 0\} \cap \Omega_{\rm e}$, from the definition of $f(\mathbf{x}')$ in (2.41), it follows that, for $\mathbf{x} \in \mathcal{S}(w)$,

$$\boldsymbol{\nu}(\mathbf{x}) = \frac{Dw(\mathbf{x})}{|Dw(\mathbf{x})|}. \tag{2.44}$$

By the definition of $\mathcal{K}_M$ and (2.39) with (2.22), $\boldsymbol{\nu}(\mathbf{x})$ can be extended to $\Omega_{\rm e}$ via formula (2.44) and

$$\|\boldsymbol{\nu} - \boldsymbol{\nu}_0\|_{1,\alpha,\Omega_{\rm e}} \leqslant CM\sigma \tag{2.45}$$

with $C = C(q^+, q^-)$. Motivated by the free boundary condition (2.31), we define a function G_w on $\Omega_{\rm e}$:

$$G_w(\mathbf{x}) := \big(\rho(|D\varphi^-(\mathbf{x})|^2) - \tilde{\rho}(|D\varphi^-(\mathbf{x})|^2)\big) D\varphi^-(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}), \tag{2.46}$$

where $\boldsymbol{\nu}(\cdot)$ is defined by (2.44).

We now solve the following fixed boundary value problem for u in domain $\Omega^+(w)$:

$$\operatorname{div} A(Du, \mathbf{x}) = F(\mathbf{x}) \qquad \text{in } \Omega^+ := \{w > 0\}, \tag{2.47}$$

$$A(Du, \mathbf{x}) \cdot \boldsymbol{\nu} = G_w(\mathbf{x}) \qquad \text{on } \mathcal{S}(w) := \partial\Omega^+(w) \backslash \partial\Omega_{\rm e}, \tag{2.48}$$

$$u = \varphi^- - q^+ \qquad \text{on } \{x_d = 1\} = \partial\Omega^+(w) \backslash \mathcal{S}(w), \tag{2.49}$$

and show that its unique solution u can be extended to the whole domain $\Omega_{\rm e}$ so that $u \in \mathcal{K}_M$.

2. **Existence and uniqueness of the solution for the fixed boundary value problem** (2.47)–(2.49). We establish the existence and uniqueness of solution u for problem (2.47)–(2.49) and show that u is close in $C^{2,\alpha}(\overline{\Omega^+(w)})$ to the unperturbed subsonic solution $\varphi^- - \varphi_0^+$: For $M \geqslant 1$, there is

$\sigma_0 > 0$, depending only on $(M, q^+, d, \gamma, \Omega)$, such that, if $\sigma \in (0, \sigma_0)$, φ^- satisfies (2.22), and $w \in \mathcal{K}_M$, there exists a unique solution $u \in C^{2,\alpha}(\overline{\Omega^+(w)})$ of problem (2.47)–(2.49) satisfying (2.36) and

$$\|u - (\varphi^- - \varphi_0^+)\|_{2,\alpha,\Omega^+(w)} \leqslant C\sigma, \tag{2.50}$$

where C depends only on (q^+, d, γ, Ω) and is independent of M, $w \in \mathcal{K}_M$, and $\sigma \in (0, \sigma_0)$.

To achieve this, it requires to combine the existence arguments with careful Schauder estimates for nonlinear oblique boundary value problems for nonlinear elliptic PDEs, based on the results in [61, 80, 81, 102] and the references cited therein. Moreover, the independence of C from M is achieved by employing a cancellation based on the structure in (2.48) with the explicit expressions of $A(Du, \mathbf{x})$ and $G_w(\mathbf{x})$ and on the Rankine-Hugoniot condition for the background solution.

3. **Construction and continuity of the iteration map**. We now construct the iteration map by an extension of the unique solution of (2.47)–(2.49), which satisfies (2.50), and show the continuity of the iteration map: Let $w \in \mathcal{K}_M$, and let $u(\mathbf{x})$ be a solution of problem (2.47)–(2.49) in domain $\Omega^+(w)$ established in Step 2 above. Then $u(\mathbf{x})$ can be extended to the whole domain Ω_{e} in such a way that this extension, denoted as $\mathcal{P}_w u(\mathbf{x})$, satisfies the following two properties:

(i) There exists $C_0 > 0$, which depends only on (q^+, d, γ, Ω) and is independent of (M, σ) and $w(\mathbf{x})$, such that

$$\|\mathcal{P}_w u - (\varphi^- - \varphi_0^+)\|_{2,\alpha,\Omega_{\mathrm{e}}} \leqslant C_0\sigma. \tag{2.51}$$

(ii) Let $\beta \in (0, \alpha)$. Let $w_j \in \mathcal{K}_M$ converge in $C^{2,\beta}(\overline{\Omega_{\mathrm{e}}})$ to $w \in \mathcal{K}_M$. Let $u_j \in C^{2,\alpha}(\overline{\Omega^+(w_j)})$ and $u \in C^{2,\alpha}(\overline{\Omega^+(w)})$ be the solutions of problems (2.47)–(2.49) for $w_j(x)$ and $w(x)$, respectively. Then $\mathcal{P}_{w_j} u_j \to \mathcal{P}_w u$ in $C^{2,\beta}(\overline{\Omega_{\mathrm{e}}})$.

Define the iteration map $J : \mathcal{K}_M \to C^{2,\alpha}(\overline{\Omega_{\mathrm{e}}})$ by

$$Jw := \mathcal{P}_w u, \tag{2.52}$$

where $u(\mathbf{x})$ is the unique solution of problem (2.47)–(2.49) for $w(\mathbf{x})$. By (ii), J is continuous in the $C^{2,\beta}(\overline{\Omega_{\mathrm{e}}})$–norm for any positive $\beta < \alpha$.

Now we denote by $u(\mathbf{x})$ both the function $u(\mathbf{x})$ in $\Omega^+(w)$ and its extension $\mathcal{P}_w u(\mathbf{x})$. Choose M to be the constant C_0 from (2.51). Then, for $w \in \mathcal{K}_M$, we see that $u := Jw \in \mathcal{K}_M$ if $\sigma > 0$ is sufficiently small, depending only on (q^+, d, γ, Ω), since M is now fixed. Thus, (2.52) defines the iteration map $J : \mathcal{K}_M \to \mathcal{K}_M$ and, from (2.51), J is continuous on $\mathcal{K}_M$ in the $C^{2,\beta}(\overline{\Omega_{\mathrm{e}}})$–norm for any positive $\beta < \alpha$.

4. **Existence of a fixed point of the iteration map**. We then prove the existence of solutions of the free boundary problem, Problem 2.2.

First, in order to solve Problem 2.3, we seek a fixed point of map J. We use the Schauder fixed point theorem (*cf.* Gilbarg-Trudinger [61, Theorem 11.1]) in the following setting:

Let $\sigma > 0$ satisfy the conditions in Step 2 above. Let $\beta \in (0, \alpha)$. Since Ω_{e} is a compact manifold with boundary and $\mathcal{K}_M$ is a bounded convex subset of $C^{2,\alpha}(\overline{\Omega_{\mathrm{e}}})$, it follows that $\mathcal{K}_M$ is a compact convex subset of $C^{2,\beta}(\overline{\Omega_{\mathrm{e}}})$. We have shown that $J(\mathcal{K}_M) \subset \mathcal{K}_M$, and J is continuous in the $C^{2,\beta}(\overline{\Omega_{\mathrm{e}}})$–norm. Then, by the Schauder fixed point theorem, J has a fixed point $\varphi \in \mathcal{K}_M$.

If $u(\mathbf{x})$ is such a fixed point, then

$$\tilde{u}(\mathbf{x}) := \max(0, u(\mathbf{x}))$$

is a classical solution of Problem 2.3, and $\mathcal{S}(u)$ is its free boundary.

It follows that $\varphi := \varphi^- - \tilde{u}$ is a solution of Problem 2.2, provided that σ is small enough so that (2.50) implies that $|D\varphi| = |D(\varphi^- - u)| < p^1_{\mathrm{sonic}} - \varepsilon$ on $\Omega^+(u)$, where $\varepsilon = \frac{p_{\mathrm{sonic}} - q^+}{2}$ defined in §2.2.1. Indeed, then (2.29) implies that $\varphi(\mathbf{x})$ lies in the non-truncated region for equation (2.30). Note also that the boundary condition $\varphi_{\boldsymbol{\nu}} = 0$ on $\partial(0,1)^{d-1} \times [-1, 1]$ is satisfied because u and φ^- satisfy (2.36) on $\mathbb{T}^{d-1} \times [-1, 1]$.

For such values of σ, if $\varphi^-(\mathbf{x})$ is a supersonic solution of (2.1) satisfying the conditions stated in Problem 2.1, the defined function $\varphi(x)$ is a solution of Problem 2.1. Indeed, $|D\varphi| = |D(\varphi^- - \tilde{u})| <$

$p^1_{\rm sonic} - \varepsilon$ on $\Omega^+(\varphi) := \{\varphi < \varphi^-\} = \{\tilde{u}(\mathbf{x}) > 0\}$ since $\tilde{u} = u$ on $\Omega^+(\tilde{u})$ and $|D\varphi| = |D\varphi^-| > p^1_{\rm sonic}$ on $\Omega\backslash\Omega^+(\varphi)$, equation (2.1) is satisfied in both $\Omega^+(\varphi)$ and $\Omega\backslash\Omega^+(\varphi)$, and the Rankine-Hugoniot conditions (2.8)–(2.9) are satisfied on $\mathcal{S} = \partial\Omega^+(\varphi)\backslash\partial\Omega$.

This completes the construction of the global solution. The uniqueness and stability of the solution of the free boundary problem are obtained by using the regularity and nondegeneracy of solutions.

Remark 2.2. *For clarity, in this section, we focus on the simplest setup of the domain as* $\Omega = (0,1)^{d-1} \times (-1,1)$*, which can be extended directly to* $\Omega_R = \Pi_{j=1}^{d-1}(0,a_j) \times (-1,R)$ *for any* $R > 0$*, then to* $\Omega_\infty = \Pi_{j=1}^{d-1}(0,a_j) \times (-1,\infty)$ *by analyzing the asymptotic behavior of the solution when* $R \to \infty$*, as well as to* $\Omega = \mathbb{R}^{d-1} \times (-1,\infty)$*; see Chen-Feldman* [29–31]. *See also* Chen [46] *for the extension to the isentropic Euler case.*

If the hyperbolic phase is C^∞*, then the solution and its corresponding free boundary in* Theorem 2.1 *are also* C^∞*. Furthermore, our results can be extended to the problem with a steady* $C^{1,\alpha}$*–perturbation of the upstream supersonic flow and/or general Dirichlet data* $h(\mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{d-1}$*, at* $x_d = 1$ *satisfying*

$$\|h - \varphi_0^+\|_{1,\alpha,\mathbb{R}^{d-1}} \leqslant C\sigma \qquad \text{for } \alpha \in (0,1).$$

Also, the Dirichlet data in Problem 2.2 *may be replaced by the corresponding Neumann data satisfying the global solvability condition.*

The global uniqueness of piecewise constant transonic shocks in straight ducts modulo translations was analyzed in [41, 58].

Remark 2.3. *The domains in the setup of* Problems 2.1–2.2 *have also been extended to M-D infinite nozzles of arbitrary cross-section in Chen-Feldman* [32]*; also see Xin-Yin* [111]*, Yuan* [113]*, and the references cited therein for the* 2*-D case with the downstream pressure exit. For the analysis of geometric effects of the nozzles on the uniqueness and stability of steady transonic shocks, see* [7, 42, 76, 86, 87] *and the references cited therein.*

Remark 2.4. *The iteration scheme can also be reformulated in a way such that the free boundary normal* $\boldsymbol{\nu}$ *is unknown in the iteration by replacing the known function* w *in* (2.44) *by the unknown* u*, that is, by replacing* $\boldsymbol{\nu}(\mathbf{x})$ *in* (2.44) *via*

$$\boldsymbol{\nu}(\mathbf{x}) = \frac{Du(\mathbf{x})}{|Du(\mathbf{x})|}. \tag{2.53}$$

Note that (2.53) *coincides with* (2.44) *at the fixed point* $u = w$*, i.e., defines the normal to* $\mathcal{S}$*. Using expression* (2.53) *for* $\boldsymbol{\nu}$ *in the iteration boundary condition, we improve the regularity and structure of the boundary condition; in particular, it is made independent of the regularity and constants in the iteration set. This is useful in many cases, see e.g.* [33]*. Moreover, this allows us to obtain the compactness of the iteration map, which has been used in* [35]*.*

This nonlinear method and related ideas and techniques described above for free boundary problems have played a key role in many recent developments in the analysis of M-D transonic shock problems, as shown in §3–§5 below.

3. Two-Dimensional Transonic Shocks and Free Boundary Problems for the Steady Full Euler Equations

We now describe how the nonlinear method and related ideas and techniques presented in §2 can be applied to establish the existence, stability, and asymptotic behavior of 2-D steady transonic flows with transonic shocks past curved wedges for the full Euler equations, by reformulating the problems as free boundary problems, via two different approaches.

The 2-D steady Euler equations for polytropic gases are of the form (*cf.* [35, 52]):

$$\begin{cases} \operatorname{div}(\rho\mathbf{u}) = 0, \\ \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = 0, \\ \operatorname{div}\big(\rho\mathbf{u}(E+\frac{p}{\rho})\big) = 0, \end{cases} \tag{3.1}$$

where $\mathbf{u} = (u_1, u_2)$ is the velocity, ρ the density, p the pressure, and $E = \frac{1}{2}|\mathbf{u}|^2 + e$ the total energy with internal energy e.

Choose pressure p and density ρ as the independent thermodynamical variables. Then the constitutive relations can be written as

$$(e, T, S) = (e(p,\rho), T(p,\rho), S(p,\rho))$$

governed by

$$TdS = de - \frac{p}{\rho^2}d\rho,$$

where T and S represent the temperature and the entropy, respectively. For a polytropic gas,

$$e = e(p,\rho) = \frac{p}{(\gamma-1)\rho}, \quad T = T(p,\rho) = \frac{p}{(\gamma-1)c_v\rho}, \quad S = S(p,\rho) = c_v\ln(\frac{p}{\kappa\rho}), \tag{3.2}$$

where $\gamma > 1$ is the adiabatic exponent, $c_v > 0$ the specific heat at constant volume, and $\kappa > 0$ any constant under scaling.

System (3.1) can be written as a first-order system of conservation laws:

$$\partial_{x_1}F(U) + \partial_{x_2}G(U) = 0, \qquad U = (\mathbf{u}, p, \rho) \in \mathbb{R}^4. \tag{3.3}$$

Solving $\det(\lambda\nabla_U F(U) - \nabla_U G(U)) = 0$ for λ, we obtain four eigenvalues:

$$\lambda_1 = \lambda_2 = \frac{u_2}{u_1}, \qquad \lambda_j = \frac{u_1u_2 + (-1)^j c\sqrt{|\mathbf{u}|^2 - c^2}}{u_1^2 - c^2} \quad \text{for } j = 3, 4,$$

where

$$c = \sqrt{\frac{\gamma p}{\rho}} \tag{3.4}$$

is the sonic speed of the flow for a polytropic gas.

The repeated eigenvalues λ_1 and λ_2 are real and correspond to the two linear degenerate characteristic families which generate vortex sheets and entropy waves, respectively. The eigenvalues λ_3 and λ_4 are real when the flow is supersonic (*i.e.*, $|\mathbf{u}| > c$), and complex when the flow is subsonic (*i.e.*, $|\mathbf{u}| < c$) in which case the corresponding two equations are elliptic.

For a transonic flow in which both the supersonic and subsonic phases occur in the flow, system (3.1) is of mixed-composite hyperbolic-elliptic type, which consists of two equations of mixed elliptic-hyperbolic type and two equations of hyperbolic type (*i.e.*, two transport-type equations).

In the regimes with $\rho|\mathbf{u}| > 0$, from the first equation in (3.1), in any domain containing the origin, there exists a unique stream function ψ such that

$$D\psi = (-\rho u_2, \rho u_1) \qquad \text{with } \psi(\mathbf{0}) = 0. \tag{3.5}$$

We use the following Lagrangian coordinate transformation:

$$(x_1, x_2) \to (y_1, y_2) = (x_1, \psi(x_1, x_2)), \tag{3.6}$$

under which the original curved streamlines become straight. In the new coordinates $\mathbf{y} = (y_1, y_2)$, we still denote the unknown variables $U(\mathbf{x}(\mathbf{y}))$ by $U(\mathbf{y})$ for simplicity of notation. Then the original Euler

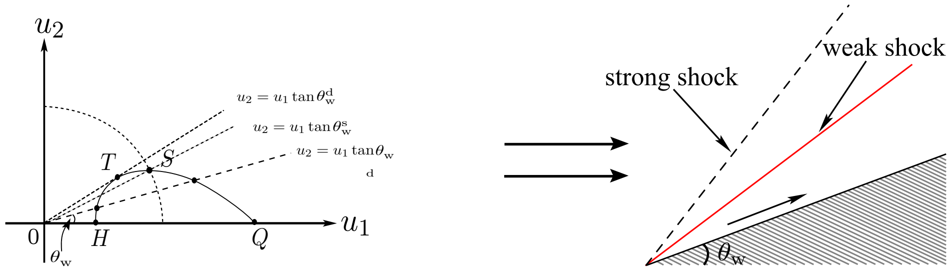

FIGURE 3.1. The shock polar in the **u**-plane and uniform steady (weak/strong) shock flows (see [22])

equations in (3.1) become the following equations in divergence form:

$$\Big(\frac{1}{\rho u_1}\Big)_{y_1} - \Big(\frac{u_2}{u_1}\Big)_{y_2} = 0, \tag{3.7}$$

$$\Big(u_1 + \frac{p}{\rho u_1}\Big)_{y_1} - \Big(\frac{p u_2}{u_1}\Big)_{y_2} = 0, \tag{3.8}$$

$$(u_2)_{y_1} + p_{y_2} = 0, \tag{3.9}$$

$$\Big(\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho}\Big)_{y_1} = 0. \tag{3.10}$$

One of the advantages of the Lagrangian coordinates is to straighten the streamlines so that the streamline may be employed as one of the coordinates to simplify the formulations, since the Bernoulli variable is conserved along the streamlines. Note that the entropy is also conserved along the streamlines in the continuous part of the flow.

3.1. **Steady Supersonic Flow onto Solid Wedges and Free Boundary Problems.** For an upstream steady uniform supersonic flow past a symmetric straight-sided wedge (see Fig. 3.1):

$$W := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \,:\, |x_2| < x_1 \tan\theta_{\mathrm{w}}, x_1 > 0\} \tag{3.11}$$

whose angle θ_{w} is less than the detachment angle $\theta_{\mathrm{w}}^{\mathrm{d}}$, there exists an oblique shock emanating from the wedge vertex. Since the upper and lower subsonic regions do not interact with each other, it suffices to study the upper part. More precisely, if the upstream steady flow is a uniform supersonic state, we can find the corresponding constant downstream flow along the straight-sided upper wedge boundary, together with a straight shock separating the two states. The downstream flow is determined by the shock polar whose states in the phase space are governed by the Rankine-Hugoniot conditions and the entropy condition; see Fig. 3.1. Indeed, Prandtl in [94] first employed the shock polar analysis to show that there are two possible steady oblique shock configurations when the wedge angle θ_{w} is less than the detachment angle $\theta_{\mathrm{w}}^{\mathrm{d}}$ — The steady weak shock with supersonic or subsonic downstream flow (determined by the wedge angle that is less or larger than the sonic angle $\theta_{\mathrm{w}}^{\mathrm{s}}$) and the steady strong shock with subsonic downstream flow, both of which satisfy the entropy condition, provided that no additional conditions are assigned at downstream. See also [9, 22, 52, 91, 94] and the references cited therein.

The fundamental issue – whether one or both of the steady weak and strong shocks are physically admissible – has been vigorously debated over the past seven decades (*cf.* [22, 52, 88, 99, 107, 108]). Experimental and numerical results have strongly indicated that the steady weak shock solution would be physically admissible, as Prandtl conjectured in [94]. One natural approach to single out the physically admissible steady shock solutions is via the stability analysis: the stable ones are physical. See Courant-Friedrichs [52] and von Neumann [107, 108]; see also [88, 99].

A piecewise smooth solution $U = (\mathbf{u}, p, \rho) \in \mathbb{R}^4$ separated by a front $\mathcal{S} := \{\mathbf{x} \,:\, x_2 = \sigma(x_1), x_1 \geqslant 0\}$ becomes a weak solution of the Euler equations (3.1) as in §2.1 if and only if the Rankine-Hugoniot conditions are satisfied along $\mathcal{S}$:

$$\begin{cases} \sigma'(x_1)[\,\rho u_1\,] = [\,\rho u_2\,], \\ \sigma'(x_1)[\,\rho u_1^2 + p\,] = [\,\rho u_1 u_2\,], \\ \sigma'(x_1)[\,\rho u_1 u_2\,] = [\,\rho u_2^2 + p\,], \\ \sigma'(x_1)[\,\rho u_1(E + \frac{p}{\rho})\,] = [\,\rho u_2(E + \frac{p}{\rho})\,], \end{cases} \tag{3.12}$$

where $[\,\cdot\,]$ denotes the jump between the quantities of two states across front $\mathcal{S}$ as before.

Such a front $\mathcal{S}$ is called a shock if the entropy condition holds along $\mathcal{S}$: *The density increases in the fluid direction across $\mathcal{S}$.*

For given state U^-, all states U that can be connected with U^- through the relations in (3.12) form a curve in the state space $\mathbb{R}^4$; the part of the curve whose states satisfy the entropy condition is called the *shock polar*. The projection of the shock polar onto the $\mathbf{u}$–plane is shown in Fig. 3.1. In particular, for an upstream uniform horizontal flow $U_0^- = (u_{10}^-, 0, p_0^-, \rho_0^-)$ past the upper part of a straight-sided wedge whose angle is $\theta_{\rm w}$, the downstream constant flow can be determined by the shock polar (see Fig. 3.1). Note that the downstream flow must be parallel to the wedge, and the upstream flow is parallel to the axis of wedge, so the angle between the upstream and downstream flow is equal to the (half) wedge angle. According to the shock polar, the two flow angles (or, equivalently, wedge angles) are particularly important:

One is the detachment angle $\theta_{\rm w}^{\rm d}$ such that line $u_2 = u_1 \tan\theta_{\rm w}^{\rm d}$ is tangential to the shock polar at point T and there is no intersection between line $u_2 = u_1 \tan\theta_{\rm w}$ and the shock polar when $\theta_{\rm w} > \theta_{\rm w}^{\rm d}$. For any wedge angle $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, there are two intersection points of line $u_2 = u_1 \tan\theta_{\rm w}$ and the shock polar: one intersection point is on arc $\widehat{TH}$ which determines velocity $\mathbf{u}^{\rm sg} = (u_1^{\rm sg}, u_2^{\rm sg})$ of the downstream flow corresponding to the strong shock, and the other intersection point is on arc $\widehat{TQ}$ which determines velocity $\mathbf{u}^{\rm wk} = (u_1^{\rm wk}, u_2^{\rm wk})$ of the downstream flow corresponding to the weak shock. Thus, for any wedge angle $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, the shock polar ensures the existence of two attached shocks at the wedge: strong versus weak.

Since each point on the shock polar defines a downstream flow that is a constant state, we can use (3.4) to compute its sonic speed c_0 and then determine whether this downstream state is subsonic or supersonic. It can be shown that there exists the unique point S on the shock polar so that all downstream states are subsonic for the points on $\widehat{HS}\backslash\{S\}$, supersonic for the points of $\widehat{SQ}\backslash\{S\}$, and sonic for the state at S. Moreover, S lies in the interior of arc TQ. Then, denoting by $\theta_{\rm w}^{\rm s}$ the angle corresponding to point S, we see that $\theta_{\rm w}^{\rm s} < \theta_{\rm w}^{\rm d}$. The wedge angle $\theta_{\rm w}^{\rm s}$ is called the sonic angle. Point T divides arc $\widehat{HS}$, which corresponds to the transonic shocks, into the two open arcs $\widehat{TS}$ and $\widehat{TH}$; see Fig. 3.1. The nature of these two cases, as well as the case for arc $\widehat{SQ}$, is very different. When the wedge angle $\theta_{\rm w}$ is between $\theta_{\rm w}^{\rm s}$ and $\theta_{\rm w}^{\rm d}$, there are two subsonic solutions (corresponding to the strong and weak shocks); while, for the wedge angle $\theta_{\rm w}$ is smaller than $\theta_{\rm w}^{\rm s}$, there are one subsonic solution (for the strong shock) and one supersonic solution (for the weak shock). Such an oblique shock $\mathcal{S}_0$ is straight, described by $x_2 = s_0 x_1$ with s_0 as its slope. The question is whether the steady oblique shock solution is stable under a perturbation of both the upstream supersonic flow and the wedge boundary.

Since we are interested in determining the downstream flow, we can restrict the domain to the first quadrant; see Fig. 3.2. Fix a constant upstream flow U_0^-, a wedge angle $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, and a constant downstream state U_0^+ which is one of the downstream states (weak or strong) determined by the shock polar for the chosen upstream flow and wedge angle. States U_0^- and U_0^+ determine the oblique shock $x_2 = s_0 x_1$, and the transonic shock solution U_0 in $\{\mathbf{x} \,:\, x_1 > 0,\, x_2 > 0\}\backslash W$ such that $U_0 = U_0^-$ in $\Omega_0^- = \{\mathbf{x} \in \mathbb{R}^2 \,:\, x_2 > s_0 x_1, x_1 > 0\}$ and $U_0 = U_0^+$ in $\Omega_0^+ = \{\mathbf{x} \in \mathbb{R}^2 \,:\, x_1 \tan\theta_{\rm w} < x_2 < s_0 x_1, x_1 > 0\}$; see Fig. 3.1. We refer to this solution as a constant transonic solution (U_0^-, U_0^+).

Assume that the perturbed upstream flow U_I^- is close to U_0^- (so that U_I^- is supersonic and almost horizontal) and that the perturbed wedge is close to a straight-sided wedge. Then, for any suitable wedge angle (smaller than the detachment angle), it is expected that there should be a shock attached

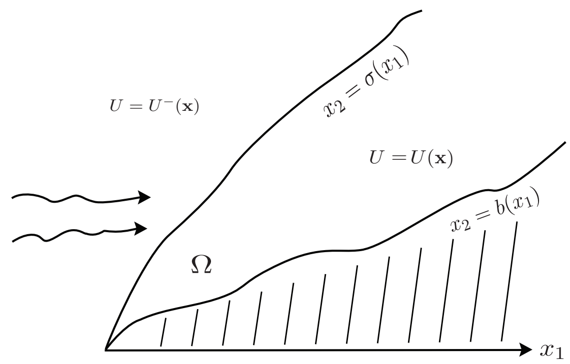

FIGURE 3.2. The leading steady shock $x_2 = \sigma(x_1)$ as a free boundary under the perturbation (see [22])

to the wedge vertex; see Fig. 3.2. We now use a function $b(x_1) \geqslant 0$ to describe the upper perturbed wedge boundary:

$$\partial W = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = b(x_1),\ x_1 > 0\} \qquad \text{with } b(0) = 0. \tag{3.13}$$

Then the wedge problem can be formulated as the following problem:

Problem 3.1 (Initial-Boundary Value Problem)**.** *Find a global solution of system* (3.1) *in* $\Omega := \{x_2 > b(x_1), x_1 > 0\}$ *such that the following conditions hold*:

(i) *Cauchy condition at* $x_1 = 0$:

$$U|_{x_1=0} = U_I^-(x_2); \tag{3.14}$$

(ii) *Boundary condition on* ∂W *as the slip boundary*:

$$\mathbf{u} \cdot \boldsymbol{\nu}_{\mathrm{w}}|_{\partial W} = 0, \tag{3.15}$$

where $\boldsymbol{\nu}_{\mathrm{w}}$ *is the outer unit normal vector to* ∂W.

Note that the background shock is the straight line given by $x_2 = \sigma_0(x_1)$ with $\sigma_0(x_1) := s_0 x_1$. When the upstream steady supersonic perturbation $U_I^-(x_2)$ at $x_1 = 0$ is suitably regular and small under some natural norm, the upstream steady supersonic smooth solution $U^-(\mathbf{x})$ exists in region $\Omega^- = \{\mathbf{x} : x_2 > \frac{s_0}{2} x_1, x_1 \geqslant 0\}$, beyond the background shock, and U^- in Ω^- is still close to U_0^-.

Assume that the shock-front $\mathcal{S}$ to be determined is

$$\mathcal{S} = \{\mathbf{x} : x_2 = \sigma(x_1),\ x_1 \geqslant 0\} \qquad \text{with } \sigma(0) = 0 \text{ and } \sigma(x_1) > 0 \text{ for } x_1 > 0. \tag{3.16}$$

The domain for the downstream flow behind $\mathcal{S}$ is denoted by

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 : b(x_1) < x_2 < \sigma(x_1), x_1 > 0\}. \tag{3.17}$$

Then Problem 3.1 can be reformulated into the following free boundary problem with $\mathcal{S}$ as a free boundary:

Problem 3.2 (Free Boundary Problem; see Fig. 3.2)**.** *Let* (U_0^-, U_0^+) *be a constant transonic solution for the wedge angle* $\theta_{\mathrm{w}} \in (0, \theta_{\mathrm{w}}^{\mathrm{d}})$ *with transonic shock* $\mathcal{S}_0 := \{x_2 = \sigma_0(x_1) : x_1 > 0\}$ *for* $\sigma_0(x_1) := s_0 x_1$. *For any upstream flow* U^- *for system* (3.1) *in domain* Ω^- *which is a small perturbation of* U_0^-, *and any wedge boundary function* $b(x_1)$ *that is a small perturbation of* $b_0(x_1) = x_1 \tan\theta_{\mathrm{w}}$, *find a shock* $\mathcal{S}$ *as a free boundary* $x_2 = \sigma(x_1)$ *and a solution* U *in* Ω, *which are small perturbations of* $\mathcal{S}_0$ *and* U_0^+ *respectively, such that*

(i) U *satisfies* (3.1) *in domain* Ω,
(ii) *The slip condition* (3.15) *holds along the wedge boundary,*
(iii) *The Rankine-Hugoniot conditions in* (3.12) *as free boundary conditions hold along the transonic shock-front* $\mathcal{S}$.

There are three subcases based on U_0^+*: For a weak supersonic shock* $\mathcal{S}_0$ *given by* U_0^+ *corresponding to a supersonic state on arc* $\widehat{SQ}$*, we denote the problem by* Problem 3.2(WS); *for a weak transonic shock* $\mathcal{S}_0$ *given by* U_0^+ *corresponding to a subsonic state on arc* $\widehat{TS}$*, we denote the problem by* Problem 3.2(WT); *finally, for a strong transonic shock* $\mathcal{S}_0$ *given by* U_0^+ *corresponding to a subsonic state on arc* $\widehat{TH}$*, we denote the problem by* Problem 3.2(ST).

In general, the uniqueness for the initial-boundary value problem (Problem 3.1) is not known (as it is a problem for a nonlinear system of a composite elliptic-hyperbolic type), so it may not yet be excluded that Problem 3.1 has solutions which are not of steady oblique shock structure, *i.e.*, are not solutions of Problem 3.2. On the other hand, the global solution of the free boundary problem (Problem 3.2) provides the global structural stability of the steady oblique shock, as well as more detailed structure of the solution.

Supersonic (*i.e.*, supersonic-supersonic) shocks correspond to arc $\widehat{SQ}$ which is a weaker shock (see Fig. 3.1). The local stability of such shocks was first established in [64,79,96]. The global stability of the supersonic shocks for potential flow past piecewise smooth perturbed curved wedges was established in Zhang [115]; also see [45, 48, 49] and the references therein. The global stability and uniqueness of the supersonic shocks for the full Euler equations, Problem 3.2(WS), were solved for more general perturbations of both the initial data and wedge boundary even in BV in Chen-Zhang-Zhu [43] and Chen-Li [39].

For transonic (*i.e.*, supersonic-subsonic) shocks, the strong shock case corresponding to arc $\widehat{TH}$ was first studied in Chen-Fang [48] for the potential flow (see Fig. 3.1). In Fang [57], the full Euler equations were studied with a uniform Bernoulli constant for both weak and strong transonic shocks. Because the framework is a weighted Sobolev space, the asymptotic behavior of the shock slope or subsonic solution was not derived. In Yin-Zhou [112], the Hölder norms were used for the estimates of solutions of the full Euler equations with the assumption on the sharpness of the wedge angle, which means that the subsonic state is near point H in the shock polar, by Approach I introduced first in [23] which is described in §3.2 below. In Chen-Chen-Feldman [24], the weaker transonic shock, which corresponds to arc $\widehat{TS}$, was first investigated by Approach I as described in §3.2 below. Then, in [25], the weak and strong transonic shocks, which correspond to arcs $\widehat{TS}$ and $\widehat{TH}$, respectively, were solved, by Approach II which is described in §3.3 below, so that the existence, uniqueness, stability, and asymptotic behavior of subsonic solutions of both Problem 3.2(WT) and Problem 3.2(ST) in a weighted Hölder space were obtained.

We now describe two approaches for the wedge problem, based on the nonlinear method and related ideas and techniques presented in §2. First, we need to introduce the weighed Hölder norms in the subsonic domain Ω, where Ω is either a truncated triangular domain or an unbounded domain with the vertex at origin O and one side as the wedge boundary. There are two weights: One is the distance function to origin O and the other is to the wedge boundary ∂W. For any $\mathbf{x}, \mathbf{x}' \in \Omega$, define

$$\delta^{\mathrm{o}}_{\mathbf{x}} := \min(|\mathbf{x}|, 1), \quad \delta^{\mathrm{o}}_{\mathbf{x},\mathbf{x}'} := \min(\delta^{\mathrm{o}}_{\mathbf{x}}, \delta^{\mathrm{o}}_{\mathbf{x}'}), \quad \delta^{\mathrm{w}}_{\mathbf{x}} := \min(\mathrm{dist}(\mathbf{x}, \partial W), 1), \quad \delta^{\mathrm{w}}_{\mathbf{x},\mathbf{x}'} := \min(\delta^{\mathrm{w}}_{\mathbf{x}}, \delta^{\mathrm{w}}_{\mathbf{x}'}),$$

$$\Delta_{\mathbf{x}} := |\mathbf{x}| + 1, \quad \Delta_{\mathbf{x},\mathbf{x}'} := \min(\Delta_{\mathbf{x}}, \Delta_{\mathbf{x}'}), \quad \widetilde{\Delta}_{\mathbf{x}} := \mathrm{dist}(\mathbf{x}, \partial W) + 1, \quad \widetilde{\Delta}_{\mathbf{x},\mathbf{x}'} := \min(\widetilde{\Delta}_{\mathbf{x}}, \widetilde{\Delta}_{\mathbf{x}'}).$$

Let $\alpha \in (0,1)$ and $l_1, l_2, \gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma_1 \geqslant \gamma_2$, and let $k \geqslant 0$ be an integer. Let $\mathbf{k} = (k_1, k_2)$ be an integer-valued vector, where $k_1, k_2 \geqslant 0$, $|\mathbf{k}| = k_1 + k_2$, and $D^{\mathbf{k}} = \partial_{x_1}^{k_1}\partial_{x_2}^{k_2}$. We define

$$[f]^{(\gamma_1;O)(\gamma_2;\partial W)}_{k,0;(l_1,l_2);\Omega} = \sup_{\substack{\mathbf{x}\in\Omega\\ |\mathbf{k}|=k}} \big\{(\delta^{\mathrm{o}}_{\mathbf{x}})^{\hat{\gamma}_0}(\delta^{\mathrm{w}}_{\mathbf{x}})^{\max\{k+\gamma_2,0\}}\,\Delta^{l_1}_{\mathbf{x}}\widetilde{\Delta}^{l_2+k}_{\mathbf{x}}|D^{\mathbf{k}}f(\mathbf{x})|\big\}, \tag{3.18}$$

$$[f]^{(\gamma_1;O)(\gamma_2;\partial W)}_{k,\alpha;(l_1,l_2);\Omega} = \sup_{\substack{\mathbf{x},\mathbf{x}'\in\Omega\\ \mathbf{x}\neq\mathbf{x}',|\mathbf{k}|=k}} \Big\{(\delta^{\mathrm{o}}_{\mathbf{x},\mathbf{x}'})^{\hat{\gamma}_\alpha}(\delta^{\mathrm{w}}_{\mathbf{x},\mathbf{x}'})^{\max\{k+\alpha+\gamma_2,0\}}\Delta^{l_1}_{\mathbf{x},\mathbf{x}'}\widetilde{\Delta}^{l_2+k+\alpha}_{\mathbf{x},\mathbf{x}'}\frac{|D^{\mathbf{k}}f(\mathbf{x})-D^{\mathbf{k}}f(\mathbf{x}')|}{|\mathbf{x}-\mathbf{x}'|^\alpha}\Big\}, \tag{3.19}$$

$$\|f\|^{(\gamma_1;O)(\gamma_2;\partial W)}_{k,\alpha;(l_1,l_2);\Omega} = \sum_{i=0}^{k}[f]^{(\gamma_1;O)(\gamma_2;\partial W)}_{i,0;(l_1,l_2);\Omega} + [f]^{(\gamma_1;O)(\gamma_2;\partial W)}_{k,\alpha;(l_1,l_2);\Omega}, \tag{3.20}$$

where $\hat{\gamma}_\beta = \max\{\gamma_1 + \min\{k+\beta, -\gamma_2\}, 0\}$ for $\beta \in [0,1)$. Similarly, the Hölder norms for a function of one variable on $\mathbb{R}^+ := (0,\infty)$ with the weight near $\{0\}$ and the decay at infinity are denoted by $\|f\|^{(\gamma_2;0)}_{k,\alpha;(l);\mathbb{R}^+}$.

For a vector-valued function $\mathbf{f} = (f_1, f_2, \cdots, f_n)$, we define

$$\|\mathbf{f}\|^{(\gamma_1;O)(\gamma_2;\partial W)}_{k,\alpha;(l_1,l_2);\Omega} = \sum_{i=1}^{n}\|f_i\|^{(\gamma_1;O)(\gamma_2;\partial W)}_{k,\alpha;(l_1,l_2);\Omega}.$$

Let

$$C^{k,\alpha;(l_1,l_2)}_{(\gamma_1;O)(\gamma_2;\partial W)}(\Omega) = \Big\{\mathbf{f} \,:\, \|\mathbf{f}\|^{(\gamma_1;O)(\gamma_2;\partial W)}_{k,\alpha;(l_1,l_2);\Omega} < \infty\Big\}. \tag{3.21}$$

The requirement $\gamma_1 \geqslant \gamma_2$ in the definition above means that the regularity up to the wedge boundary is no worse than the regularity up to the wedge vertex. When $\gamma_1 = \gamma_2$, the δ^{o}–terms disappear so that (γ_1, O) can be dropped in the superscript. If there is no weight $(\gamma_2, \partial W)$ in the superscript, the δ–terms for the weights should be understood as $(\delta^{\mathrm{o}}_{\mathbf{x}})^{\max\{k+\gamma_1,0\}}$ and $(\delta^{\mathrm{o}}_{\mathbf{x}})^{\max\{k+\alpha+\gamma_1,0\}}$ in (3.18) and (3.19), respectively. Moreover, when no weight appears in the superscripts of the seminorms in (3.18) and (3.19), it means that neither δ^{o} nor δ^{w} is present. For a function of one variable defined on $(0,\infty)$, the weighted norm $\|f\|^{(\gamma_1;0)}_{k,\alpha;(l);\mathbb{R}^+}$ is understood in the same as the definition above with the weight to $\{0\}$ and the decay at infinity.

In the study of Problem 3.2 for a transonic solution (U_0^-, U_0^+) with wedge angle θ_{w}, the variables in U are expected to have different levels of regularity. Thus, we distinguish these variables by defining

$$U_1 = (\mathbf{u}\cdot\boldsymbol{\tau}^0_{\mathrm{w}}, \rho), \qquad U_2 = (w, p) \ \text{ with } w = \tfrac{\mathbf{u}\cdot\boldsymbol{\nu}^0_{\mathrm{w}}}{\mathbf{u}\cdot\boldsymbol{\tau}^0_{\mathrm{w}}}, \tag{3.22}$$

where $\boldsymbol{\nu}^0_{\mathrm{w}} = (-\sin\theta_{\mathrm{w}}, \cos\theta_{\mathrm{w}})$ and $\boldsymbol{\tau}^0_{\mathrm{w}} = (\cos\theta_{\mathrm{w}}, \sin\theta_{\mathrm{w}})$. We note that, for the solutions under our consideration, the denominator in the definition of w is strictly positive, since it is a positive constant for the background solution.

Note that $U^+_{10} = (|\mathbf{u}^+_0|, \rho^+_0)$ and $U^+_{20} = (0, p^+_0)$ are the corresponding quantities for the background subsonic state. Moreover, $\boldsymbol{\nu}^0_{\mathrm{w}}$ is the interior (for Ω_0) unit normal to ∂W_0, and $\boldsymbol{\tau}^0_{\mathrm{w}}$ is the tangential unit vector to ∂W_0, where ∂W_0 and Ω_0 are defined by (3.13) and (3.17) for the background solution (U_0^-, U_0^+), *i.e.*, $\mathbf{u}\cdot\boldsymbol{\tau}^0_{\mathrm{w}}$ and $\mathbf{u}\cdot\boldsymbol{\nu}^0_{\mathrm{w}}$ are the components u_1 and u_2 of $\mathbf{u}$ in the coordinates rotated clockwise by angle θ_{w}, so that the background downstream flow becomes horizontal.

Theorem 3.1 (Chen-Chen-Feldman [25]). *Let (U_0^-, U_0^+) be a constant transonic solution for the wedge angle $\theta_{\mathrm{w}} \in (0, \theta^{\mathrm{d}}_{\mathrm{w}})$. There are positive constants α, β, C_0, and ε, depending only on the background states (U_0^-, U_0^+), such that*

(i) *If (U_0^-, U_0^+) corresponds to the state on arc $\widehat{TS}$, and*

$$\|U^- - U_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|b' - \tan\theta_{\mathrm{w}}\|^{(-\alpha;0)}_{1,\alpha;(1+\beta);\mathbb{R}^+} < \varepsilon, \tag{3.23}$$

then there exist a solution (U,σ) *of* Problem 3.2(WT) *and a function* $U^\infty(\mathbf{x}) = (u_1^\infty, 0, p_0^+, \rho^\infty)(\mathbf{x}) = Z^\infty(-x_1\sin\theta_{\rm w} + x_2\cos\theta_{\rm w})$ *with* $Z^\infty : [0,\infty) \to \mathbb{R}^4$ *of form* $Z^\infty = (z_1, 0, p_0^+, z_4)$ *such that* U_1 *and* U_2 *defined by* (3.22) *satisfy*

$$\begin{aligned}&\|U_1 - U_1^\infty\|^{(-\alpha;\partial W)}_{2,\alpha;(\beta,1);\Omega} + \|U_2 - U_{20}^+\|^{(-\alpha;O)(-1-\alpha;\partial W)}_{2,\alpha;(1+\beta,0);\Omega} + \|\sigma' - s_0\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} + \|Z_1^\infty - U_{10}^+\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);[0,\infty)}\\ &\leqslant C_0\left(\|U^- - U_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|b' - \tan\theta_{\rm w}\|^{(-\alpha;0)}_{1,\alpha;(1+\beta);\mathbb{R}^+}\right),\end{aligned}\tag{3.24}$$

where we have denoted $U_1^\infty := (\mathbf{u}^\infty\cdot\boldsymbol{\tau}_{\rm w}^0, \rho^\infty) = (u_1^\infty\cos\theta_{\rm w}, \rho^\infty)$ *and* $Z_1^\infty := (z_1\cos\theta_{\rm w}, z_4)$;

(ii) *If* (U_0^-, U_0^+) *corresponds to the state on arc* $\widehat{TH}$ *and*

$$\|U^- - U_0^-\|_{2,\alpha;(\beta,0);\Omega^-} + \|b' - \tan\theta_{\rm w}\|^{(-\alpha-1;0)}_{2,\alpha;(\beta);\mathbb{R}^+} < \varepsilon,\tag{3.25}$$

then there exists a solution (U,σ) *of* Problem 3.2(ST) *such that* U_1 *and* U_2 *defined by* (3.22) *satisfy*

$$\begin{aligned}&\|U_1 - U_{10}^+\|^{(-1-\alpha;\partial W)}_{2,\alpha;(0,\beta);\Omega} + \|U_2 - U_{20}^+\|^{(-1-\alpha;O)}_{2,\alpha;(\beta,0);\Omega} + \|\sigma' - s_0\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+}\\ &\leqslant C_0\left(\|U^- - U_0^-\|_{2,\alpha;(\beta);\Omega^-} + \|b' - \tan\theta_{\rm w}\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+}\right).\end{aligned}\tag{3.26}$$

The solution (U,σ) *is unique within the class of solutions for each of* Problem 3.2(WT) *and* Problem 3.2(ST) *when the left-hand sides of* (3.24) *for* Problem 3.2(WT) *and* (3.26) *for* Problem 3.2(ST) *are less than* $C_0\varepsilon$ *correspondingly.*

The dependence of constants α, β, C_0, and ε in Theorem 3.1 is as follows: α and β depend on (U_0^-, U_0^+) but are independent of (C_0, ε), C_0 depends on $(U_0^-, U_0^+, \alpha, \beta)$ but is independent of ε, and ε depends on all $(U_0^-, U_0^+, \alpha, \beta, C_0)$.

The difference in the results of the two problems is that the solution of Problem 3.2(WT) has less regularity at corner O and decays faster with respect to $|\mathbf{x}|$ (or the distance from the wedge boundary) than the solution of Problem 3.2(ST).

Notice that part (i) of Theorem 3.1 gives the asymptotics of solution U as $|\mathbf{x}| \to \infty$ within Ω, and U^∞ is an asymptotic profile. Moreover, the convergence of U_2 to $U_2^\infty = U_{20}^+$ as $|\mathbf{x}| \to \infty$ is of polynomial rate $|\mathbf{x}|^{-(\beta+1)}$ that is faster than the convergence rate of U_1, which is $|\mathbf{x}|^{-\beta}$. However, as $x_2 \to \infty$, both U_1 and U_2 decay to U_{10}^+ and U_{20}^+, respectively, with the decay rate $x_2^{-(\beta+1)}$, which can be seen by combining the estimates of the first and last terms on the right-hand side of (3.24) for U_1. Part (ii) of Theorem 3.1 does not give the asymptotic limit of U_1 as $|\mathbf{x}| \to \infty$, while U_2 converges to U_{20}^+ with the decay rate $|\mathbf{x}|^{-\beta}$. Also, as $x_2 \to \infty$, both U_1 and U_2 decay to U_{10}^+ and U_{20}^+, respectively, with the decay rate $x_2^{-\beta}$ for part (ii).

Furthermore, for both parts (i) and (ii) of Theorem 3.1, the asymptotic profile in the Lagrangian coordinates is given in Theorem 3.3.

3.2. **Approach I for Problem 3.2(WT).** We now describe Approach I for solving Problem 3.2(WT). We work in the Lagrangian coordinates introduced in (3.6). From the slip condition (3.15) on the wedge boundary ∂W, it follows that ∂W is a streamline so that ∂W becomes the half-line $\mathcal{L}_1 = \{(y_1,y_2) : y_1 > 0, y_2 = 0\}$ in the Lagrangian coordinates. Let $\mathcal{S} = \{y_2 = \hat\sigma(y_1)\}$ be a shock-front. Then, from equations (3.7)–(3.10), we can derive the Rankine-Hugoniot conditions along $\mathcal{S}$:

$$\hat\sigma'(y_1)\big[\frac{1}{\rho u_1}\big] = -\big[\frac{u_2}{u_1}\big],\tag{3.27}$$

$$\hat\sigma'(y_1)\big[u_1 + \frac{p}{\rho u_1}\big] = -\big[\frac{pu_2}{u_1}\big],\tag{3.28}$$

$$\hat\sigma'(y_1)\big[\,u_2\,\big] = \big[\,p\,\big],\tag{3.29}$$

$$\big[\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho}\big] = 0.\tag{3.30}$$

The background shock-front in the Lagrangian coordinates is $\mathcal{S}_0 = \{y_2 = s_1 y_1\}$ with $s_1 = \rho_0^+ u_{10}^+ (s_0 - \tan\theta_0) > 0$.

Without loss of generality, we assume that, in the Lagrangian coordinates, the supersonic solution U^- exists in domain $\mathbb{D}^-$ defined by

$$\mathbb{D}^- = \Big\{\mathbf{y} \,:\, y_2 > \frac{s_1}{2} y_1,\, y_1 > 0\Big\}. \tag{3.31}$$

For a given shock function $\hat{\sigma}(y_1)$, let

$$\mathbb{D}_{\hat{\sigma}}^- = \big\{\mathbf{y} : y_2 > \hat{\sigma}(y_1),\, y_1 > 0\big\}, \tag{3.32}$$

$$\mathbb{D}_{\hat{\sigma}} = \big\{\mathbf{y} : 0 < y_2 < \hat{\sigma}(y_1),\, y_1 > 0\big\}. \tag{3.33}$$

Then **Approach I** consists of three steps:

1. **Potential function** $\phi(\mathbf{y})$. We first use a potential function to reduce the full Euler equations (3.7)–(3.10) to a scalar second-order nonlinear elliptic PDE in the subsonic region. This method was first proposed in [23] in which the advantage of the conservation properties of the Euler system is taken for the reduction.

More precisely, since $\rho u_1 \neq 0$ in either the supersonic or subsonic region, it follows from (3.7) that there exists a potential function of the vector field $(\frac{u_2}{u_1}, \frac{1}{\rho u_1})$ such that

$$D\phi = (\frac{u_2}{u_1}, \frac{1}{\rho u_1}) \qquad \text{with } \phi(\mathbf{0}) = 0. \tag{3.34}$$

Equation (3.10) implies the Bernoulli law:

$$\frac{1}{2}q^2 + \frac{\gamma p}{(\gamma-1)\rho} = B(y_2), \tag{3.35}$$

where $q = |\mathbf{u}| = \sqrt{u_1^2 + u_2^2}$, and $B = B(y_2)$ is completely determined by the given incoming flow U^- at the initial position $\mathcal{I}$ because of the Rankine-Hugoniot condition (3.30).

From equations (3.7)–(3.10), we find

$$\Big(\frac{p}{\rho^\gamma}\Big)_{y_1} = 0, \tag{3.36}$$

which implies

$$p = A(y_2)\rho^\gamma \qquad \text{in the subsonic region } \mathbb{D}_{\hat{\sigma}}. \tag{3.37}$$

With equations (3.34) and (3.37), we can rewrite the Bernoulli law (3.35) as

$$\frac{\phi_{y_1}^2 + 1}{2\phi_{y_2}^2} + \frac{\gamma}{\gamma - 1} A \rho^{\gamma+1} = B\rho^2. \tag{3.38}$$

In the subsonic region, $q = |\mathbf{u}| < c = \sqrt{\frac{\gamma p}{\rho}}$. Therefore, the Bernoulli law (3.35) implies

$$\rho^{\gamma-1} > \frac{2(\gamma-1)B}{\gamma(\gamma+1)A}. \tag{3.39}$$

Condition (3.39) guarantees that ρ can be solved from (3.38) as a smooth function of $(D\phi, A, B)$.

Assume that $A = A(y_2)$ has been determined. Then $(\mathbf{u}, p, \rho)$ can be expressed as functions of $D\phi$:

$$\rho = \rho(D\phi, A, B), \qquad \mathbf{u} = (\frac{1}{\rho\phi_{y_2}}, \frac{\phi_{y_1}}{\rho\phi_{y_2}}), \qquad p = A\rho^\gamma, \tag{3.40}$$

since $B = B(y_2)$ is given by the incoming flow.

Similarly, in the supersonic region $\mathbb{D}^-$, we employ the corresponding variables (ϕ^-, A^-, B) to replace U^-, where B is the same as in the subsonic region because of the Rankine-Hugoniot condition (3.30).

We now choose (3.9) to derive a *second-order nonlinear elliptic equation for* ϕ so that the full Euler system (3.7)–(3.10) is reduced to the following nonlinear PDE in the subsonic region:

$$\big(N^1(D\phi, A, B)\big)_{y_1} + \big(N^2(D\phi, A, B)\big)_{y_2} = 0, \tag{3.41}$$

where $(N^1, N^2)(D\phi, A, B) = (u_2, p)(D\phi, A, B)$ are given by

$$N^1(D\phi, A, B) = \frac{\phi_{y_1}}{\phi_{y_2}\rho(D\phi, A(y_2), B(y_2))}, \quad N^2(D\phi, A, B) = A(y_2)\big(\rho(D\phi, A(y_2), B(y_2))\big)^{\gamma}. \tag{3.42}$$

Then a careful calculation shows that

$$N^1_{\phi_{y_1}} N^2_{\phi_{y_2}} - N^1_{\phi_{y_2}} N^2_{\phi_{y_1}} = \frac{c^2\rho^2 u_1^2}{c^2 - q^2} > 0 \tag{3.43}$$

in the subsonic region with $\rho u_1 \neq 0$. Therefore, when ϕ is sufficiently close to ϕ_0^+ (determined by the subsonic background state U_0^+) in the C^1–norm, equation (3.41) is uniformly elliptic, and the Euler system (3.7)–(3.10) is reduced to the elliptic equation (3.41) in domain $\mathbb{D}_{\hat\sigma}$, where $\hat\sigma$ is the function for the free boundary (transonic shock).

The boundary condition for ϕ on the wedge boundary $\{y_2 = 0\}$ is derived from the fact that $\phi(y_1, y_2) = x_2(y_1, y_2)$ by (3.5)–(3.6) and (3.34). Then, recalling that $\partial W = \{\mathbf{x} : x_2 = b(x_1), x_1 > 0\}$ in the $\mathbf{x}$–coordinates which is $\{\mathbf{y} : y_2 = 0, y_1 > 0\}$ in the $\mathbf{y}$–coordinates and using $y_1 = x_1$ by (3.6), we obtain

$$\phi(y_1, 0) = b(y_1). \tag{3.44}$$

The condition on $\mathcal{S}$ is derived from the Rankine-Hugoniot conditions (3.27)–(3.29). Condition (3.27) is equivalent to the continuity of ϕ across $\mathcal{S}$:

$$[\phi]|_{\mathcal{S}} = 0, \tag{3.45}$$

which, by (3.34), gives

$$\hat\sigma'(y_1) = -\frac{[\phi_{y_1}]}{[\phi_{y_2}]}(y_1, \hat\sigma(y_1)). \tag{3.46}$$

Replacing $\hat\sigma'(y_1)$ in (3.28)–(3.29) by (3.46) and using (3.37) give rise to the conditions on $\mathcal{S}$:

$$G(D\phi, A, U^-) \equiv [\phi_{y_1}]\Big[\frac{1}{\rho\phi_{y_2}} + A\rho^\gamma\phi_{y_2}\Big] - [\phi_{y_2}][A\rho^\gamma\phi_{y_1}] = 0, \tag{3.47}$$

$$H(D\phi, A, U^-) \equiv [\phi_{y_1}][N^1] + [\phi_{y_2}][N^2] = 0. \tag{3.48}$$

We now combine the above two conditions into the boundary condition for (3.41) by eliminating A. Taking the partial derivative of G and H with respect to A respectively and making careful calculation, we have

$$\begin{aligned} G_A &= [\phi_{y_1}]\Big(\frac{N^1_A}{\phi_{y_1}} + \phi_{y_2}N^2_A\Big) - [\phi_{y_2}]\phi_{y_1}N^2_A \\ &= \frac{u_2\rho^\gamma(q^2 + \frac{c^2}{\gamma-1})}{u_1(c^2-q^2)}\left[\frac{1}{\rho u_1}\right] - \frac{\rho^{\gamma-1}}{u_1(c^2-q^2)}\left(u_2^2 + \frac{c^2-u_1^2}{\gamma-1}\right)\left[\frac{u_2}{u_1}\right] < 0, \end{aligned}$$

and

$$H_A = N^1_A[\phi_{y_1}] + N^2_A[\phi_{y_2}] = \frac{\gamma}{\gamma-1}\frac{\rho^{\gamma-1}u_2}{c^2-q^2}\left[\frac{u_2}{u_1}\right] - \frac{\rho^\gamma(q^2+\frac{c^2}{\gamma-1})}{c^2-q^2}\left[\frac{1}{\rho u_1}\right] > 0,$$

since $[\frac{1}{\rho u_1}] < 0$ and u_2^- is close to 0. Therefore, both equations (3.47) and (3.48) can be solved for A to obtain $A = g_1(D\phi, U^-)$ and $A = g_2(D\phi, U^-)$, respectively. With these, we obtain our desired condition on the free boundary (*i.e.*, the shock-front):

$$\bar g(D\phi, U^-) := (g_2 - g_1)(D\phi, U^-) = 0. \tag{3.49}$$

Then the original free boundary problem, Problem 3.2, is reduced to the following free boundary problem for the elliptic equation (3.41):

Problem 3.3 (Free Boundary Problem)**.** *Seek* $(\hat{\sigma}, \phi, A)$ *such that* ϕ *is a solution of the elliptic equation* (3.41) *in the region with the fixed boundary condition* (3.44) *and conditions* (3.45) *and* (3.47)*–*(3.49) *on* $\mathcal{S}$*.*

2. **Hodograph transformation and fixed boundary value problem**. In order to solve the free boundary problem, we employ the hodograph transformation so that the shock-front becomes a fixed boundary by using the free boundary conditions (3.45) and (3.49). This allows us to find ϕ for each A from an appropriately chosen set. After that, we only need to perform an iteration for the unknown function A to satisfy (3.47)–(3.48).

Note that the expected solutions in Theorem 3.1 satisfy that $\|U - U_0^+\|_{L^\infty(\Omega)} \leqslant C_0\varepsilon$. Then, denoting by ϕ_0^+ the potential function (3.34) for the subsonic background state U_0^+, we obtain that ϕ is close to ϕ_0^+ in C^1 on the closure of the subsonic region. On the iteration, we consider (and eventually obtain) solutions U for which the same property holds. Thus, we assume that ϕ is close to ϕ_0^+ in $C^1(\overline{\mathbb{D}_{\hat{\sigma}}})$ below; see (3.33).

We now extend the domain of ϕ^- from $\mathbb{D}^-$ to the first quadrant $\mathbb{D}^- \cup \mathbb{D}_{\hat{\sigma}}$. Let

$$\phi_0^- = \frac{1}{\rho_0^- u_{20}^-} y_2,$$

which is the potential function (3.34) for the supersonic background state U_0^-. Then ϕ^- is close to ϕ_0^- in $C^1(\overline{\mathbb{D}^-})$ since U^- is close to U_0^- in L^∞ (and in the stronger norm; see Theorem 3.1). We can extend ϕ^- into $\mathbb{D}^- \cup \mathbb{D}_{\hat{\sigma}}$ so that it remains close to ϕ_0^- in C^1 on the closure of $\mathbb{D}^- \cup \mathbb{D}_{\hat{\sigma}}$. We then use the following partial hodograph transformation:

$$(y_1, y_2) \to (z_1, z_2) = (\phi - \phi^-, y_2). \tag{3.50}$$

Note that $\partial_{y_1}(\phi_0^+ - \phi_0^-) = \frac{u_{20}^+}{u_{10}^+} > 0$ by using (3.34), where (u_{10}^+, u_{20}^+) is the velocity of the background subsonic state U_0^+ and the fact that $u_{20}^- = 0$ has been used. Since ϕ and ϕ^- are close to ϕ_0^+ and ϕ_0^- in the C^1–norm respectively, transformation (3.50) is invertible, so that y_1 is a function of $\mathbf{z} := (z_1, z_2)$, denoted as $y_1 = \varphi(\mathbf{z})$.

Let

$$M^1(D\phi, A, U^-) = N^1(D\phi, A, B) + N^2(D\phi, A, B)\frac{[\phi_{y_2}]}{[\phi_{y_1}]}, \quad M^2(D\phi, A, U^-) = \frac{N^2(D\phi, A, B)}{[\phi_{y_1}]},$$

and

$$\overline{M}^i(D\varphi, \varphi, A, \mathbf{z}) = -M^i(\partial_{y_1}\phi^-(\varphi, z_2) + \frac{1}{\varphi_{z_1}}, \partial_{y_2}\phi^-(\varphi, z_2) - \frac{\varphi_{z_2}}{\varphi_{z_1}}, A, U^-(\varphi, z_2)), \qquad i = 1, 2.$$

Then equation (3.41) becomes

$$\big(\overline{M}^1(D\varphi, \varphi, A, \mathbf{z})\big)_{z_1} + \big(\overline{M}^2(D\varphi, \varphi, A, \mathbf{z})\big)_{z_2} = 0. \tag{3.51}$$

Notice that

$$\overline{M}^1_{\varphi_{z_1}}\overline{M}^2_{\varphi_{z_2}} - \frac{1}{4}\big(\overline{M}^1_{\varphi_{z_2}} + \overline{M}^2_{\varphi_{z_1}}\big)^2 = [\phi_{y_1}]^2\big(N^1_{\phi_{y_1}}N^2_{\phi_{y_2}} - (N^1_{\phi_{y_2}})^2\big) > 0,$$

which implies that equation (3.51) is uniformly elliptic, for any solution φ that is close to φ_0^+ (determined by (3.50) with $\phi = \phi_0^+$) in the C^1–norm.

Under transform (3.50), the unknown shock-front $\mathcal{S}$ becomes a fixed boundary, which is the z_2-axis, where we have used that ϕ is close in C^1 to ϕ_0^+ in $\overline{\mathbb{D}_{\hat{\sigma}}}$ and to ϕ_0^- in $\overline{\mathbb{D}_{\hat{\sigma}}^-}$ in order to conclude that ϕ is

Lipschitz across $\mathcal{S}$ from (3.34) and then that $\phi = \phi^-$ on $\mathcal{S}$ but $\phi \neq \phi^-$ in $\overline{\mathbb{D}_{\hat\sigma}}\backslash\mathcal{S}$. Along the z_2-axis, condition (3.49) is now

$$\begin{aligned}\tilde g(D\varphi,\varphi,\mathbf{z}) &:= \bar g(\partial_{y_1}\phi^-(\varphi,z_2)+\frac{1}{\varphi_{z_1}},\partial_{y_2}\phi^-(\varphi,z_2)-\frac{\varphi_{z_2}}{\varphi_{z_1}},U^-(\varphi,z_2))\\ &= 0 \qquad \text{on } \{z_1=0,\, z_2>0\}.\end{aligned} \tag{3.52}$$

We also convert condition (3.48) into the $\mathbf{z}$–coordinates: Along the z_2-axis,

$$\widetilde H(D\varphi,\varphi,A,\mathbf{z}) := H(\partial_{y_1}\phi^-(\varphi,z_2)+\frac{1}{\varphi_{z_1}},\, \partial_{y_2}\phi^-(\varphi,z_2)-\frac{\varphi_{z_2}}{\varphi_{z_1}},A,U^-(\varphi,z_2))=0. \tag{3.53}$$

The condition on the z_1-axis can be derived from (3.44) as follows: Restricted on $z_2=0$, the coordinate transformation (3.50) becomes

$$z_1 = b(y_1)-\phi_-(y_1,0).$$

Then y_1 can be expressed in terms of z_1 as $y_1=\tilde b(z_1)$ so that $\varphi(z_1,0)=y_1$ becomes

$$\varphi(z_1,0)=\tilde b(z_1) \qquad \text{on } \mathcal{L}_1 := \{z_2=0,\, z_1>0\}. \tag{3.54}$$

Therefore, the original wedge problem has now been reduced to the following problem on the first quadrant $\mathbb{Q}$.

Problem 3.4 (Fixed Boundary Value Problem)**.** *Seek* (φ,A) *such that* φ *is a solution of the second-order nonlinear elliptic equation* (3.51) *in the unbounded domain* $\mathbb{Q}$ *with the boundary conditions* (3.52) *and* (3.54)*, and such that* (3.53) *holds.*

3. **Solution to the fixed boundary value problem – Problem 3.4**. Through the shock polar, we can determine the values of U at the origin so that $A(0)$ is fixed, depending on the values of $U^-(\mathbf{0})$ and $b'(0)$. Then we solve (3.53) to obtain a unique solution $\tilde A = h(\mathbf{z},\phi,D\phi)$ that defines the iteration map.

This is achieved by the following fixed point argument. Consider a Banach space:

$$X=\{\mathcal{A}\,:\,\mathcal{A}(0)=0,\ \|\mathcal{A}\|^{(-\alpha);\{0\}}_{1,\alpha;(1+\beta);\mathbb{R}^+}<\infty\}.$$

Then we define our iteration map $\mathcal{J}: X \longrightarrow X$ through the following:

First, we define a smooth cutoff function χ on $[0,\infty)$ such that

$$\chi(s)=\begin{cases}1 & \text{for } 0\leqslant s<1,\\ 0 & \text{for } s>2.\end{cases}$$

Set

$$A(0):=t(\omega(0),b'(0)) \qquad \text{for } \omega=U^- - U_0^-, \tag{3.55}$$

where t is a function determined by the Rankine-Hugoniot conditions (3.47)–(3.48). Then we define $w_t(z_2)$ as

$$w_t(z_2):=A_0^+ + \big(t(\omega(0),b'(0))-A_0^+\big)\chi(z_2), \tag{3.56}$$

where $A_0^+=\frac{p_0^+}{(\rho_0^+)^\gamma}$.

Consider any $A=A(z_2)$ so that $A-w_t\in X$ satisfying

$$\|A-A_0^+\|^{(-\alpha);\{0\}}_{1,\alpha;(1+\beta);\mathbb{R}^+}\leqslant C_0\varepsilon \tag{3.57}$$

for some fixed constant $C_0>0$. With this A, we solve equation (3.51) for $\varphi=\varphi_A$ in the unbounded domain $\mathbb{Q}$ with the boundary conditions (3.52) and (3.54), and with the asymptotic condition $\varphi\to\varphi^\infty$ as $\mathbf{x}\to\infty$, where the limit is understood in the appropriate sense, φ^∞ is the solution of

$$z_1=(\phi^\infty-\phi^-)(\varphi^\infty,z_2), \tag{3.58}$$

with $\phi^\infty = \frac{u_{20}^+}{u_{10}^+} y_1 + l(y_2)$, and $l(y_2)$ is determined by the Bernoulli law (3.38), via replacing ϕ and ρ by their asymptotic values ϕ^∞ and $\rho^\infty(y_2) = \big(\frac{p_0^+}{A(y_2)}\big)^{\frac{1}{\gamma}}$ and noting that $B = B(y_2)$ is determined by the upstream state U^-. More specifically, we show the existence of a solution φ of (3.51)–(3.52) and (3.54) in the set:

$$\Sigma_\delta = \big\{\varphi : \|\varphi - \varphi^\infty\|^{(-1-\alpha);\mathcal{L}_1}_{2,\alpha;(\beta,0);\mathbb{Q}} \leqslant \delta\big\} \qquad \text{for sufficiently small } \delta > 0,$$

which is a compact and convex subset of the Banach space:

$$\mathcal{B} = \big\{\varphi \,:\, \|\varphi - \varphi^\infty\|^{(-1-\alpha');\mathcal{L}_1}_{2,\alpha';(\beta',0);\mathbb{Q}} < \infty\big\} \qquad \text{with } 0 < \alpha' < \alpha \text{ and } 0 < \beta' < \beta.$$

For $\varphi \in \Sigma_\delta$, equation (3.51) is uniformly elliptic if $\delta > 0$ is small. This allows us to solve the problem for $\varphi = \varphi_A \in \Sigma_\delta$ by the Schauder fixed point theorem if the perturbation is small, *i.e.*, if ε is small in (3.57) and the conditions of Theorem 3.1. Then, with this $\varphi = \varphi_A$, we solve (3.53) to obtain a unique $\tilde{A}$ that defines the iteration map J by $\mathcal{J}(A - w_t) := \tilde{A} - w_t$.

Finally, by the implicit function theorem, we prove that $\mathcal{J}$ has a fixed point $A - w_t$, for which A satisfies (3.57).

For more details for this approach, see Chen-Chen-Feldman [23, 24]. This approach can also be applied to Problem 3.2(ST); see [112] for the case when the wedge angle is sufficiently small.

3.3. **Approach II for Problem 3.2(ST) and Problem 3.2(WT).** We now describe the second approach, Approach II. It allows us to handle both cases in Theorem 3.1: Problem 3.2(WT) and Problem 3.2(ST). In particular, for Problem 3.2(WT), this approach yields a better asymptotic decay rate, as stated in (3.24).

It is convenient to rotate the $\mathbf{x}$–coordinates clockwise by the wedge angle $\theta_{\rm w}$, so that the background downstream flow becomes horizontal, as discussed in the paragraph before Theorem 3.1. We still use the same notations in the rotated coordinates when no confusion arises; in particular, we write $\mathbf{x} = (x_1, x_2)$ and $\mathbf{u} = (u_1, u_2)$ in the rotated basis. Then, in the new coordinates,

$$\frac{u_{20}^-}{u_{10}^-} = -\tan\theta_{\rm w}, \quad U_0^- = (u_{10}^-,\, -u_{10}^- \tan\theta_{\rm w},\, p_0^-,\, \rho_0^-), \quad U_0^+ = (u_{10}^+,\, 0,\, p_0^+,\, \rho_0^+). \tag{3.59}$$

Since the velocity components (u_1, u_2) are now in the basis $(\boldsymbol{\tau}_{\rm w}^0, \boldsymbol{\nu}_{\rm w}^0)$, *i.e.*, $u_1 = \mathbf{u}\cdot\boldsymbol{\tau}_{\rm w}^0$ and $u_2 = \mathbf{u}\cdot\boldsymbol{\nu}_{\rm w}^0$, we see that, by (3.22),

$$U_1 = (u_1, \rho), \qquad U_2 = (w, p) \ \text{ with } w = \tfrac{u_2}{u_1} \tag{3.60}$$

in the new coordinates. Furthermore, we obtain from (3.13) and (3.23) or (3.25) with small ε that, in the rotated coordinates,

$$\partial W = \{\mathbf{x} \in \mathbb{R}^2 \,:\, x_2 = b_{\rm rot}(x_1),\ b_{\rm rot}(0) = 0\}, \tag{3.61}$$

and function $b_{\rm rot}(x_1)$ satisfies the estimates in (3.63) or (3.65) below, respectively, with $C\varepsilon$ instead of ε when ε is small, where C depends only on $b(\cdot)$. For the background solution, $b_{\rm rot,0} = 0$, *i.e.*, ∂W_0 is the positive x_1-axis.

We now construct a solution with a shock-front $\mathcal{S}$ expressed as (3.16) in the rotated coordinates with a function $\tilde{\sigma}(x_1)$. The background shock is now expressed as $\mathcal{S}_0 := \{x_2 = \tilde{\sigma}_0(x_1) \,:\, x_1 > 0\}$ for $\tilde{\sigma}_0(x_1) := \tilde{s}_0 x_1$, where $\tilde{s}_0 = \tan(\arctan s_0 - \theta_{\rm w})$. Then the subsonic region of the solution has the form:

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 \,:\, b_{\rm rot}(x_1) < x_2 < \tilde{\sigma}(x_1), x_1 > 0\}. \tag{3.62}$$

We can assume that the upstream steady supersonic smooth solution $U^-(\mathbf{x})$ exists in region $\Omega^- = \{\mathbf{x} : \frac{\tilde{s}_0}{2} x_1 < x_2 < 2\tilde{s}_0 x_1, x_1 \geqslant 0\}$, beyond the background shock, but is still close to U_0^-. Moreover, in part (i) of Theorem 3.1, U^∞ is independent of x_1 and $U^\infty = Z^\infty$ in the rotated coordinates.

More specifically, we establish the following theorem in the rotated coordinates:

Theorem 3.2 (Chen-Chen-Feldman [25]). *Let (U_0^-, U_0^+), given by (3.59), be a constant transonic solution for the wedge angle $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$. There are positive constants α, β, C_0, and ε depending only on the background states (U_0^-, U_0^+) such that*

(i) *If (U_0^-, U_0^+) corresponds to the state on arc $\widehat{TS}$ and*

$$\|U^- - U_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|b'_{\rm rot}\|^{(-\alpha;0)}_{1,\alpha;(1+\beta);\mathbb{R}^+} < \varepsilon, \tag{3.63}$$

then there exist a solution $(U, \tilde\sigma)$ of Problem 3.2(WT) *and a function*

$$U^\infty(x_2) = (u_1^\infty(x_2), 0, p_0^+, \rho^\infty(x_2))$$

so that U_1 and U_2 defined by (3.22) satisfy

$$\begin{aligned} &\|U_1 - U_1^\infty\|^{(-\alpha;\partial W)}_{2,\alpha;(\beta,1);\Omega} + \|U_2 - U_{20}^+\|^{(-\alpha;O)(-1-\alpha;\partial W)}_{2,\alpha;(1+\beta,0);\Omega} + \|\tilde\sigma' - \tilde s_0\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} + \|U_1^\infty - U_{10}^+\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);[0,\infty)} \\ &\leqslant C_0\left(\|U^- - U_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|b'_{\rm rot}\|^{(-\alpha;0)}_{1,\alpha;(1+\beta);\mathbb{R}^+}\right), \end{aligned} \tag{3.64}$$

where $U_1^\infty = (u_1^\infty, \rho^\infty)$.

(ii) *If (U_0^-, U_0^+) corresponds to the state on arc $\widehat{TH}$, and*

$$\|U^- - U_0^-\|_{2,\alpha;(\beta,0);\Omega^-} + \|b'_{\rm rot}\|^{(-\alpha-1;0)}_{2,\alpha;(\beta);\mathbb{R}^+} < \varepsilon, \tag{3.65}$$

then there exists a solution $(U, \tilde\sigma)$ of Problem 3.2(ST) *so that U_1 and U_2 defined by (3.22) satisfy*

$$\begin{aligned} &\|U_1 - U_{10}^+\|^{(-1-\alpha;\partial W)}_{2,\alpha;(0,\beta);\Omega} + \|U_2 - U_{20}^+\|^{(-1-\alpha;O)}_{2,\alpha;(\beta,0);\Omega} + \|\tilde\sigma' - \tilde s_0\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+} \\ &\leqslant C_0\left(\|U^- - U_0^-\|_{2,\alpha;(\beta);\Omega^-} + \|b'_{\rm rot}\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+}\right). \end{aligned} \tag{3.66}$$

The solution $(U, \tilde\sigma)$ is unique within the class of solutions for each of Problem 3.2(WT) *and* Problem 3.2(ST) *when the left-hand sides of* (3.24) *for* Problem 3.2(WT) *and* (3.26) *for* Problem 3.2(ST) *are less than $C_0\varepsilon$ correspondingly.*

Clearly, Theorem 3.1 follows from Theorem 3.2 if ε is small so that, from the estimates of $\tilde\sigma$ in (3.64) or (3.66), the shock remains a graph of C^1 function: $x_2 = \sigma(x_1)$ after rotating the coordinates back.

To prove Theorem 3.2, we work in the Lagrangian coordinates (3.6) defined for the rotated coordinates $\mathbf{x} = (x_1, x_2)$. Then, as in the previous case, using the fact that the wedge boundary ∂W is a streamline due to the slip condition (3.15) on ∂W, we obtain that, in the present Lagrangian coordinates, ∂W becomes the half-line:

$$\mathcal{L}_1 = \{(y_1, y_2) \,:\, y_1 > 0, y_2 = 0\}.$$

The background shock-front $\mathcal{S}_0$ is now given by $\mathcal{S}_0 = \{y_2 = s_1 y_1, y_1 > 0\}$ with $s_1 = \rho_0^+ u_{10}^+ \tilde s_0$. We can assume that, in the Lagrangian coordinates, the supersonic solution U^- exists in domain $\mathbb{D}^-$ defined by (3.31). Shock $\mathcal{S}$ is given by $y_2 = \hat\sigma(y_1)$ for $y_1 > 0$, where function $\hat\sigma$ differs from the one in Approach I because the Lagrangian coordinates are now defined differently. The supersonic region $\mathbb{D}^-_{\hat\sigma}$ and the subsonic region $\mathbb{D}_{\hat\sigma}$ of the solution are given by (3.32) and (3.33) respectively, with the present function $\hat\sigma$.

We first present the existence and estimates of the solution in the Lagrangian coordinates:

Theorem 3.3. *Let (U_0^-, U_0^+) be a constant transonic solution for the wedge angle $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$. There are positive constants α, β, C_0, and ε depending only on the background states (U_0^-, U_0^+) such that, if ∂W in (3.61) and U^- satisfy*

(i) (3.63) *for* Problem 3.2(WT),

(ii) (3.65) *for* Problem 3.2(ST),

then there exist a transonic shock $\mathcal{S}_L = \{y_2 = \hat{\sigma}(y_1),\, y_1 > 0\}$ *and a subsonic solution* $U = U(\mathbf{y})$ *of* (3.7)–(3.10) *in* $\mathbb{D}_{\hat{\sigma}}$*, satisfying the Rankine-Hugoniot conditions* (3.27)–(3.30) *along* $\mathcal{S}_L$ *with* U^- *expressed in the Lagrangian coordinates in* $\mathbb{D}^-_{\hat{\sigma}}$ *and the slip condition* $w_{|\mathcal{L}_1} = b'_{\rm rot}$*, as well as there exists a function* $\mathcal{U}^\infty(y_2) = (u_1^\infty(y_2), 0, p_0^+, \rho^\infty(y_2))$*, such that* $U(\mathbf{y})$ *satisfies the following estimates:*

(i) *For* Problem 3.2(WT)*,*

$$\begin{aligned}&\|U_1-\mathcal{U}_1^\infty\|^{(-\alpha;\mathcal{L}_1)}_{2,\alpha;(1+\beta,0);\mathbb{D}_{\hat{\sigma}}}+\|U_2-U_{20}^+\|^{(-\alpha;O)(-1-\alpha;\mathcal{L}_1)}_{2,\alpha;(1+\beta,0);\mathbb{D}_{\hat{\sigma}}}+\|\hat{\sigma}'-s_1\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+}+\|\mathcal{U}_1^\infty-U_{10}^+\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+}\\&\leqslant C_0\left(\|U^--U_0^-\|_{2,\alpha;(1+\beta,0);\mathbb{D}^-_{\hat{\sigma}}}+\|b'_{\rm rot}\|^{(-\alpha;0)}_{1,\alpha;(1+\beta);\mathbb{R}^+}\right);\end{aligned}\tag{3.67}$$

(ii) *For* Problem 3.2(ST)*,*

$$\begin{aligned}&\|U_1-\mathcal{U}_1^\infty\|^{(-1-\alpha;\mathcal{L}_1)}_{2,\alpha;(\beta,0);\mathbb{D}_{\hat{\sigma}}}+\|U_2-U_{20}^+\|^{(-1-\alpha;O)}_{2,\alpha;(\beta,0);\mathbb{D}_{\hat{\sigma}}}+\|\hat{\sigma}'-s_1\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+}+\|\mathcal{U}_1^\infty-U_{10}^+\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+}\\&\leqslant C_0\left(\|U^--U_0^-\|_{2,\alpha;(\beta);\mathbb{D}^-_{\hat{\sigma}}}+\|b'_{\rm rot}\|^{(-1-\alpha;0)}_{2,\alpha;(\beta);\mathbb{R}^+}\right),\end{aligned}\tag{3.68}$$

where $\mathcal{U}_1^\infty(y_2) := (u_1^\infty(y_2), \rho^\infty(y_2))$.

The solution U *is unique within the class of solutions for each of* Problem 3.2(WT) *and* Problem 3.2(ST) *when the left-hand sides of* (3.67) *for* Problem 3.2(WT) *and* (3.68) *for* Problem 3.2(ST) *are less than* $C_0\varepsilon$ *correspondingly.*

We remark that function $\mathcal{U}^\infty(y_2)$ can be understood as the asymptotic limit of $U(\mathbf{y})$ as $y_1 \to \infty$.

Now we describe the proof of Theorem 3.3, which is the main part of **Approach II**. Rewrite system (3.7)–(3.10) into the following nondivergence form for $U = (\mathbf{u}, p, \rho) \in \mathbb{R}^4$:

$$A(U)U^\top_{y_1} + B(U)U^\top_{y_2} = 0,\tag{3.69}$$

where

$$A(U)=\begin{bmatrix}-\frac{1}{\rho u_1^2} & 0 & 0 & -\frac{1}{\rho^2 u_1}\\ 1-\frac{p}{\rho u_1^2} & 0 & \frac{1}{\rho u_1} & -\frac{p}{\rho^2 u_1}\\ 0 & 1 & 0 & 0\\ u_1 & u_2 & \frac{\gamma}{(\gamma-1)\rho} & -\frac{\gamma p}{(\gamma-1)\rho^2}\end{bmatrix},\qquad B(U)=\begin{bmatrix}\frac{u_2}{u_1^2} & -\frac{1}{u_1} & 0 & 0\\ \frac{pu_2}{u_1^2} & -\frac{p}{u_1} & -\frac{u_2}{u_1} & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Solving $\det(\lambda A - B) = 0$ for λ, we obtain four eigenvalues:

$$\lambda_1 = \lambda_2 = 0\,(\text{real}),\qquad \lambda_j = -\frac{c\rho}{c^2-u_1^2}\big(cu_2 + (-1)^j u_1\sqrt{c^2-q^2}\,i\big)\ \text{for } j = 3,4\quad(\text{complex}),$$

where $q = \sqrt{u_1^2+u_2^2} < c$ in the subsonic region and $i = \sqrt{-1}$. The corresponding left eigenvectors are

$$\mathbf{l}_1 = (0,0,0,1),\qquad \mathbf{l}_2 = (-pu_1, u_1, u_2, -1),$$

$$\mathbf{l}_{3,4} = \Big(\frac{p(\gamma p-\rho u_1^2)}{(\gamma-1)\rho u_1}\lambda_{3,4} + \frac{\gamma p^2 u_2}{(\gamma-1)u_1},\ -\big(u_1+\frac{\gamma p}{(\gamma-1)\rho u_1}\big)\lambda_{3,4} - \frac{\gamma p u_2}{(\gamma-1)u_1},\ \frac{\gamma p}{\gamma-1} - u_2\lambda_{3,4},\ \lambda_{3,4}\Big).$$

Then

(i) Multiplying equations (3.69) from the left by $\mathbf{l}_1$ leads to the same equation (3.10). This, together with the Rankine-Hugoniot condition (3.30), implies the Bernoulli law (3.35) to be held in both supersonic and subsonic domains, as well as across the shock-front. Therefore, $B(y_2)$ can be computed from the upstream flow U^-. If u_1 is a small perturbation of u_{10}^+, then $u_1 > 0$. Therefore, we can solve (3.35) for u_1:

$$u_1 = \frac{\sqrt{2\big(B - \frac{2\gamma p}{(\gamma-1)\rho}\big)}}{\sqrt{1+w^2}}\qquad \text{with } w = \tfrac{u_2}{u_1}.\tag{3.70}$$

(ii) Multiplying system (3.69) from the left by $\mathbf{l}_2$ also gives (3.36).

(iii) Multiplying equations (3.69) from the left by $\mathbf{l}_3$ and separating the real and imaginary parts of the equation lead to the elliptic system:

$$\begin{aligned} &D_R w + h D_I p = 0, \\ &D_I w - h D_R p = 0, \end{aligned} \tag{3.71}$$

where $D_R = \partial_{y_1} + \lambda_R \partial_{y_2}$, $D_I = \lambda_I \partial_{y_2}$, $\lambda_R = -\frac{c^2\rho u_2}{c^2-u_1^2}$, $\lambda_I = \frac{c\rho u_1\sqrt{c^2-q^2}}{c^2-u_1^2}$, and $h = \frac{\sqrt{c^2-q^2}}{c\rho u_1^2}$.

Therefore, system (3.7)–(3.10) is decomposed into (3.70)–(3.71).

We solve this problem by iteration. Given U^- that is close to U_0^- as defined in Theorem 3.3, we solve the problem for U in the Lagrangian coordinates. However, since $\mathcal{U}^\infty$ is not known, we cannot directly solve Problem 3.2(WT) for U satisfying (3.67), or Problem 3.2(ST) for U satisfying (3.68). Instead, we solve Problem 3.2(ST) for U that is close to U_0^+ as in (3.26), and Problem 3.2(WT) for U in similar norms with appropriate growths, but using these norms in the Lagrangian coordinates (more precisely, the $\mathbf{z} = (z_1, z_2)$–coordinates defined by (3.75) below). Note that these norms are weaker than the ones in (3.67) or (3.68) respectively; in particular, they do not determine any limit for $U_1 = (u_1, \rho)$ as $|\mathbf{y}| \to \infty$ within the subsonic region. On the other hand, these norms determine that (w, p) have the limit: $(0, p_0^+)$ at infinity within the subsonic region; this asymptotic condition is sufficient to make the iteration problem well-defined (in fact, we use only the asymptotic decay of w) and to obtain the existence and uniqueness for the iteration problem. After the unique solution U of the problem stated in Theorem 3.3 is obtained by iteration, we identify $\mathcal{U}_1^\infty = (\rho^\infty, u_1^\infty)$ and show the faster convergence of (ρ, u_1) to $(\rho^\infty, u_1^\infty)$, which lead to (3.67) and (3.68), respectively. Note that, in the estimates discussed above, $U - U_0^+$ (rather than U itself) lies in the weighted spaces (3.21). For this reason, it is convenient to perform the iteration in terms of

$$\delta U_1 = U_1 - U_{10}^+, \qquad \delta U_2 = U_2 - U_{20}^+, \qquad \delta\hat{\sigma} = \hat{\sigma} - \hat{\sigma}_0 = \hat{\sigma} - s_1 y_1, \tag{3.72}$$

where U_1 and U_2 are defined by (3.60). Then we follow the steps below to solve this problem:

1. **Introduce a linear boundary value problem for the iteration.** For a given shock-front $\hat{\sigma}$, the subsonic domain $\mathbb{D}^{\hat{\sigma}}$ is fixed, depending on $\hat{\sigma}$. We make the coordinate transformation to transform the domain from $\mathbb{D}^{\hat{\sigma}}$ to $\mathbb{D}$, where $\mathbb{D} = \mathbb{D}^{\hat{\sigma}_0}$ with $\hat{\sigma}_0(y_2) = s_1 y_1$ is the domain corresponding to the background solution:

$$\mathbb{D} = \big\{\mathbf{y} : 0 < y_2 < s_1 y_1\big\} \tag{3.73}$$

with $\partial\mathbb{D} = \overline{\mathcal{L}_1} \cup \mathcal{L}_2$, where

$$\mathcal{L}_1 = \{(y_1, y_2) \,:\, y_1 > 0, y_2 = 0\}, \qquad \mathcal{L}_2 = \{(y_1, y_2) \,:\, y_1 > 0, y_2 = s_1 y_1\}. \tag{3.74}$$

This transformation is:

$$\mathbf{y} = (y_1, y_2) \,\to\, \mathbf{z} = (z_1, z_2) := (y_1, y_2 - \delta\hat{\sigma}(y_1)), \tag{3.75}$$

where $\delta\hat{\sigma}(y_1) = \hat{\sigma}(y_1) - \hat{\sigma}_0(y_1)$. In the $\mathbf{z}$–coordinates, $\mathcal{L}_1$ corresponds to ∂W, and $\mathcal{L}_2$ corresponds to $\partial\mathcal{S}$. Also, $U(\mathbf{y})$ becomes $U_{\hat{\sigma}}(\mathbf{z})$ depending on $\hat{\sigma}$. Then the upstream flow U^- involves an unknown variable explicitly depending on $\hat{\sigma}$:

$$U_{\hat{\sigma}}^-(\mathbf{z}) = U^-(z_1, z_2 + \delta\hat{\sigma}(z_1)), \tag{3.76}$$

where U^- is the given upstream flow in the $\mathbf{y}$–coordinates. Equations (3.71) in $\mathbb{D}$ in the $\mathbf{z}$-coordinates are:

$$\begin{cases} \widetilde{D}_R w + h\widetilde{D}_I p = 0, \\ \widetilde{D}_I w - h\widetilde{D}_R p = 0, \end{cases} \tag{3.77}$$

where $\widetilde{D}_R = \partial_{z_1} + (\lambda_R - \delta\hat{\sigma}')\partial_{z_2}$ and $\widetilde{D}_I = \lambda_I \partial_{z_2}$. Since U_0^+ is a constant vector and $w_0^+ = 0$, the same system holds for $(\delta p, \delta w)$, where we have used notation (3.72). Moreover, since the iteration:

$(\delta U, \delta w) \to (\delta\tilde{U}, \delta\tilde{w})$ is considered, we use $U := U_0^+ + \delta U$ to determine the coefficients in (3.77) and $(\delta\tilde{p}, \delta\tilde{w})$ for the unknown functions. Thus, we have

$$\begin{cases} \widetilde{D}_R\delta\tilde{w} + h\widetilde{D}_I\delta\tilde{p} = 0, \\ \widetilde{D}_I\delta\tilde{w} - h\widetilde{D}_R\delta\tilde{p} = 0. \end{cases} \tag{3.78}$$

We use system (3.78) in $\mathbb{D}$ as a linear system for the iteration.

In the $\mathbf{z}$–coordinates, the Rankine-Hugoniot conditions (3.27)–(3.30) keep the same form, except that $\hat{\sigma}'(y_1)$ is replaced by $\hat{\sigma}'(z_1)$ and U^- is replaced by $U^-_{\hat{\sigma}}$ along line $z_2 = s_1z_1$. Among the four Rankine-Hugoniot conditions, (3.30) is used in the Bernoulli law. From condition (3.29), we have

$$\hat{\sigma}'(z_1) = \frac{[p]}{[u_1w]}(z_1, s_1z_1), \tag{3.79}$$

which is used to update the shock-front later. Now, because of (3.70), we can use $\bar{U} = (w, p, \rho)$ as the unknown variables along $z_2 = s_1z_1$. Using (3.79) to eliminate $\hat{\sigma}'$ in conditions (3.27)–(3.28) gives

$$G_1(U^-_{\hat{\sigma}}, \bar{U}) := [p]\Big[\frac{1}{\rho u_1}\Big] + [w][u_1w] = 0, \tag{3.80}$$

$$G_2(U^-_{\hat{\sigma}}, \bar{U}) := [p]\Big[u_1 + \frac{p}{\rho u_1}\Big] + [pw][u_1w] = 0, \tag{3.81}$$

on $\mathcal{L}_2$. We use conditions (3.80)–(3.81) to define the linear conditions for the iteration: $\bar{U} \to \tilde{\bar{U}}$ such that, at a fixed point $\bar{U} = \tilde{\bar{U}}$, these iteration conditions imply that the original conditions (3.80)–(3.81) hold. Specifically, we define the conditions:

$$\nabla_{\bar{U}}G_i(U_0^-, \bar{U}_0^+) \cdot \delta\tilde{\bar{U}} = \nabla_{\bar{U}}G_i(U_0^-, \bar{U}_0^+) \cdot \delta\bar{U} - G_i(U^-_{\hat{\sigma}}, \bar{U}) \qquad \text{on } \mathcal{L}_2 \tag{3.82}$$

for $i = 1, 2$, which can be written as

$$b_{i1}\delta\tilde{w} + b_{i2}\delta\tilde{p} + b_{i3}\delta\tilde{\rho} = g_i(U^-_{\hat{\sigma}}, \bar{U}) \qquad \text{on } \mathcal{L}_2, \tag{3.83}$$

where $(b_{i1}, b_{i2}, b_{i3}) := \nabla_{\bar{U}}G_i(U_0^-, \bar{U}_0^+)$ and

$$g_i(U^-_{\hat{\sigma}}, \bar{U}) := \nabla_{\bar{U}}G_i(U_0^-, \bar{U}_0^+) \cdot \delta\bar{U} - G_i(U^-_{\hat{\sigma}}, \bar{U}) \qquad \text{for } i = 1, 2.$$

Since there are two conditions in (3.83), $i = 1, 2$, we can eliminate $\delta\tilde{\rho}$ to obtain

$$\delta\tilde{w} + b_1\delta\tilde{p} = g_3 \qquad \text{on } \mathcal{L}_2, \tag{3.84}$$

where

$$b_1 = \frac{b_{12}b_{23} - b_{22}b_{13}}{b_{11}b_{23} - b_{21}b_{13}}, \qquad g_3 = \frac{b_{23}g_1 - b_{13}g_2}{b_{11}b_{23} - b_{21}b_{13}} \tag{3.85}$$

with

$$b_{11}b_{23} - b_{21}b_{13} = (-u_{20}^-)[p_0]\left(\frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^2u_{10}^+} + \frac{p_0^-}{u_{10}^-}\big(\frac{1}{(\rho_0^+)^2} + \frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^3(u_{10}^+)^2}\big)\right) > 0.$$

Notice that the shock polar is a one-parameter curve determined by the Rankine-Hugoniot conditions. If p is used as the parameter, by equation (3.84), we obtain that $\delta w = -b_1\delta p + g_3(\delta p)$, which shows that $-b_1\delta p$ is the linear term and $g_3(\delta p)$ is the higher order term. We know from Fig. 3.3 that $w(p)$ is decreasing in p on arc $\widehat{TH}$ and increasing on $\widehat{TS}$. Therefore, it is easy to see that

$b_1 > 0$ *corresponds to the state on arc* $\widehat{TH}$, $b_1 < 0$ *to* $\widehat{TS}$, *and* $b_1 = 0$ *at the tangent point* T. (3.86)

This difference in the sign of b_1 is the reason for the different rates of decay at infinity and near the origin in the two different cases (i) and (ii) of Theorems 3.1 and 3.3.

It can be checked that

$$b_{13} = -[p_0]\left(\frac{p_0^+}{(\rho_0^+)^2u_{10}^+} + \frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^3(u_{10}^+)^3}\right) < 0.$$

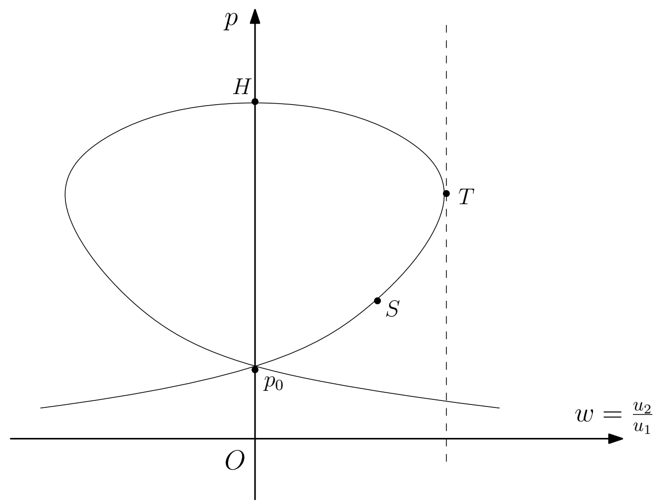

FIGURE 3.3. The shock polar in the (w, p)–variables (*cf.* [25])

Thus, condition (3.83) for $i = 1$ can be rewritten as

$$\delta\tilde{\rho} = g_4 - b_2 \delta\tilde{w} - b_3 \delta\tilde{p} \qquad \text{on } \mathcal{L}_2, \tag{3.87}$$

where $g_4 = \frac{g_1}{b_{13}}, b_2 = \frac{b_{11}}{b_{13}}$, and $b_3 = \frac{b_{12}}{b_{13}}$.

We notice that conditions (3.84)–(3.87) are equivalent to conditions (3.83) for $i = 1, 2$. The boundary condition on $\mathcal{L}_1$ comes from the slip condition (3.15) on ∂W. Specifically, using (3.15) and (3.61), we obtain that $w = b'_{\text{rot}}$ on ∂W. Then, in the $\mathbf{z}$–coordinates, this must hold on $\mathcal{L}_1$. Also, for the background solution, $b_{\text{rot}} = 0$ by (3.61). Then we prescribe

$$\delta\tilde{w} = b'_{\text{rot}} \qquad \text{on } \mathcal{L}_1. \tag{3.88}$$

2. **Design the iteration map $\mathcal{Q}$ and prove the existence of a fixed point for $\mathcal{Q}$.** We perform the iteration in terms of δU_k, $k = 1, 2$, and $\delta\hat{\sigma}$ as defined by (3.72), in the $\mathbf{z}$-coordinates defined in (3.75). In fact, for $\hat{\sigma}$, we only need $\hat{\sigma}'$ since $\hat{\sigma}(0) = 0$, *i.e.*, the shock is attached to the wedge vertex. Note also that $\delta\hat{\sigma}' = \hat{\sigma}' - s_1$. We thus denote $V = (U_1, U_2, \delta\hat{\sigma}')$ and perform the following iteration: $\delta V \to \delta\tilde{V}$. For a given δV, we determine $V := \delta V + V_0^+$. Then we find $\tilde{V}$ by solving the linear system (3.78) in $\mathbb{D}$ with the boundary conditions (3.84) and (3.88), to determine $(\tilde{w}, \tilde{p})$ and then determine u_1 from (3.70) and ρ from (3.36) (which holds in the $\mathbf{z}$–coordinates without change), and the boundary condition (3.87). The final step is to use solution $(\delta u_1, \delta\rho, \delta w, \delta p)$ and $U^-_{\hat{\sigma}}$ defined by (3.76) on the right-hand side of (3.79) to update $\delta\hat{\sigma}'$. This defines the iteration map $\mathcal{Q}$ from V to $\tilde{V}$, except that we discuss below how the boundary value problem for (3.78) with the boundary conditions (3.84) and (3.88) is solved in $\mathbb{D}$.

As we discussed above, we perform the iteration in the spaces from (3.68) for Problem 3.2(ST) and similar norms with appropriate growths for Problem 3.2(WT), expressed in the $\mathbf{z}$–coordinates (3.75). We focus below on the case of Problem 3.2(WT), since the other case is similar.

For $\tau > 0$, define

$$\begin{aligned}
&\Sigma_1^\tau = \{v \,:\, \|v\|^{(-\alpha;\mathcal{L}_1)}_{2,\alpha;(0,1+\beta);\mathbb{D}} + \|v_{z_1}\|^{(1-\alpha;\mathcal{L}_1)}_{2,\alpha;(1+\beta,1);\mathbb{D}} \leqslant \tau\},\\
&\Sigma_2^\tau = \{v \,:\, \|v\|^{(-\alpha;O)(-1-\alpha;\mathcal{L}_1)}_{2,\alpha;(1+\beta,0);\mathbb{D}} \leqslant \tau\}, \qquad \Sigma_3^\tau = \{v \,:\, \|v\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} \leqslant \tau\},\\
&\Sigma^\tau = \{(\delta U_1, \delta U_2, \delta\hat{\sigma}') \,:\, \delta U_1 \in \Sigma_1^\tau \times \Sigma_1^\tau,\ \ \delta U_2 \in \Sigma_2^\tau \times \Sigma_2^\tau,\ \ \delta\hat{\sigma}' \in \Sigma_3^\tau\}.
\end{aligned} \tag{3.89}$$

The condition on v_{z_1} in Σ_1^τ is added for technical reasons.

It remains to discuss how we find $(\delta\tilde{w}, \delta\tilde{p}) \in \Sigma_2^{C_0\varepsilon} \times \Sigma_2^{C_0\varepsilon}$ that solves (3.78) in $\mathbb{D}$ with the boundary conditions (3.84) and (3.88). From system (3.78), we obtain

$$(\delta\tilde{p})_{z_1} = \frac{\lambda_R - \delta\hat{\sigma}'}{h\lambda_I}(\delta\tilde{w})_{z_1} + \frac{(\lambda_R - \delta\hat{\sigma}')^2 + \lambda_I^2}{h\lambda_I}(\delta\tilde{w})_{z_2}, \tag{3.90}$$

$$(\delta\tilde{p})_{z_2} = -\frac{1}{h\lambda_I}(\delta\tilde{w})_{z_1} - \frac{\lambda_R - \delta\hat{\sigma}'}{h\lambda_I}(\delta\tilde{w})_{z_2}. \tag{3.91}$$

Now, differentiating and subtracting the equations, we eliminate $\delta\tilde{p}$ to obtain a second-order PDE for $\delta\tilde{w}$ of the form:

$$\sum_{i,j=1}^{2} (a_{ij}(\delta\tilde{w})_{z_j})_{z_i} = 0, \tag{3.92}$$

where the coefficients are computed explicitly from (3.90)–(3.91). Note that, at the subsonic background solution (3.59), we have

$$\lambda_{R0} = 0, \quad \lambda_{I0} > 0, \quad h_0 > 0,$$

where the left-hand sides are constants and $\delta\hat{\sigma}_0 = 0$. Then, computing the coefficients at the background solution, equation (3.92) becomes

$$\frac{1}{\lambda_{I0}}(\delta\tilde{w})_{z_1z_1} + \lambda_{I0}(\delta\tilde{w})_{z_2z_2} = 0,$$

that is, the equation is uniformly elliptic. Then, for the coefficients computed at $(U_{10}^+ + \delta U_1, U_{20}^+ + \delta U_2, \delta\hat{\sigma}')$ for $(\delta U_1, \delta U_2, \delta\hat{\sigma}') \in \Sigma^{C_0\varepsilon}$, equation (3.92) is uniformly elliptic if ε is small. This allows us to obtain the unique solution $\delta\tilde{w} \in \Sigma_2^{C_0\varepsilon}$ of (3.92) in $\mathbb{D}$ with the boundary conditions (3.84) and (3.88). Note that the inclusion $\delta\tilde{w} \in \Sigma_2^{C_0\varepsilon}$ involves the asymptotic condition at infinity, which makes the boundary value problem well-defined and allows us to prove the uniqueness. After $\delta\tilde{w}$ is determined, we determine $\delta\tilde{p}$ by the z_2–integration from (3.91) with the initial condition (3.84), where it can be shown that $b_1 \neq 0$. Then we show that $\delta p \in \Sigma_2^{C_0\varepsilon}$. This completes the definition of the iteration map.

The iteration set for Problem 3.2(WT) is $\Sigma^{C_0\varepsilon}$. We first show that $\mathcal{Q}(\Sigma^{C_0\varepsilon}) \subset \Sigma^{C_0\varepsilon}$ when ε is small, and then obtain a fixed point by the Schauder fixed point theorem, via showing that $\Sigma^{C_0\varepsilon}$ is a compact subset in the Banach space defined by replacing α via $\alpha' \in (0, \alpha)$ in the norms used in the definition of Σ^τ and showing that map $\mathcal{Q}$ is continuous in this norm.

3. **Asymptotic limit of the fixed point in the y–coordinates**. Let $(\delta U_1, \delta U_2, \delta\hat{\sigma}') \in \Sigma^{C_0\varepsilon}$ be a fixed point of the iteration map, and let $(U_1, U_2, \hat{\sigma}') = (U_{10}^+ + \delta U_1, U_{20}^+ + \delta U_2, \delta\hat{\sigma}')$.

We change from the **z**– to **y**–coordinates by inverting (3.75):

$$\mathbf{z} := (z_1, z_2) \rightarrow \mathbf{y} := (y_1, y_2) = (z_1, z_2 + \delta\hat{\sigma}(z_1)).$$

Since $\delta\hat{\sigma}' \in \Sigma_3^{C_0\varepsilon}$, then both (3.75) and its inverse are close to the identity map in $C^{2,\alpha}(\mathbb{D}_{\hat{\sigma}}; \mathbb{R}^2)$ and $C^{2,\alpha}(\mathbb{D}; \mathbb{R}^2)$, respectively. Then it follows that, in the **y**–coordinates, $(\delta U_1, \delta U_2, \delta\hat{\sigma}') \in \tilde{\Sigma}^{2C_0\varepsilon}$ if ε is small, where

$$\begin{aligned} &\tilde{\Sigma}_1^\tau = \{v \,:\, \|v\|^{(-\alpha;\mathcal{L}_1)}_{2,\alpha;(0,1+\beta);\mathbb{D}_{\hat{\sigma}}} + \|v_{z_1}\|^{(1-\alpha;\mathcal{L}_1)}_{2,\alpha;(1+\beta,1);\mathbb{D}_{\hat{\sigma}}} \leqslant \tau\}, \quad \tilde{\Sigma}_2^\tau = \{v \,:\, \|v\|^{(-\alpha;O)(-1-\alpha;\mathcal{L}_1)}_{2,\alpha;(1+\beta,0);\mathbb{D}_{\hat{\sigma}}} \leqslant \tau\}, \\ &\tilde{\Sigma}^\tau = \{(\delta U_1, \delta U_2, \delta\hat{\sigma}') \,:\, \delta U_1 \in \tilde{\Sigma}_1^\tau \times \tilde{\Sigma}_1^\tau, \;\; \delta U_2 \in \tilde{\Sigma}_2^\tau \times \tilde{\Sigma}_2^\tau, \;\; \delta\hat{\sigma}' \in \Sigma_3^\tau\}. \end{aligned} \tag{3.93}$$

In particular, this leads to the estimates of the second and third terms on the left-hand side of (3.67).

Note that, if $v \in \tilde{\Sigma}_2^\tau$, then $v \to 0$ as $|\mathbf{y}| \to \infty$ in $\mathbb{D}_{\hat{\sigma}}$, with decay rate $|\mathbf{y}|^{-(\beta+1)}$. However, for $v \in \tilde{\Sigma}_1^\tau$, no asymptotic limit as $|\mathbf{y}| \to \infty$ in $\mathbb{D}_{\hat{\sigma}}$ is defined.

Then, from (3.59)–(3.60), it follows that $U_2 = (w, p) \to (0, p_0^+)$ as $|\mathbf{y}| \to \infty$ in $\mathbb{D}$; however, for $U_1 = (u_1, \rho)$, the limit is not determined by space $\tilde{\Sigma}_1^\tau$, and (u_1, ρ) does not converge to (u_{10}^+, ρ_0^+) in general, as we see below. Thus, we have to determine the limiting profiles $(u_1^\infty(y_2), \rho^\infty(y_2))$.

To determine $\rho^\infty(y_2)$, we first obtain (3.36) from (3.7)–(3.10), which implies (3.37). Since $\hat\sigma(y_1)$ is determined, then $A(y_2)$ in (3.37) is determined by the upstream state $U^-(\mathbf{y})$ from the Rankine-Hugoniot conditions (3.27)–(3.30). Then, noting that $p \to p_0^\infty$, we obtain formally

$$\rho \to \rho^\infty(y_2) = \Big(\frac{p_0^+}{A(y_2)}\Big)^{\frac{1}{\gamma}} \qquad \text{as } |\mathbf{y}| \to \infty \text{ in } \mathbb{D}_{\hat\sigma}.$$

Similarly, we use (3.70) to obtain

$$u_1 \to u_1^\infty(y_2) = \sqrt{2\Big(B(y_2) - \frac{\gamma p_0^+}{(\gamma-1)\rho^\infty(y_2)}\Big)} \qquad \text{as } |\mathbf{y}| \to \infty \text{ in } \mathbb{D}_{\hat\sigma}.$$

Defining $\mathcal{U}^\infty(y_2) = (u_1^\infty(y_2), 0, p_0^+, \rho^\infty(y_2))$, we can show that the estimates of the first and the last terms on the left-hand side of (3.67) hold. This completes the argument for case (i) of Theorem 3.3.

Case (ii) is handled similarly. Note that the slower decay at infinity for case (ii), *i.e.*, $|\mathbf{y}|^{-\beta}$, is from the elliptic estimates, even if the faster decay at infinity in (3.25) is required. The reason for the difference in the rates for cases (i) and (ii) is (3.86).

4. **Return to the x-coordinates**. We obtain Theorem 3.2 directly from Theorem 3.3 by changing the coordinates. Recall that, when the Lagrangian coordinates are defined for Theorem 3.3, we have used the rotated coordinates $\mathbf{x}$ in (3.6); see the discussion in the paragraph before Theorem 3.3.

From the estimates in Theorem 3.3, it follows that, in the Lagrangian coordinates, $|U - U_0^+| \leqslant C\varepsilon$ in $\mathbb{D}_{\hat\sigma}$, where C depends only on (U_0^-, U_0^+). Thus, the same is true in the $\mathbf{x}$–coordinates in Ω. Then it follows from (3.5)–(3.6) and (3.59) for positive u_{10}^+ and ρ_0^+ that the change of coordinates $\mathbf{x} \to \mathbf{y}$ given by (3.6) is bi-Lipschitz. Then (3.66) follows from (3.68) directly.

Similarly, the estimates of the second and third terms on the left-hand side of (3.64) follow from (3.67) directly. In order to obtain the estimates of the remaining terms on the left-hand side of (3.64), we need to identify $U^\infty(x_2)$.

Note that, on shock $\mathcal{S}$, using (3.6) and the estimate of the third term on the left-hand side of (3.64), we see that, for small ε,

$$\partial_{\boldsymbol{\tau}_{\mathcal{S}}}\psi = \rho\mathbf{u}\cdot\boldsymbol{\nu}_{\mathcal{S}} \geqslant \rho\mathbf{u}_0^+\cdot\boldsymbol{\nu}_{\mathcal{S}_0} - C\varepsilon \geqslant \frac{1}{2}\rho\mathbf{u}_0^+\cdot\boldsymbol{\nu}_{\mathcal{S}_0} > 0.$$

Recall also that $\psi(\mathbf{0}) = 0$ by (3.5). Then, for each $y_2 > 0$, there exists a unique $\mathbf{x}^{\rm in}(y_2) = (x_1^{\rm in}(y_2), x_2^{\rm in}(y_2)) \in \mathcal{S}$ such that $\psi(\mathbf{x}^{\rm in}(y_2)) = y_2$ and

$$\|\mathbf{x}^{\rm in}\|_{C^{2,\alpha}([0,\infty))} \leqslant C, \qquad (\mathbf{x}^{\rm in})' \geqslant C^{-1} > 0 \quad \text{on } [0,\infty).$$

From this and (3.5), it follows that, for each $y_2 > 0$,

$$\Omega \cap \{\mathbf{x} \,:\, \psi(\mathbf{x}) = y_2\} = \{(x_1, x_2^*(x_1; y_2)) \;:\; x_1 > x_1^{\rm in}(y_2)\},$$

where $x_2^*(\cdot; y_2)$ is the solution of the initial value problem for the differential equation:

$$\begin{cases} \partial_{x_1} x_2^*(x_1; y_2) = w(x_1, x_2^*(x_1; y_2)), \\ x_2^*(x_1^{\rm in}(y_2); y_2) = x_2^{\rm in}(y_2), \end{cases} \tag{3.94}$$

where $w = \frac{u_2}{u_1}$ (*cf.* (3.60)). Since we have obtained the estimate of the second term on the left-hand side of (3.24), using (3.59), we have

$$|D^k w(\mathbf{x})| \leqslant C_0\varepsilon(1+|\mathbf{x}|)^{-1-\beta} \qquad \text{in } \Omega, \quad \text{for } k = 0, 1, 2. \tag{3.95}$$

In particular, for each $y_2 \geqslant 0$ and $k = 0, 1, 2$,

$$\int_{x_1^{\rm in}(y_2)}^\infty |D^k w(x_1, x_2^*(x_1; y_2))|\,{\rm d}x_1 \leqslant C_0\varepsilon \int_0^\infty (1+x_1)^{-1-\beta}\,{\rm d}x_1 \leqslant C\varepsilon. \tag{3.96}$$

Applying this with $k = 1$, we conclude that $\lim_{x_1\to\infty} x_2^*(x_1; y_2)$ exists for each $y_2 \geqslant 0$, which is denoted as $x_2^\infty(y_2)$.

Recall the structure of Ω in (3.62), where $b_{\rm rot}(x_1) \to 0$ and $\tilde{\sigma}(x_1) \to \infty$ as $x_1 \to \infty$ by (3.63) and the estimate of the third term in (3.64). Differentiating (3.94) twice with respect to y_2 and using the C^2–estimate of $\mathbf{x}^{\rm in}$ and (3.96), we obtain that $\|x_2^*(x_1;\cdot)\|_{C^2([b_{\rm rot}(x_1),\tilde{\sigma}(x_1)])} \leqslant C$ for each $x_1 > 0$. From this, we have

$$x_2^*(x_1;\cdot) \to x_2^\infty(\cdot) \text{ in } C^1 \text{ on compact subsets on } [0,\infty) \text{ as } x_1 \to \infty, \tag{3.97}$$

with $\|x_2^\infty\|_{C^2([0,\infty))} \leqslant C$. Also, by a similar argument, using the $C^{2,\alpha}$–regularity of $\mathbf{x}^{\rm in}$ and the estimate of w in the second term in (3.64), we obtain that $x_2^\infty \in C^{2,\alpha}([0,\infty))$.

Furthermore, for the background solution, the potential functions ψ_0^- of U_0^-, ψ_0^+ of U_0^+, and ψ_0 of the transonic shock solution (U_0^-, U_0^+) in $\{x_1 > 0, x_2 > 0\}$ are:

$$\psi_0^-(\mathbf{x}) = \rho_0^- u_{10}^-(x_1 - x_2 \tan\theta_{\rm w}), \quad \psi_0^+(\mathbf{x}) = \rho_0^+ u_{10}^+ x_1, \quad \psi_0(\mathbf{x}) = \begin{cases} \psi_0^-(\mathbf{x}) & \text{if } x_2 < \tilde{s}x_1, \\ \psi_0^+(\mathbf{x}) & \text{if } x_2 > \tilde{s}x_1, \end{cases}$$

by using (3.59), where ψ_0 is Lipschitz. Then, estimating $\psi - \psi_0^-$ in Ω^- via (3.63) (where the polynomial decay is of degree $-(1+\beta)$ so that the calculations similar to (3.96) can be used) and using the Rankine-Hugoniot conditions on $\mathcal{S}$, we obtain

$$|(\mathbf{x}^{\rm in})' - (\mathbf{x}_0^{\rm in})'| \leqslant C\varepsilon \qquad \text{on } [0,\infty),$$

where $\mathbf{x}_0^{\rm in}(y_2) = \frac{y_2}{\rho_0^+ u_{10}^+}(1, \tilde{s}_0)$ that is the corresponding function $\mathbf{x}^{\rm in}$ of the background solution.

Denote by $x_{20}^*(x_1; y_2)$ the corresponding function $x_2^*(x_1; y_2)$ of the background solution. Then

$$x_{20}^*(x_1; y_2) = \frac{y_2}{\rho_0^+ u_{10}^+} \qquad \text{on } x_1 > \tfrac{y_2}{\rho_0^+ u_{10}^+ \tilde{s}_0} \text{ for each } y_2 \geqslant 0.$$

Thus, $x_{20}^*(x_1; y_2)$ is independent of x_1, so that $x_{20}^*(x_1; y_2) = x_{20}^*(y_2)$. Then, denoting

$$g(x_1; y_2) = x_2^*(x_1; y_2) - x_{20}^*(y_2),$$

we see that g satisfies

$$\begin{cases} \partial_{x_1} g(x_1; y_2) = w(x_1, x_2^*(x_1; y_2)), \\ |g(x_1^{\rm in}(y_2); y_2)| \leqslant C\varepsilon. \end{cases} \tag{3.98}$$

From this and (3.96)–(3.97), we obtain that $|(x_2^\infty)' - (x_{20}^\infty)'| \leqslant C\varepsilon$, where $(x_{20}^\infty)'(y_2) = (x_{20}^*)'(y_2) = \frac{1}{\rho_0^+ u_{10}^+}$. Therefore, we have

$$(x_2^\infty)' \geqslant \frac{1}{2\rho_0^+ u_{10}^+} \qquad \text{on } [0,\infty),$$

if ε is small. In particular, noting that $x_2^\infty(0) = 0$ since ∂W is a streamline corresponding to $\psi = 0$ and $\lim_{x_1\to\infty} b_{\rm rot}(x_1) = 0$ by (3.63), we obtain $x_2^\infty([0,\infty)) = [0,\infty)$. Then there exists the inverse function $y_2^* : [0,\infty) \to [0,\infty)$ to $x_2^\infty(\cdot)$ such that $y_2^* \in C^{2,\alpha}([0,\infty))$ with $y_2^*(0) = 0$ and $(y_2^*)' \geqslant \frac{1}{C} > 0$.

Finally, defining $U^\infty(x_2) = \mathcal{U}^\infty(y_2^*(x_2))$, we obtain (3.64) from (3.67).

For more details, see Chen-Chen-Feldman [25].

Remark 3.1. *For the global stability of weak transonic shocks for the* 3*-D wedge problem, see* [26, 28]*; also see the instability phenomenon for strong transonic shocks for the* 3*-D wedge problem in* [77]*. For the global stability of conical shocks for the M-D conic problem, see* [27] *for the transonic shock case and* [38, 49, 82] *for the supersonic shock case.*

4. Two-Dimensional Transonic Shocks and Free Boundary Problems for the Self-Similar Euler Equations for Potential Flow

In §2–§3, we have discussed the free boundary problems for steady transonic shock solutions of the compressible Euler equations. Now we discuss free boundary problems for time-dependent solutions.

Time-dependent solutions with shocks of the Cauchy problem for the compressible Euler system may exhibit non-uniqueness in general; see [50,71] and the references cited therein. On the other hand, many fundamental physical phenomena, including shock reflection/diffraction, are determined by the time-dependent solutions of self-similar structure; moreover, the uniqueness can be established in a carefully chosen class of self-similar solutions with shocks. In this section, we focus on this case; more precisely, we describe transonic shocks and free boundary problems for self-similar shock reflection/diffraction for the Euler equations for potential flow.

The two-dimensional compressible potential flow is governed by the conservation law of mass and the Bernoulli law for the density function ρ and the velocity potential Φ (*i.e.*, $\mathbf{u} = \nabla\Phi$):

$$\partial_t\rho + \nabla_{\mathbf{x}} \cdot (\rho\nabla_{\mathbf{x}}\Phi) = 0, \tag{4.1}$$

$$\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2 + h(\rho) = B \tag{4.2}$$

for $t \in \mathbb{R}^+ := (0,\infty)$ and $\mathbf{x} \in \mathbb{R}^2$, where B is the Bernoulli constant, and $h(\rho)$ is given by

$$h(\rho) = \frac{\rho^{\gamma-1}-1}{\gamma-1} \qquad \text{for the adiabatic exponent } \gamma > 1. \tag{4.3}$$

By (4.2)–(4.3), ρ can be expressed as

$$\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi) = h^{-1}(B - \partial_t\Phi - \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2). \tag{4.4}$$

Then system (4.1)–(4.2) can be rewritten as the following second-order nonlinear wave equation:

$$\partial_t\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi) + \nabla_{\mathbf{x}} \cdot \big(\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)\nabla_{\mathbf{x}}\Phi\big) = 0 \tag{4.5}$$

with $\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)$ determined by (4.4).

Note that equation (4.4) is invariant under the self-similar scaling:

$$(t,\mathbf{x}) \to (\alpha t, \alpha\mathbf{x}), \quad \Phi \to \frac{\Phi}{\alpha} \qquad \text{for } \ \alpha \neq 0, \tag{4.6}$$

and thus it admits self-similar solutions in the form of

$$\Phi(t,\mathbf{x}) = t\phi(\boldsymbol{\xi}) \qquad \text{for } \ \boldsymbol{\xi} = \frac{\mathbf{x}}{t}. \tag{4.7}$$

Then the pseudo-potential function

$$\varphi(\boldsymbol{\xi}) = \phi(\boldsymbol{\xi}) - \frac{1}{2}|\boldsymbol{\xi}|^2$$

satisfies the following equation:

$$\operatorname{div}(\rho(|D\varphi|^2,\varphi)D\varphi) + 2\rho(|D\varphi|^2,\varphi) = 0 \tag{4.8}$$

for

$$\rho(|D\varphi|^2,\varphi) = \big(B_0 - (\gamma-1)(\frac{1}{2}|D\varphi|^2 + \varphi)\big)^{\frac{1}{\gamma-1}}, \tag{4.9}$$

where $B_0 = (\gamma-1)B + 1$, and the divergence div and gradient D are with respect to $\boldsymbol{\xi} \in \mathbb{R}^2$.

Equation (4.8) written in the non-divergence form is

$$(c^2 - \varphi_{\xi_1}^2)\varphi_{\xi_1\xi_1} - 2\varphi_{\xi_1}\varphi_{\xi_2}\varphi_{\xi_1\xi_2} + (c^2 - \varphi_{\xi_2}^2)\varphi_{\xi_2\xi_2} + 2c^2 - |D\varphi|^2 = 0, \tag{4.10}$$

where the sonic speed $c = c(|D\varphi|^2,\varphi)$ is determined by

$$c^2(|D\varphi|^2,\varphi) = \rho^{\gamma-1}(|D\varphi|^2,\varphi) = B_0 - (\gamma-1)\big(\frac{1}{2}|D\varphi|^2 + \varphi\big). \tag{4.11}$$

Another form of (4.10), which uses both the potential ϕ and the pseudo-potential φ, is

$$(c^2 - \varphi_{\xi_1}^2)\phi_{\xi_1\xi_1} - 2\varphi_{\xi_1}\varphi_{\xi_2}\phi_{\xi_1\xi_2} + (c^2 - \varphi_{\xi_2}^2)\phi_{\xi_2\xi_2} = 0. \tag{4.12}$$

Equation (4.8) is a nonlinear PDE of mixed elliptic-hyperbolic type. It is elliptic at $\boldsymbol{\xi}$ if and only if

$$|D\varphi| < c(|D\varphi|^2, \varphi) \qquad \text{at } \boldsymbol{\xi}, \tag{4.13}$$

and is hyperbolic if the opposite inequality holds. This can be seen more clearly from the rotational invariance of (4.10), by fixing $\boldsymbol{\xi}$ and choosing coordinates (ξ_1, ξ_2) so that ξ_1 is along the direction of $D\varphi(\boldsymbol{\xi})$.

Moreover, from (4.10)–(4.11), equation (4.8) satisfies the Galilean invariance property: If $\varphi(\boldsymbol{\xi})$ is a solution, then its shift $\varphi(\boldsymbol{\xi}-\boldsymbol{\xi}_0)$ for any constant vector $\boldsymbol{\xi}_0$ is also a solution. Furthermore, $\varphi(\boldsymbol{\xi}) + \mathit{const.}$ is a solution of (4.8) with adjusted constant B correspondingly in (4.9) and (4.11).

One class of solutions of (4.8) is that of *constant states* that are the solutions with constant velocity $\mathbf{v} \in \mathbb{R}^2$. This implies that the pseudo-potential of a constant state satisfies $D\varphi = \mathbf{v} - \boldsymbol{\xi}$ so that

$$\varphi(\boldsymbol{\xi}) = -\frac{1}{2}|\boldsymbol{\xi}|^2 + \mathbf{v} \cdot \boldsymbol{\xi} + C, \tag{4.14}$$

where C is a constant. For such φ, the expressions in (4.9) and (4.11) imply that the density and sonic speed are positive constants ρ and c, *i.e.*, independent of $\boldsymbol{\xi}$. Then, from (2.4) and (4.14), the ellipticity condition for the constant state is

$$|\boldsymbol{\xi} - \mathbf{v}| < c.$$

Thus, for a constant state $\mathbf{v}$, equation (4.8) is elliptic inside the *sonic circle*, with center $\mathbf{v}$ and radius c, and hyperbolic outside this circle.

We also note that, if density ρ is a constant, then the solution is a constant state; that is, the corresponding pseudo-potential φ is of form (4.14).

Since the problem involves transonic shocks, we have to consider weak solutions of equation (4.8), which admit shocks. As in [33], the weak solutions are defined in the distributional sense in a domain Λ in the $\boldsymbol{\xi}$–coordinates.

Definition 4.1. *A function $\varphi \in W^{1,1}_{\mathrm{loc}}(\Lambda)$ is called a weak solution of* (4.8) *if*

(i) $B_0 - (\gamma - 1)(\frac{1}{2}|D\varphi|^2 + \varphi) \geqslant 0$ *a.e. in* Λ,

(ii) $(\rho(|D\varphi|^2, \varphi), \rho(|D\varphi|^2, \varphi)|D\varphi|) \in (L^1_{\mathrm{loc}}(\Lambda))^2$,

(iii) *For every* $\zeta \in C^\infty_{\mathrm{c}}(\Lambda)$,

$$\int_\Lambda \big(\rho(|D\varphi|^2, \varphi) D\varphi \cdot D\zeta - 2\rho(|D\varphi|^2, \varphi)\zeta\big) \mathrm{d}\boldsymbol{\xi} = 0. \tag{4.15}$$

A shock is a curve across which $D\varphi$ is discontinuous. If Λ^+ and $\Lambda^-(:= \Lambda \backslash \overline{\Lambda^+})$ are two nonempty open subsets of a domain $\Lambda \subset \mathbb{R}^2$, and $\mathcal{S} := \partial\Lambda^+ \cap \Lambda$ is a C^1-curve where $D\varphi$ has a jump, then $\varphi \in C^1(\Lambda^\pm \cup \mathcal{S}) \cap C^2(\Lambda^\pm)$ is a global weak solution of (4.8) in Λ if and only if φ is in $W^{1,\infty}_{\mathrm{loc}}(\Lambda)$ and satisfies equation (4.8) and the Rankine-Hugoniot condition on $\mathcal{S}$:

$$\rho(|D\varphi|^2, \varphi) D\varphi \cdot \boldsymbol{\nu}|_{\Lambda^+ \cap \mathcal{S}} = \rho(|D\varphi|^2, \varphi) D\varphi \cdot \boldsymbol{\nu}|_{\Lambda^- \cap \mathcal{S}}. \tag{4.16}$$

Note that the condition $\varphi \in W^{1,\infty}_{\mathrm{loc}}(\Lambda)$ requires that

$$\varphi_{\Lambda^+ \cap \mathcal{S}} = \varphi_{\Lambda^- \cap \mathcal{S}}, \tag{4.17}$$

which is consistent with $\mathrm{curl}(D\varphi) = 0$ in the distributional sense.

A piecewise smooth solution with the discontinuities is called an *entropy solution* of (4.8) if it satisfies the entropy condition: density ρ increases in the pseudo-flow direction of $D\varphi_{\Lambda^+ \cap \mathcal{S}}$ across any discontinuity. Then such a discontinuity is called a shock.

4.1. **The von Neumann Problem for Shock Reflection-Diffraction.** We now describe the von Neumann problem for shock reflection-diffraction, proposed for mathematical analysis first in [105–107]. When a vertical planar shock perpendicular to the flow direction x_1 and separating two uniform states (0) and (1), with constant velocities $\mathbf{u}_0 = (0, 0)$ and $\mathbf{u}_1 = (u_1, 0)$ and constant densities $\rho_0 < \rho_1$ (state (0) is ahead or to the right of the shock, and state (1) is behind the shock), hits a symmetric wedge:

$$W := \{(x_1, x_2) : |x_2| < x_1 \tan\theta_{\rm w}, x_1 > 0\}$$

head-on at time $t = 0$, a reflection-diffraction process takes place when $t > 0$. Then a fundamental question is what types of wave patterns of reflection-diffraction configurations may be formed around the wedge. The complexity of reflection-diffraction configurations was first reported by Ernst Mach [89] in 1878, who first observed two patterns of reflection-diffraction configurations: Regular reflection (two-shock configuration; see *e.g.* Figs. 4.1–4.2) and Mach reflection (three-shock/one-vortex-sheet configuration); also see [8, 35, 52, 103]. The issues remained dormant until the 1940s when John von Neumann [105–107], as well as other mathematical/experimental scientists (*cf.* [8, 35, 52, 63, 103] and the references cited therein), began extensive research into all aspects of shock reflection-diffraction phenomena, due to its importance in applications. It has been found that the situations are much more complicated than what Mach originally observed: The Mach reflection can be further divided into more specific sub-patterns, and various other patterns of shock reflection-diffraction configurations may occur such as the double Mach reflection, the von Neumann reflection, and the Guderley reflection; see [8,35,52,63,65,103] and the references cited therein. Then the fundamental scientific issues include:

(i) Structures of the shock reflection-diffraction configurations;
(ii) Transition criteria among the different patterns of shock reflection-diffraction configurations;
(iii) Dependence of the patterns upon the physical parameters such as the wedge angle $\theta_{\rm w}$, the incident-shock-wave Mach number, and the adiabatic exponent $\gamma > 1$.

In particular, several transition criteria among the different patterns of shock reflection-diffraction configurations have been proposed, including the sonic conjecture and the detachment conjecture by von Neumann [105–107].

A careful asymptotic analysis has been made for various reflection-diffraction configurations in Lighthill [83, 84], Keller-Blank [69], Hunter-Keller [68], Harabetian [67], Morawetz [93], and the references cited therein; also see Glimm-Majda [63]. Large or small scale numerical simulations have been also performed; *cf.* [8,63,110] and the references cited therein. However, most of the fundamental issues for shock reflection-diffraction phenomena have not been understood, especially the global structure and transition between the different patterns of shock reflection-diffraction configurations. This is partially because physical and numerical experiments are hampered by many difficulties and have not yielded clear transition criteria between the different patterns. In particular, numerical dissipation or physical viscosity smear the shocks and cause boundary layers that interact with the reflection-diffraction patterns and can cause spurious Mach steams; *cf.* [110]. Furthermore, some different patterns occur when the wedge angles are only fractions of a degree apart, a resolution as yet unreachable even by sophisticated experiments (*cf.* [**?**, 8]). For this reason, it is impossible to distinguish experimentally between the sonic and detachment criteria clearly, as pointed out in [8]. In this regard, the necessary approach to understand fully the shock reflection-diffraction phenomena, especially the transition criteria, is via rigorous mathematical analysis. To achieve this, it is essential to formulate the shock reflection-diffraction problem as a free boundary problem and establish the global existence, regularity, and structural stability of its solution.

Mathematically, the shock reflection-diffraction problem is a two-dimensional lateral Riemann problem in domain $\mathbb{R}^2\backslash\overline{W}$.

Problem 4.1 (Two-Dimensional Lateral Riemann Problem)**.** *Piecewise constant initial data, consisting of state* (0) *with velocity* $\mathbf{u}_0 = (0, 0)$ *on* $\{x_1 > 0\}\backslash\overline{W}$ *and state* (1) *with velocity* $\mathbf{u}_1 = (u_1, 0)$ *on* $\{x_1 < 0\}$ *connected by a shock at* $x_1 = 0$*, are prescribed at* $t = 0$*. Seek a solution of the Euler system* (4.1)–(4.2) *for* $t \geqslant 0$ *subject to these initial data and the boundary condition* $\nabla\Phi \cdot \boldsymbol{\nu} = 0$ *on* ∂W*.*

In order to define the notion of weak solutions of Problem 4.1, it is noted that the boundary condition can be written as $\rho\nabla\Phi\cdot\boldsymbol{\nu}=0$ on ∂W, which is spatial conormal to equation (4.5). Then we have

Definition 4.2 (Weak Solutions of Problem 4.1)**.** *A function* $\Phi\in W^{1,1}_{\rm loc}(\mathbb{R}_+\times(\mathbb{R}^2\backslash W))$ *is called a weak solution of* Problem 4.1 *if* Φ *satisfies the following properties*:

(i) $B-\big(\partial_t\Phi+\frac12|\nabla_{\mathbf{x}}\Phi|^2\big)\geqslant h(0+)$ *a.e. in* $\mathbb{R}_+\times(\mathbb{R}^2\backslash W)$.

(ii) *For* $\rho(\partial_t\Phi,\nabla_{\mathbf{x}}\Phi)$ *determined by* (4.4),
$$(\rho(\partial_t\Phi,|\nabla_{\mathbf{x}}\Phi|^2),\rho(\partial_t\Phi,|\nabla_{\mathbf{x}}\Phi|^2)|\nabla_{\mathbf{x}}\Phi|)\in(L^1_{\rm loc}(\overline{\mathbb{R}}_+\times\overline{\mathbb{R}^2\backslash W}))^2.$$

(iii) *For every* $\zeta\in C^\infty_c(\overline{\mathbb{R}_+}\times\mathbb{R}^2)$,
$$\int_0^\infty\int_{\mathbb{R}^2\backslash W}\Big(\rho(\partial_t\Phi,|\nabla_{\mathbf{x}}\Phi|^2)\partial_t\zeta+\rho(\partial_t\Phi,|\nabla_{\mathbf{x}}\Phi|^2)\nabla\Phi\cdot\nabla\zeta\Big)\,\mathrm{d}\mathbf{x}\mathrm{d}t+\int_{\mathbb{R}^2\backslash W}\rho(0,\mathbf{x})\zeta(0,\mathbf{x})\mathrm{d}\mathbf{x}=0,$$

where
$$\rho|_{t=0}=\begin{cases}\rho_0 & \text{for } |x_2|>x_1\tan\theta_{\rm w} \text{ and } x_1>0,\\ \rho_1 & \text{for } x_1<0.\end{cases}$$

Remark 4.3. *Since* ζ *does not need to be zero on* $\partial\Lambda$*, the integral identity in Definition* 4.2 *is a weak form of equation* (4.5) *and the boundary condition* $\rho\nabla\Phi\cdot\boldsymbol{\nu}=0$ *on* ∂W.

Remark 4.4. *A weak solution is called an entropy solution if it satisfies the entropy condition that is consistent with the second law of thermodynamics* (*cf.* [35, 52, 53, 73])*. In particular, a piecewise smooth solution is an entropy solution if the discontinuities are all shocks.*

Notice that Problem 4.1 is invariant under scaling (4.6), so it admits self-similar solutions determined by equation (4.8) with (4.9), along with the appropriate boundary conditions, through (4.7). We now show how such solutions in self-similar coordinates $\boldsymbol{\xi}=(\xi_1,\xi_2)=\frac{\mathbf{x}}{t}$ can be constructed.

First, by the symmetry of the problem with respect to the ξ_1–axis, we consider only the upper half-plane $\{\xi_2>0\}$ and prescribe the boundary condition: $\varphi_{\boldsymbol{\nu}}=0$ on the symmetry line $\{\xi_2=0\}$. Note that state (1) satisfies this condition. Then Problem 4.1 is reformulated as a boundary value problem in the unbounded domain:
$$\Lambda:=\mathbb{R}^2_+\backslash\{\boldsymbol{\xi}\,:\,|\xi_2|\leqslant\xi_1\tan\theta_{\rm w},\xi_1>0\}$$
in the self-similar coordinates $\boldsymbol{\xi}=(\xi_1,\xi_2)$, where $\mathbb{R}^2_+:=\mathbb{R}^2\cap\{\xi_2>0\}$. The incident shock in the self-similar coordinates is the half-line $\mathcal{S}_0=\{\xi_1=\xi_1^0\}\cap\Lambda$, where
$$\xi_1^0=\rho_1\sqrt{\frac{2(c_1^2-c_0^2)}{(\gamma-1)(\rho_1^2-\rho_0^2)}}=\frac{\rho_1u_1}{\rho_1-\rho_0},\tag{4.18}$$
which is determined by the Rankine-Hugoniot conditions between states (0) and (1) on $\mathcal{S}_0$. Then Problem 4.1 for self-similar solutions becomes the following problem:

Problem 4.2 (Boundary Value Problem)**.** *Seek a solution* φ *of equation* (4.8)–(4.9) *in the self-similar domain* Λ *with the slip boundary condition* $D\varphi\cdot\boldsymbol{\nu}|_{\partial\Lambda}=0$ *and the asymptotic boundary condition at infinity*:
$$\varphi\to\bar\varphi=\begin{cases}\varphi_0 & \text{for } \xi_1>\xi_1^0,\xi_2>\xi_1\tan\theta_{\rm w},\\ \varphi_1 & \text{for } \xi_1<\xi_1^0,\ \xi_2>0,\end{cases}\qquad \text{when } |\boldsymbol{\xi}|\to\infty,$$
where $\varphi_0=-\frac12|\boldsymbol{\xi}|^2$ *and* $\varphi_1=-\frac12|\boldsymbol{\xi}|^2+u_1(\xi_1-\xi_1^0)$.

A weak solution of Problem 4.2 is obtained by the following modification of Definition 4.1: (4.15) is now required to hold for all $\zeta\in C^\infty_c(\mathbb{R}^2)$. As discussed in Remark 4.3, with such a choice of function ζ, the integral identity (4.15) includes both equation (4.8) and the boundary condition of conormal form: $\rho D\varphi\cdot\boldsymbol{\nu}=0$ on $\partial\Lambda$. A weak solution is called entropy solution if it satisfies the entropy condition: density ρ increases in the pseudo-flow direction of $D\varphi|_{\Lambda^+\cap\mathcal{S}}$ across any discontinuity curve (*i.e.*, shock).

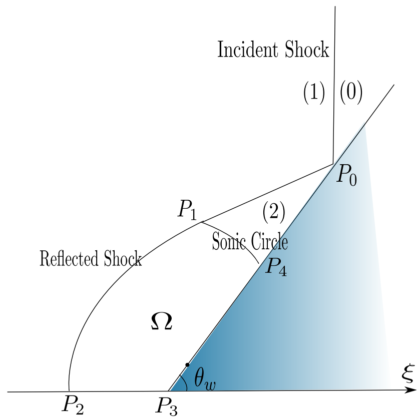

FIGURE 4.1. Supersonic regular shock reflection-diffraction configuration

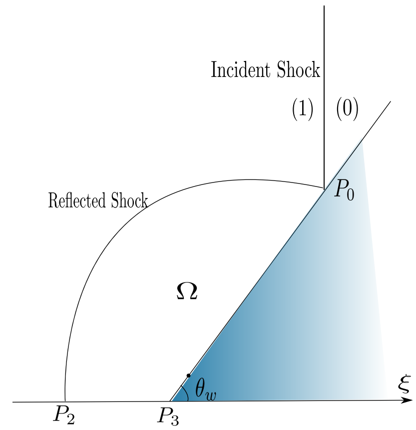

FIGURE 4.2. Subsonic regular shock reflection-diffraction configuration

Now we describe the more detailed structure of the regular reflection-diffraction configurations as shown in Figs. 4.1–4.2. If a solution has one of the regular shock reflection-diffraction configurations, and if its pseudo-potential φ is C^1 in the subregion $\widehat{\Omega}$ between the wedge and the reflected shock, then, at P_0, it should satisfy both the slip boundary condition on the wedge and the Rankine-Hugoniot conditions with state (1) across the flat shock $\mathcal{S}_1 = \{\varphi_1 = \varphi_2\}$, which passes through point P_0 where the incident shock meets the wedge boundary. Define the uniform state (2) with pseudo-potential $\varphi_2(\boldsymbol{\xi})$ such that

$$\varphi_2(P_0) = \varphi(P_0), \qquad D\varphi_2(P_0) = \lim_{P\to P_0,\ P\in\widehat{\Omega}} D\varphi(P).$$

Then the constant density ρ_2 of state (2) is equal to $\rho(|D\varphi|^2, \varphi)(P_0)$ defined by (4.8):

$$\rho_2 = \rho(|D\varphi_2|^2, \varphi_2)(P_0).$$

From the properties of φ discussed above, it follows that $D\varphi_2 \cdot \boldsymbol{\nu} = 0$ on the wedge boundary and the Rankine-Hugoniot conditions (4.16)–(4.17) hold on the flat shock $\mathcal{S}_1 = \{\varphi_1 = \varphi_2\}$ between states (1) and (2), which passes through P_0. In particular, φ_2 satisfies the following three conditions at P_0:

$$D\varphi_2 \cdot \boldsymbol{\nu}_{\rm w} = 0, \quad \varphi_2 = \varphi_1, \quad \rho(|D\varphi_2|^2, \varphi_2) D\varphi_2 \cdot \boldsymbol{\nu}_{\mathcal{S}_1} = \rho_1 D\varphi_1 \cdot \boldsymbol{\nu}_{\mathcal{S}_1} \qquad \text{for } \boldsymbol{\nu}_{\mathcal{S}_1} = \tfrac{D(\varphi_1-\varphi_2)}{|D(\varphi_1-\varphi_2)|}. \tag{4.19}$$

where $\boldsymbol{\nu}_{\rm w}$ is the outward normal to the wedge boundary.

The entropy solution φ, correspondingly state (2), can be either supersonic or subsonic at P_0. This determines the supersonic or subsonic type of regular shock reflection-diffraction configurations. The regular reflection solution in the supersonic region is expected to consist of the constant states separated by straight shocks (*cf.* [98, Theorem 4.1]). Then, when state (2) is supersonic at P_0, it can be shown that the constant state (2), extended up to arc P_1P_4 of the sonic circle of state (2) between the wall and the straight shock $P_0P_1 \subset \mathcal{S}_1$ separating it from state (1), as shown in Fig. 4.1, satisfies equation (4.8) in the region, the Rankine-Hugoniot condition (4.16)–(4.17) on the straight shock P_0P_1, and the slip boundary condition: $D\varphi_2 \cdot \boldsymbol{\nu}_{\rm w} = 0$ on the wedge P_0P_4, and is expected to be a part of the regular shock reflection-diffraction configuration. Then the supersonic regular shock reflection-diffraction configuration on Fig. 4.1 consists of three uniform states (0), (1), (2), and a non-uniform state in domain $\Omega = P_1P_2P_3P_4$ where equation (4.8) is elliptic. The reflected shock $P_0P_1P_2$ has a straight part P_0P_1. The elliptic domain Ω is separated from the hyperbolic region $P_0P_1P_4$ of state (2) by the sonic arc P_1P_4 which lies on the sonic circle of state (2), and the ellipticity in Ω degenerates on the sonic arc P_1P_4. The subsonic regular shock reflection-diffraction configuration as shown in Fig. 4.2 consists of two uniform

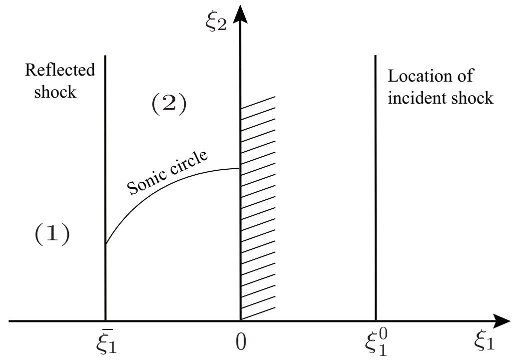

FIGURE 4.3. Normal reflection configuration (*cf.* [35])

states (0) and (1), and a non-uniform state in domain $\Omega = P_0P_2P_3$, where the equation is elliptic, and $\varphi_{|\Omega}(P_0) = \varphi_2(P_0)$ and $D(\varphi_{|\Omega})(P_0) = D\varphi_2(P_0)$.

For the supersonic regular shock reflection-diffraction configurations in Fig. 4.1, we use $\Gamma_{\rm sonic}$, $\Gamma_{\rm shock}$, $\Gamma_{\rm wedge}$, and $\Gamma_{\rm sym}$ for the sonic arc P_1P_4, the curved part of the reflected shock P_1P_2, the wedge boundary P_3P_4, and the symmetry line segment P_2P_3, respectively.

For the subsonic regular shock reflection-diffraction configurations in Fig. 4.2, $\Gamma_{\rm shock}$, $\Gamma_{\rm wedge}$, and $\Gamma_{\rm sym}$ denote P_0P_2, P_0P_3, and P_2P_3, respectively. We unify the notations with the supersonic reflection case by introducing points P_1 and P_4 for the subsonic reflection case as

$$P_1 := P_0, \quad P_4 := P_0, \quad \overline{\Gamma_{\rm sonic}} := \{P_0\}. \tag{4.20}$$

The corresponding solution for $\theta_{\rm w} = \frac{\pi}{2}$ is called *normal reflection*. In this case, the incident shock normally reflects from the flat wall; see Fig. 4.3. The reflected shock is also a plane $\{\xi_1 = \bar{\xi}_1\}$, where $\bar{\xi}_1 < 0$.

From the discussion above, it follows that a necessary condition for the existence of a regular reflection solution is the existence of the uniform state (2) with pseudo-potential φ_2 determined by the boundary condition $D\varphi_2 \cdot \boldsymbol{\nu} = 0$ on the wedge and the Rankine-Hugoniot conditions (4.16)–(4.17) across the flat shock $\mathcal{S}_1 = \{\varphi_1 = \varphi_2\}$ separating it from state (1), and satisfying the entropy condition: $\rho_2 > \rho_1$. These conditions lead to the system of algebraic equations (4.19) for the constant velocity $\mathbf{u}_2$ and density ρ_2 of state (2). System (4.19) has solutions for some but not all of the wedge angles. More specifically, for any fixed densities $0 < \rho_0 < \rho_1$ of states (0) and (1), there exist a sonic angle $\theta_{\rm w}^{\rm s}$ and a detachment angle $\theta_{\rm w}^{\rm d}$ satisfying

$$0 < \theta_{\rm w}^{\rm d} < \theta_{\rm w}^{\rm s} < \frac{\pi}{2}$$

such that the algebraic system (4.19) has two solutions for each $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, which become equal when $\theta_{\rm w} = \theta_{\rm w}^{\rm d}$. Thus, for each $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, there exist two states (2), called weak and strong, with densities $\rho_2^{\rm weak} < \rho_2^{\rm strong}$. The weak state (2) is supersonic at the reflection point $P_0(\theta_{\rm w})$ for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$, sonic for $\theta_{\rm w} = \theta_{\rm w}^{\rm s}$, and subsonic for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \hat{\theta}_{\rm w}^{\rm s})$ for some $\hat{\theta}_{\rm w}^{\rm s} \in (\theta_{\rm w}^{\rm d}, \theta_{\rm w}^{\rm s}]$. The strong state (2) is subsonic at $P_0(\theta_{\rm w})$ for all $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$.

There had been a long debate to determine which of the two states (2) for $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, weak or strong, is physical for the local theory; see [8,35,52] and the references cited therein. It was conjectured that the strong shock reflection-diffraction configuration would be non-physical; indeed, it is shown in Chen-Feldman [33, 35] that the weak shock reflection-diffraction configuration tends to the unique normal reflection in Fig. 4.3, but the strong reflection-diffraction configuration does not, when the wedge angle $\theta_{\rm w}$ tends to $\frac{\pi}{2}$. The entropy condition and the definition of weak and strong states (2) imply that $0 < \rho_1 < \rho_2^{\rm weak} < \rho_2^{\rm strong}$, which shows that the strength of the corresponding reflected shock near P_0 in the weak shock reflection-diffraction configuration is relatively weak, compared to the other shock given by the strong state (2).

If the weak state (2) is supersonic, the propagation speeds of the solution are finite, and state (2) is completely determined by the local information: state (1), state (0), and the location of point P_0. That is, any information from the reflection-diffraction region, particularly the disturbance at corner P_3, cannot travel towards the reflection point P_0. However, if it is subsonic, the information can reach P_0 and interact with it, potentially altering the subsonic reflection-diffraction configuration. This argument motivated the following conjecture by von Neumann in [105, 106]:

The Sonic Conjecture: *There exists a supersonic regular shock reflection-diffraction configuration when $\theta_{\rm w} \in (\theta_{\rm w}^{\rm s}, \frac{\pi}{2})$ for $\theta_{\rm w}^{\rm s} > \theta_{\rm w}^{\rm d}$. That is, the supersonicity of the weak state* (2) *implies the existence of a supersonic regular reflection solution, as shown in Fig.* 4.1.

Another conjecture is that the global regular shock reflection-diffraction configuration is possible whenever the local regular reflection at the reflection point is possible:

The von Neumanm Detachment Conjecture: *There exists a regular shock reflection-diffraction configuration for any wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$. That is, the existence of state* (2) *implies the existence of a regular reflection solution, as shown in Figs.* 4.1–4.2.

It is clear that the supersonic/subsonic regular shock reflection-diffraction configurations are not possible without a local two-shock configuration at the reflection point on the wedge, so the detachment conjecture is the weakest possible criterion for the existence of supersonic/subsonic regular shock reflection-diffraction configurations.

From now on, for the given wedge angle $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, state (2) represents the unique weak state (2) and φ_2 is its pseudo-potential. We now show how the solutions of regular shock reflection-diffraction configurations can be constructed. This provides a solution to the von Neumann conjectures for potential flow. Note that state (2) is obtained from the algebraic conditions described above, which determine line $\mathcal{S}_1$ and the sonic arc P_1P_4 when state (2) is supersonic at P_0, and the slope of $\Gamma_{\rm shock}$ at P_0 (arc P_1P_4 on the boundary of Ω becomes a corner point P_0) when state (2) is subsonic at P_0. Thus, the unknowns are domain Ω (or equivalently, the curved part of the reflected shock $\Gamma_{\rm shock}$) and the pseudo-potential φ in Ω. Then, from (4.16)–(4.17), in order to construct a solution of Problem 4.2 of the supersonic or subsonic regular shock reflection-diffraction configuration, it suffices to solve the following problem:

Problem 4.3 (Free Boundary Problem)**.** *For $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$, find a free boundary* (*curved reflected shock*) $\Gamma_{\rm shock} \subset \Lambda \cap \{\xi_1 < \xi_{1P_1}\}$ ($\Gamma_{\rm shock} = P_1P_2$ *on Fig.* 4.1 *and* $\Gamma_{\rm shock} = P_0P_2$ *on Fig.* 4.2) *and a function φ defined in region Ω as shown in Figs.* 4.1–4.2 *such that*

(i) *Equation* (4.8) *is satisfied in Ω, and the equation is strictly elliptic for φ in $\overline{\Omega}\backslash\overline{\Gamma_{\rm sonic}}$,*
(ii) $\varphi = \varphi_1$ *and* $\rho D\varphi \cdot \boldsymbol{\nu}_{\rm s} = \rho_1 D\varphi_1 \cdot \boldsymbol{\nu}_{\rm s}$ *on the free boundary* $\Gamma_{\rm shock}$*,*
(iii) $\varphi = \varphi_2$ *and* $D\varphi = D\varphi_2$ *on* P_1P_4 *in the supersonic case as shown in Fig.* 4.1 *and at* P_0 *in the subsonic case as shown in Fig.* 4.1*,*
(iv) $D\varphi \cdot \boldsymbol{\nu}_{\rm w} = 0$ *on* $\Gamma_{\rm wedge}$*, and* $D\varphi \cdot \boldsymbol{\nu}_{\rm sym} = 0$ *on* $\Gamma_{\rm sym}$*,*

where $\boldsymbol{\nu}_{\rm s}$, $\boldsymbol{\nu}_{\rm w}$, and $\boldsymbol{\nu}_{\rm sym}$ are the interior unit normals to Ω on $\Gamma_{\rm shock}$, $\Gamma_{\rm wedge}$, and $\Gamma_{\rm sym}$, respectively.

Indeed, if φ is a solution of Problem 4.3, we define its extension from Ω to Λ by setting:

$$\varphi = \begin{cases} \varphi_0 & \text{for } \xi_1 > \xi_1^0 \text{ and } \xi_2 > \xi_1 \tan\theta_{\rm w}, \\ \varphi_1 & \text{for } \xi_1 < \xi_1^0 \text{ and above curve } P_0P_1P_2, \\ \varphi_2 & \text{in region } P_0P_1P_4, \end{cases} \tag{4.21}$$

where we have used the notational convention (4.20) for the subsonic reflection case, in which region $P_0P_1P_4$ is one point and curve $P_0P_1P_2$ is P_0P_2; see Figs. 4.1–4.2. Also, ξ_1^0 used in (4.21) is the location of the incident shock (*cf.* (4.18)), and the extension by (4.21) is well-defined because of the requirement that $\Gamma_{\rm shock} \subset \Lambda \cap \{\xi_1 < \xi_{1P_1}\}$ in Problem 4.3.

Note that the conditions in Problem 4.3(ii) are the Rankine-Hugoniot conditions (4.16)–(4.17) on $\Gamma_{\rm shock}$ between $\varphi_{|\Omega}$ and φ_1. Since $\Gamma_{\rm shock}$ is a free boundary and equation (4.8) is strictly elliptic for φ in $\overline{\Omega}\backslash\overline{\Gamma_{\rm sonic}}$, then two conditions — the Dirichlet and oblique derivative conditions — on $\Gamma_{\rm shock}$ are consistent with one-phase free boundary problems for nonlinear elliptic PDEs of second order (*cf.* [1,3]).

In the supersonic case, the conditions in Problem 4.3(iii) are the Rankine-Hugoniot conditions on $\Gamma_{\rm sonic}$ between $\varphi_{|\Omega}$ and φ_2. Indeed, since state (2) is sonic on $\Gamma_{\rm sonic}$, then it follows from (4.16)–(4.17) that no gradient jump occurs on $\Gamma_{\rm sonic}$. Then, if φ is a solution of Problem 4.3, its extension by (4.21) is a weak solution of Problem 4.2. From now on, we consider a solution of Problem 4.3 to be a function defined in Λ by extension via (4.21).

Since $\Gamma_{\rm sonic}$ is not a free boundary (its location is fixed), it is not possible in general to prescribe two conditions given in Problem 4.3(iii) on $\Gamma_{\rm sonic}$ for a second-order elliptic PDE. In the iteration problem, we prescribe the condition: $\varphi=\varphi_2$ on $\Gamma_{\rm sonic}$, and then prove that $D\varphi=D\varphi_2$ on $\Gamma_{\rm sonic}$ by exploiting the elliptic degeneracy on $\Gamma_{\rm sonic}$, as we describe below.

We observe that the key obstacle to prove the existence of regular shock reflection-diffraction configurations as conjectured by von Neumann [105, 106] is an additional possibility that, for some wedge angle $\theta_{\rm w}^{\rm a}\in(\theta_{\rm w}^{\rm d},\frac{\pi}{2})$, shock P_0P_2 may attach to the wedge vertex P_3, as observed by experimental results (*cf.* [103, Fig. 238]). To describe the conditions of such an attachment, we note that

$$\rho_1>\rho_0,\qquad u_1=(\rho_1-\rho_0)\sqrt{\frac{2(\rho_1^{\gamma-1}-\rho_0^{\gamma-1})}{\rho_1^2-\rho_0^2}},\qquad c_1=\rho_1^{\frac{\gamma-1}{2}}.$$

Then it follows from the explicit expressions above that, for each ρ_0, there exists $\rho^{\rm c}>\rho_0$ such that

$$u_1\leqslant c_1\quad\text{if }\rho_1\in(\rho_0,\rho^{\rm c}];\qquad u_1>c_1\quad\text{if }\rho_1\in(\rho^{\rm c},\infty).$$

If $u_1\leqslant c_1$, we can rule out the solution with a shock attached to the wedge vertex. This is based on the fact that, if $u_1\leqslant c_1$, then the wedge vertex $P_3=(0,0)$ lies within the sonic circle $\overline{B_{c_1}((u_1,0))}$ of state (1), and $\Gamma_{\rm shock}$ does not intersect $\overline{B_{c_1}((u_1,0))}$, as we show below.

If $u_1>c_1$, there would be a possibility that the reflected shock could be attached to the wedge vertex as the experiments show (*e.g.*, [103, Fig. 238]).

Thus, in [33, 35], we have obtained the following results:

Theorem 4.1. *There are two cases*:

(i) *If ρ_0 and ρ_1 are such that $u_1\leqslant c_1$, then the supersonic/subsonic regular reflection solution exists for each wedge angle $\theta_{\rm w}\in(\theta_{\rm w}^{\rm d},\frac{\pi}{2})$. That is, for each $\theta_{\rm w}\in(\theta_{\rm w}^{\rm d},\frac{\pi}{2})$, there exists a solution φ of* Problem 4.3 *such that*

$$\Phi(t,\mathbf{x})=t\,\varphi(\frac{\mathbf{x}}{t})+\frac{|\mathbf{x}|^2}{2t}\qquad \textit{for }\frac{\mathbf{x}}{t}\in\Lambda,\ t>0$$

with

$$\rho(t,\mathbf{x})=\Big(\rho_0^{\gamma-1}-(\gamma-1)\big(\Phi_t+\frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2\big)\Big)^{\frac{1}{\gamma-1}}$$

is a global weak solution of Problem 4.1 *in the sense of Definition* 4.2 *satisfying the entropy condition*; *that is, $\Phi(t,\mathbf{x})$ is an entropy solution.*

(ii) *If ρ_0 and ρ_1 are such that $u_1>c_1$, then there exists $\theta_{\rm w}^{\rm a}\in\big[\theta_{\rm w}^{\rm d},\frac{\pi}{2}\big)$ so that the regular reflection solution exists for each wedge angle $\theta_{\rm w}\in(\theta_{\rm w}^{\rm a},\frac{\pi}{2})$, and the solution is of self-similar structure described in* (i) *above. Moreover, if $\theta_{\rm w}^{\rm a}>\theta_{\rm w}^{\rm d}$, then, for the wedge angle $\theta_{\rm w}=\theta_{\rm w}^{\rm a}$, there exists an attached solution, i.e., φ is a solution of* Problem 4.3 *with $P_2=P_3$.*

The type of regular shock reflection-diffraction configurations (*supersonic as in Fig.* 4.1 *or subsonic as in Fig.* 4.2) *is determined by the type of state* (2) *at P_0*:

(a) *For the supersonic and sonic reflection case, the reflected shock* P_0P_2 *is* $C^{2,\alpha}$*–smooth for some* $\alpha \in (0,1)$ *and its curved part* P_1P_2 *is* C^∞ *away from* P_1*. The solution* φ *is in* $C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$*, and is* $C^{1,1}$ *across the sonic arc which is optimal; that is,* φ *is* not C^2 *across sonic arc.*
(b) *For the subsonic reflection case (Fig.* 4.2*), the reflected shock* P_0P_2 *and solution* φ *in* Ω *is in* $C^{1,\alpha}$ *near* P_0 *and* P_3 *for some* $\alpha \in (0,1)$*, and* C^∞ *away from* $\{P_0, P_3\}$*.*

Moreover, the regular reflection solution tends to the unique normal reflection (as in Fig. 4.3*) when the wedge angle* $\theta_{\rm w}$ *tends to* $\frac{\pi}{2}$*. In addition, for both supersonic and subsonic reflection cases,*

$$\varphi_2 < \varphi < \varphi_1 \qquad \textit{in } \Omega. \tag{4.22}$$

Furthermore, φ *is an admissible solution in the sense of Definition* 4.8 *below, so that* φ *satisfies further properties listed in Definition* 4.8*.*

Theorem 4.1 is proved by solving Problem 4.3. The first results on the existence of global solutions of the free boundary problem (Problem 4.3) were obtained for the wedge angles sufficiently close to $\frac{\pi}{2}$ in Chen-Feldman [33]. Later, in Chen-Feldman [35], these results were extended up to the detachment angle as stated in Theorem 4.1. For this extension, the techniques developed in [33], notably the estimates near the sonic arc, were the starting point.

Case I: The wedge angles close to $\frac{\pi}{2}$. Let us first discuss the techniques in [33], where we employ the approach of Chen-Feldman [29] to develop an iteration scheme for constructing a global solution of Problem 4.3, when the wedge angle $\theta_{\rm w}$ is close to $\frac{\pi}{2}$. For this case, the solutions are of the supersonic regular shock reflection-diffraction configuration as in Fig. 4.1. The general procedure is similar to the one described in §2.2, which can be presented in the following four steps:

1. Fix $\theta_{\rm w}$ sufficiently close to $\frac{\pi}{2}$ so that various constants in the argument can be controlled. The iteration set consists of functions defined on a region $\mathcal{D}$, where $\mathcal{D}$ contains all possible Ω for the fixed $\theta_{\rm w}$. Specifically, an important property of the regular shock reflection-diffraction configurations is (4.22), which implies that $\Omega \subset \{\varphi_2 < \varphi_1\}$; that is, Ω lies *below* line $\mathcal{S}_1$ passing through P_0 and P_1 on Fig. 4.1. Note that, when $\theta_{\rm w}$ is close to $\frac{\pi}{2}$, this line is close to the vertical reflected shock of normal reflection on Fig. 4.3. Then $\mathcal{D}$ is defined as a region bounded by $\mathcal{S}_1$, $\Gamma_{\rm sonic} = P_1P_4$, $\Gamma_{\rm wedge} = P_3P_4$, and the symmetry line $\xi_2 = 0$. The iteration set is a set of functions φ on $\mathcal{D}$, defined by $\varphi \geqslant \varphi_2$ on $\mathcal{D}$ and the bound of norm of $\varphi - \varphi_2$ on $\mathcal{D}$ in the scaled and weighted $C^{2,\alpha}$ space defined in (4.38) below. Such functions satisfy

$$\|\varphi - \varphi_2\|_{C^{1,\alpha}(\overline{\mathcal{D}})} \leqslant C(\frac{\pi}{2} - \theta_{\rm w}),$$

which is small when $\frac{\pi}{2} - \theta_w \ll 1$, and

$$\|\varphi - \varphi_2\|_{C^{1,1}(\overline{\mathcal{D}} \cap \mathcal{N}_\varepsilon(\Gamma_{\rm sonic}))} < \infty.$$

However, $\|\varphi - \varphi_2\|_{C^{1,1}(\overline{\mathcal{D}} \cap \mathcal{N}_\varepsilon(\Gamma_{\rm sonic}))}$ is not small even if $\frac{\pi}{2} - \theta_{\rm w}$ is small; the reasons for that will be discussed below.

Given a function $\hat{\varphi}$ from the iteration set, we define domain $\Omega(\hat{\varphi}) := \{\hat{\varphi} < \varphi_1\}$ so that the iteration free boundary is $\Gamma_{\rm shock}(\hat{\varphi}) = \partial\Omega(\hat{\varphi}) \cap \mathcal{D}$. This is similar to (2.41), and the corresponding non-degeneracy similar to (2.40) in the present case is:

$$\partial_{\xi_1}(\varphi_1 - \varphi_2 - \phi) \geqslant \frac{u_1}{2} \qquad \text{in } \mathcal{D} \quad \text{if } \|\phi\|_{C^1(\overline{\mathcal{D}})} \text{ and } \tfrac{\pi}{2} - \theta_{\rm w} \text{ are small.}$$

Then we define the iteration equation by using form (4.12) of equation (4.8), by making an elliptic truncation (which is somewhat different from Step 1 in §2.2) and substituting $\hat{\varphi}$ in some terms of the coefficients of (4.12). The iteration boundary condition on $\Gamma_{\rm shock}(\hat{\varphi})$ is an oblique derivative condition obtained by combining two conditions in Problem 4.3(ii) and making some truncations. On $\Gamma_{\rm sonic}$, we prescribe $\varphi = \varphi_2$, *i.e.*, one of two conditions in Problem 4.3(iii). On $\Gamma_{\rm wedge}$ and $\Gamma_{\rm sym}(\hat{\varphi})$, we prescribe the conditions given in Problem 4.3(iv). The iteration map: $\hat{\varphi} \to \varphi$ is defined by solving the iteration problem to obtain φ and then extending φ from $\Omega(\hat{\varphi})$ to $\mathcal{D}$.

The fundamental differences between the iteration procedure in the shock reflection-diffraction problem and the previous procedures on transonic shocks in the steady case in §2–§3 (such as [29,30,32,111] and follow-up papers) include:

(i) The procedures on steady transonic shocks in §2–§3 are for the perturbation case. In particular, the ellipticity of the iteration equation and the removal of the elliptic cutoff are achieved by making the iteration set sufficiently close to the background solution in C^1 or a stronger norm. For the regular reflection problem, this cannot be done because of the elliptic degeneracy near the sonic arc.
(ii) Only one condition on $\Gamma_{\rm sonic}$ can be prescribed; however, both $\varphi = \varphi_2$ and $D\varphi = D\varphi_2$ on $\Gamma_{\rm sonic}$ are needed to be matched to obtain a global entropy solution. This is resolved by exploiting the elliptic degeneracy on $\Gamma_{\rm sonic}$.

2. In order to see the elliptic degeneracy on $\Gamma_{\rm sonic}$ more explicitly, we fix the wedge angle $\theta_{\rm w}$ and the corresponding pseudo-potential $\varphi_2 = \varphi_2^{(\theta_{\rm w})}$ of the weak state (2), and rewrite equation (4.10) in terms of the function:

$$\psi = \varphi - \varphi_2$$

in the following coordinates flattening $\Gamma_{\rm sonic}$:

$$x = c_2 - r, \qquad y = \theta - \theta_{\rm w}, \tag{4.23}$$

where (r, θ) are the polar coordinates centered at $O_2 = \mathbf{u}_2$ of the sonic circle of state (2). Then

$$\Omega_\varepsilon := \Omega \cap \mathcal{N}_\varepsilon(\Gamma_{\rm sonic}) \subset \{x > 0\} \ \text{ for small } \varepsilon > 0, \qquad \Gamma_{\rm sonic} \subset \{x = 0\}.$$

In what follows, we always assume that $\varphi \in C^{1,1}(\overline{\Omega_\varepsilon})$ as in Theorem 4.1 for the supersonic case. Then, by the conditions in Problem 4.3(iii) and the definition of ψ,

$$\psi = 0 \qquad \text{on } \Gamma_{\rm sonic}, \tag{4.24}$$

$$D\psi = 0 \qquad \text{on } \Gamma_{\rm sonic}. \tag{4.25}$$

Moreover, we *a priori* assume that solution φ satisfies (4.22) in Ω to derive the required estimates of the solution; with these estimates, we then construct such a solution and verify that it satisfies (4.22). The heuristic motivation of (4.22) is the following: From Figs. 4.1–4.2, it appears that $\Gamma_{\rm shock}$ (and hence Ω) is located *below* line $\mathcal{S}_1$, *i.e.*, in the half-plane $\{\varphi_1 > \varphi_2\}$. Thus, $\varphi = \varphi_1 > \varphi_2$ on $\Gamma_{\rm shock}$, and $\varphi_1 > \varphi_2 = \varphi$ on $\Gamma_{\rm sonic}$. Also, the potential functions ϕ_1 and ϕ_2 of states (1) and (2) are linear functions, thus they satisfy equation (4.12) with coefficients determined by φ, considered as a linear equation for ϕ. Taking into account the inequalities on $\Gamma_{\rm shock}$ and $\Gamma_{\rm sonic}$ noted above, and the oblique boundary conditions on $\Gamma_{\rm wedge}$ and $\Gamma_{\rm sym}$, we obtain (4.22) by the maximum principle. Then, from (4.22), we have

$$\psi > 0 \qquad \text{in } \Omega. \tag{4.26}$$

Even though the previous argument is heuristic, the fact that it comes from the structure of the problem allows us to include the condition that $\psi \geqslant 0$ in the definition of the iteration set and close the iteration argument for constructing the solutions within this set.

Equation (4.10) in $\Omega \cap \mathcal{N}_\varepsilon(\Gamma_{\rm sonic})$ for ψ in the (x, y)–coordinates (4.23) is

$$\big(2x - (\gamma+1)\psi_x + O_1\big)\psi_{xx} + O_2\psi_{xy} + (\frac{1}{c_2} + O_3)\psi_{yy} - (1 + O_4)\psi_x + O_5\psi_y = 0, \tag{4.27}$$

where

$$
\begin{aligned}
&O_1(D\psi,\psi,x) = -\frac{x^2}{c_2} + \frac{\gamma+1}{2c_2}(2x-\psi_x)\psi_x - \frac{\gamma-1}{c_2}\Big(\psi + \frac{1}{2(c_2-x)^2}\psi_y^2\Big),\\
&O_2(D\psi,\psi,x) = -\frac{2(\psi_x+c_2-x)\psi_y}{c_2(c_2-x)^2},\\
&O_3(D\psi,\psi,x) = \frac{1}{c_2(c_2-x)^2}\Big(x(2c_2-x) - (\gamma-1)\big(\psi + (c_2-x)\psi_x + \frac{1}{2}\psi_x^2\big) - \frac{\gamma+1}{2(c_2-x)^2}\psi_y^2\Big),\\
&O_4(D\psi,\psi,x) = \frac{1}{c_2-x}\Big(x - \frac{\gamma-1}{c_2}\big(\psi + (c_2-x)\psi_x + \frac{1}{2}\psi_x^2 + \frac{(\gamma+1)\psi_y^2}{2(\gamma-1)(c_2-x)^2}\big)\Big),\\
&O_5(D\psi,\psi,x) = -\frac{2(\psi_x+c_2-x)\psi_y}{c_2(c_2-x)^3}.
\end{aligned}
\tag{4.28}
$$

Since $\psi \in C^{1,1}(\overline{\Omega}_\varepsilon)$, it follows from (4.24)–(4.25) that $|\psi(x,y)| \leqslant Cx^2$ and

$$|D\psi(x,y)| \leqslant Cx \qquad \text{in } \Omega_\varepsilon, \tag{4.29}$$

so that

$$|O_1(D\psi,\psi,x)| \leqslant N|x|^2, \quad |O_k(D\psi,\psi,x)| \leqslant N|x| \qquad \text{for } k=2,\dots,5. \tag{4.30}$$

Using (4.30), we can show that $O_k(D\psi,\psi,x)$ are small perturbations of the leading terms of equation (4.27) in $\Omega_\varepsilon = \Omega \cap \mathcal{N}_\varepsilon(\Gamma_{\rm sonic})$. Also, if (4.29) holds, equation (4.27) is strictly elliptic in $\overline{\Omega}_\varepsilon \backslash \overline{\Gamma_{\rm sonic}}$ if

$$\psi_x(x,y) \leqslant \frac{2\mu}{\gamma+1}x \tag{4.31}$$

for $\mu \in (0,1)$, when $\varepsilon = \varepsilon(\mu, N)$ is small. For $\theta_{\rm w}$ close to $\frac{\pi}{2}$, it can be shown that any solution of Problem 4.3 (with some natural regularity properties) satisfies that, for any small $\delta > 0$,

$$|\psi_x(x,y)| \leqslant \frac{1+\delta}{\gamma+1}x \qquad \text{in } \Omega_\varepsilon \text{ for small } \varepsilon = \varepsilon(\delta), \tag{4.32}$$

which verifies (4.31) with any $\mu \in (\frac{1}{2}, 1)$ (*e.g.*, with $\mu = \frac{2}{3}$) if δ is correspondingly small.

3. The iteration equation near $\Gamma_{\rm sonic}$ is defined based on the above facts. The iteration set $\mathcal{K}_M$ used in [33] is such that every $\hat{\psi} = \hat{\varphi} - \varphi_2 \in \mathcal{K}_M$ satisfies (4.24) and (4.29) for some $C, \varepsilon > 0$. Then the iteration equation for ψ is

$$\big(2x - (\gamma+1)x\eta(\frac{\psi_x}{x}) + O_1^{(\hat{\psi})}\big)\psi_{xx} + O_2^{(\hat{\psi})}\psi_{xy} + (\frac{1}{c_2} + O_3^{(\hat{\psi})})\psi_{yy} - (1 + O_4^{(\hat{\psi})})\psi_x + O_5^{(\hat{\psi})}\psi_y = 0, \tag{4.33}$$

where the cutoff function $\eta \in C^\infty(\mathbb{R})$ satisfies $|\eta| \leqslant \frac{5}{3(\gamma+1)}$, $\eta' \geqslant 0$, and $\eta(s) = s$ if $|s| \leqslant \frac{4}{3(\gamma+1)}$, and some other technical conditions. The terms $O_k^{(\hat{\psi})}$, $k = 1,\dots,5$, are obtained from O_k by substituting $\hat{\psi}$ into certain terms in (4.28) and performing the cutoff in the remaining terms, so that estimates (4.30) hold. Then (4.33) is strictly elliptic in $\overline{\Omega}_\varepsilon \backslash \overline{\Gamma_{\rm sonic}}$ for small ε, and its ellipticity degenerates on $\Gamma_{\rm sonic}$. Since the solution of Problem 4.3 satisfies equation (4.27) and inequality (4.32) with $\delta = \frac{1}{3}$ in Ω_ε for small ε, then it satisfies equation (4.33) in Ω_ε with $\hat{\psi} = \psi$. Indeed, we have the estimate: $|\psi_x| \leqslant \frac{4}{3(\gamma+1)}x$, so that $x\eta(\frac{\psi_x}{x}) = \psi_x$; and the cutoffs in the terms of $O_k^{(\hat{\psi})}$ are removed similarly.

We also note that the degenerate ellipticity structure of equation (4.33) is the following: Writing (4.33) in the form

$$\sum_{i,j=1}^{2} A_{ij}(D\psi,\psi,x)D_{ij}\psi + \sum_{i=1}^{2} A_i(D\psi,\psi,x)D_i\psi = 0 \tag{4.34}$$

with $A_{12} = A_{21}$, we see that, for any $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$,

$$\lambda|\boldsymbol{\xi}|^2 \leqslant A_{11}(\mathbf{p},z,x)\frac{\xi_1^2}{x} + 2A_{12}(\mathbf{p},z,x)\frac{\xi_1\xi_2}{x^{1/2}} + A_{22}(\mathbf{p},z,x)\xi_2^2 \leqslant \frac{1}{\lambda}|\boldsymbol{\xi}|^2 \tag{4.35}$$

for all $(\mathbf{p}, z) \in \mathbb{R}^2 \times \mathbb{R}$ and $\mathbf{x} \in (0, \varepsilon)$.

We consider solutions of (4.33) in Ω_ε satisfying (4.24) and (4.26). Since condition (4.25) can not be prescribed in the iteration problem as discussed above, we have to obtain (4.25) from the estimates of the solutions by exploiting the elliptic degeneracy. The estimates of the positive solutions of (4.33) with (4.24) in Ω_ε are based on the fact that, for any $\delta > 0$, the function:

$$w_\delta(x, y) = \frac{1+\delta}{2(\gamma+1)} x^2$$

is a supersolution of (4.33) in Ω_ε if $\varepsilon = \varepsilon(\delta)$ is small; that is, $\mathcal{N}(w_\delta) < 0$ in Ω_ε, where $\mathcal{N}(\cdot)$ denotes the operator determined by the left-hand side of (4.33). Using this, the boundary conditions on Γ_{shock} and Γ_{wedge}, and (4.26), we obtain by the comparison principle that

$$0 \leqslant \psi \leqslant Cx^2 \qquad \text{in } \Omega_\varepsilon, \tag{4.36}$$

where ε and C are uniform with respect to the wedge angles near $\frac{\pi}{2}$. Note that $-w_\delta$ is *not* a subsolution of (4.27) so that it cannot be used to bound ψ from below. Thus, property (4.26), which is derived from the global structure of the solution, is crucially used in this argument. Then, in (4.36), the upper bound is from the local estimates near Γ_{sonic}, while the lower bound is from the global structure of the problem.

In particular, (4.36) implies that $D\psi = 0$ on Γ_{sonic}, which resolves the issue described in (ii) above. Furthermore, from (4.36), using the non-isotropic *parabolic* rescaling corresponding to the elliptic degeneracy (4.35) of equation (4.33) near $x = 0$, we obtain the estimates in the appropriately weighted and scaled Hölder norm in Ω_ε, which also imply the uniform $C^{1,1}$ estimates:

$$|D^2\psi| \leqslant C \qquad \text{in } \Omega_\varepsilon. \tag{4.37}$$

More precisely, we denote this norm by $\|\psi\|^{(\text{par})}_{2,\alpha,\Omega_\varepsilon}$ and define it as follows: Denote $\mathbf{z} = (x, y)$ and $\tilde{\mathbf{z}} = (\tilde{x}, \tilde{y})$ with $x, \tilde{x} \in (0, 2\varepsilon)$ and

$$\delta^{(\text{par})}_\alpha(\mathbf{z}, \tilde{\mathbf{z}}) := \left(|x-\tilde{x}|^2 + \max(x, \tilde{x})|y-\tilde{y}|^2\right)^{\alpha/2}.$$

Then, for $\psi \in C^2(\Omega_\varepsilon) \cap C^{1,1}(\overline{\Omega_\varepsilon})$ written in the (x, y)–coordinates, we define

$$\begin{aligned}
\|\psi\|^{(\text{par})}_{2,0,\Omega_\varepsilon} &:= \sum_{0 \leqslant k+l \leqslant 2} \sup_{\mathbf{z} \in \Omega_\varepsilon} \left(x^{k+l/2-2} |\partial_x^k \partial_y^l \psi(\mathbf{z})| \right), \\
[\psi]^{(\text{par})}_{2,\alpha,\Omega_\varepsilon} &:= \sum_{k+l=2} \sup_{\mathbf{z}, \tilde{\mathbf{z}} \in \Omega_\varepsilon, \mathbf{z} \neq \tilde{\mathbf{z}}} \left(\min(x^{k+l/2-2}, \tilde{x}^{k+l/2-2}) \frac{|\partial_x^k \partial_y^l \psi(\mathbf{z}) - \partial_x^k \partial_y^l \psi(\tilde{\mathbf{z}})|}{\delta^{(\text{par})}_\alpha(\mathbf{z}, \tilde{\mathbf{z}})} \right), \\
\|\psi\|^{(\text{par})}_{2,\alpha,\Omega_\varepsilon} &:= \|\psi\|^{(\text{par})}_{2,0,\Omega_\varepsilon} + [\psi]^{(\text{par})}_{2,\alpha,\Omega_\varepsilon}.
\end{aligned} \tag{4.38}$$

Now we obtain the required estimates in the norm in (4.38), under the assumption that (4.36) holds in $\Omega_{2\varepsilon}$. For every $\mathbf{z}_0 = (x_0, y_0) \in \overline{\Omega}_\varepsilon \backslash \Gamma_{\text{sonic}}$ (so that $x_0 \in (0, \varepsilon]$), we define

$$R_{\mathbf{z}_0} = \left\{ (x, y) \;\;:\;\; |x - x_0| < \frac{x_0}{10},\; |y - y_0| < \frac{\sqrt{x_0}}{10} \right\} \cap \Omega. \tag{4.39}$$

Note that $\text{dist}(R_{\mathbf{z}_0}, \Gamma_{\text{sonic}}) = \frac{9}{10}x_0 > 0$. We rescale the rectangle in (4.39) to the unit square $Q_1 = (-1, 1)^2$:

$$Q_1^{(\mathbf{z}_0)} := \left\{ (S, T) \in Q_1 \;:\; (x_0 + \frac{x_0}{10}S,\; y_0 + \frac{\sqrt{x_0}}{10}T) \in \Omega \right\}, \tag{4.40}$$

and define the scaled version of ψ in the (S, T)–coordinates in $Q_1^{(\mathbf{z}_0)}$:

$$\psi^{(\mathbf{z}_0)}(S, T) := \frac{1}{x_0^2} \psi(x_0 + \frac{x_0}{10}S,\; y_0 + \frac{\sqrt{x_0}}{10}T) \qquad \text{for } (S, T) \in Q_1^{(z_0)}. \tag{4.41}$$

Note that this rescaling is non-isotropic with respect to the two variables x and y. By (4.36), we have

$$\|\psi^{(\mathbf{z}_0)}\|_{L^\infty(\overline{Q_1^{(\mathbf{z}_0)}})} \leqslant C \qquad \text{for any } \mathbf{z}_0 = (x_0, y_0) \in \overline{\Omega}_\varepsilon \backslash \overline{\Gamma_{\text{sonic}}}. \tag{4.42}$$

Rewriting equation (4.33) in terms of $\psi^{(\mathbf{z}_0)}$ in the (S,T)–coordinates and noting the degenerate ellipticity structure (4.35), we find that $\psi^{(\mathbf{z}_0)}$ satisfies a uniformly elliptic equation in $Q_1^{(\mathbf{z}_0)}$ with the ellipticity constants and certain Hölder norms of the coefficients independent of $\mathbf{z}_0$. We also rescale the boundary conditions on $\Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon$ and $\Gamma_{\text{wedge}} \cap \partial\Omega_\varepsilon$ in a similar way, when $\mathbf{z}_0$ is on the corresponding part of the boundary. Then we apply the local elliptic $C^{2,\alpha}$–estimates for $\psi^{(\mathbf{z}_0)}$ in $Q_1^{(\mathbf{z}_0)}$ in the following cases:

(i) Interior rectangles $R_{\mathbf{z}_0}$, *i.e.*, all $\mathbf{z}_0$ such that $Q_1^{(\mathbf{z}_0)} = Q_1$ holds,

(ii) Rectangles $R_{\mathbf{z}_0}$ centered on the shock: $\mathbf{z}_0 \in \Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon$,

(iii) Rectangles $R_{\mathbf{z}_0}$ centered on the wedge: $\mathbf{z}_0 \in \Gamma_{\text{wedge}} \cap \partial\Omega_\varepsilon$,

where, in the last two cases, we use the local estimates for the corresponding boundary value problems. Using (4.42), we obtain

$$\|\psi^{(\mathbf{z}_0)}\|_{C^{2,\alpha}(\overline{Q_{1/2}^{(\mathbf{z}_0)}})} \leqslant C \qquad \text{with } C \text{ independent of } \mathbf{z}_0,$$

where $Q_{1/2}^{(\mathbf{z}_0)} = Q_1^{(\mathbf{z}_0)} \cap (-\frac{1}{2},\ \frac{1}{2})^2$. Rewriting in terms of ψ in the (x,y)–coordinates and combining the estimates for all $\mathbf{z}_0$ as above, we obtain the estimate: $\|\psi\|^{(\text{par})}_{2,\alpha,\Omega_\varepsilon} \leqslant C$ in norm (4.38), which also implies the $C^{1,1}$–estimates (4.37).

Remark 4.5. *Note that* $\psi^{(\mathbf{z}_0)}_{SS}(S,T) = \frac{1}{100}\psi_{xx}(x_0 + \frac{x_0}{10}S,\ y_0 + \frac{\sqrt{x_0}}{10}T)$. *It follows that* $\|D^2\psi\|_{L^\infty}$ *cannot be made small by choosing the parameters, e.g., choosing* ε *small or* θ_{w} *close to* $\frac{\pi}{2}$.

Remark 4.6. *The above argument, starting from* (4.39), *is also used for the a priori estimates of the positive solutions of* (4.27)–(4.28) *with condition* (4.24), *satisfying* (4.29) *and the ellipticity condition* (4.31) *with some* $\mu \in (0,1)$. *Note that* (4.24), (4.29), *and* $\psi \geqslant 0$ *imply* (4.36), *which is used in the argument.*

Remark 4.7. *Remark* 4.6 *applies only to the positive solutions of* (4.27) *with condition* (4.24). *For the negative solutions of* (4.27) *with condition* (4.24), *the equation is* uniformly *elliptic up to* $\{x = 0\}$ *and, similar to Hopf's lemma, the negative solutions have* linear *growth*: $|\psi(x,y)| \geqslant \frac{1}{C}x$, *in a contrast with* (4.36). *This feature is used in the proof of certain geometric properties of the free boundary for the wedge angles away from* $\frac{\pi}{2}$, *where we note that* $\varphi - \varphi_1 < 0$ *by* (4.22).

4. In order to remove the ellipticity cutoff in (4.33), *i.e.*, to show that the fixed point solution of (4.33) (*i.e.*, with $\psi = \hat{\psi}$) actually satisfies (4.27), we need to show that $|\psi_x| \leqslant \frac{4}{3(\gamma+1)}x$, as we have discussed right after (4.33). Combining (4.37) with $D\psi = 0$ on Γ_{sonic}, we obtain that $|D\psi(x,y)| \leqslant Cx$ in Ω_ε, which does not remove the ellipticity cutoff, unless we show the explicit bound $C \leqslant \frac{4}{3(\gamma+1)}$. However, this bound does not follow from the estimates discussed above (*cf.* Remark 4.5).

Note that the only explicit solution we know is the normal reflection for $\theta_{\text{w}} = \frac{\pi}{2}$, for which $\varphi = \varphi_2^{(\frac{\pi}{2})}$, *i.e.*, $\psi = 0$ in Ω. Also, the analysis in Bae-Chen-Feldman [5] has shown that the solutions of Problem 4.3 for the supersonic regular shock reflection-diffraction configuration satisfy that, for small ε,

$$\psi_x \sim \frac{x}{\gamma+1} \qquad \text{in } \Omega_\varepsilon \cap \{(x,y)\ :\ \text{dist}\,((x,y),\Gamma_{\text{shock}}) > \sqrt{x}\},$$

but

$$D\psi = o(x) \qquad \text{in } \Omega_\varepsilon \cap \{(x,y)\ :\ \text{dist}\,((x,y),\Gamma_{\text{shock}}) < x^2\}.$$

This shows that the convergence of solutions $\varphi^{(\theta_{\text{w}})}$ of Problem 4.3 to $\varphi^{(\frac{\pi}{2})}$ as $\theta_{\text{w}} \to \frac{\pi}{2}^-$ does not hold in C^2 up to the sonic arc Γ_{sonic} (but holds in $C^{1,\alpha}$) after mapping $\Omega^{(\theta_{\text{w}})}$ to a fixed domain for all θ_{w}. Moreover, the difference between the behaviors of $D\psi$ near Γ_{shock} and away from Γ_{shock} within Ω_ε

shows that there is no clear background solution such that the appropriate iteration set would lie in its small neighborhood in the norm sufficiently strong to remove the ellipticity cutoff in (4.33) by the smallness of the norm. Then, in order to remove the ellipticity cutoff for the fixed point of the iteration, we derive an equation for ψ_x in Ω_ε and boundary conditions on $\Gamma_{\text{shock}} \cap \{x < \varepsilon\}$ and $\Gamma_{\text{wedge}} \cap \{x < \varepsilon\}$, and prove that

$$\psi_x \leqslant \frac{4}{3(\gamma+1)}x$$

from this boundary value problem, if the wedge angle $\theta_{\rm w}$ is sufficiently close to $\frac{\pi}{2}$. The estimate from below:

$$\psi_x \geqslant -\frac{4}{3(\gamma+1)}x$$

is proved from the global setting of Problem 4.3 under the same condition on $\theta_{\rm w}$. This use of the local and global structure is similar to that in the proof of (4.36).

Note that, in this argument for the wedge angles near $\frac{\pi}{2}$, the non-perturbative nature of the problem is seen only in the estimates of the solution near Γ_{sonic}, specifically in the fact that $D^2\psi$ on Γ_{sonic} does not tend to zero as $\theta_{\rm w} \to \frac{\pi}{2}$. The free boundary Γ_{shock} in this case is near $\mathcal{S}_1(\theta_{\rm w})$, and also close to the reflected shock of the normal reflection as in Fig. 4.3, which is the vertical line $\mathcal{S}_1(\frac{\pi}{2})$. Also, $\|\varphi - \varphi_2^{(\theta_{\rm w})}\|_{C^1(\Omega)} \leqslant C(\frac{\pi}{2} - \theta_{\rm w})$, which is small. Thus, away from Γ_{sonic}, the argument is perturbative for the wedge angles near $\frac{\pi}{2}$. In the case of general wedge angles in Theorem 4.1, the free boundary Γ_{shock} is no longer close to a line, its structure is not known *a priori*, thus the study of geometric properties of the free boundary is a part of the argument.

Case II. General wedge angles up to the detachment angle. For the general case and the proof of Theorem 4.1, we follow the approach introduced in Chen-Feldman [35]. Similar to the case of wedge angles near $\frac{\pi}{2}$ where we have restricted our consideration to the class of solutions satisfying $\psi \geqslant 0$ in Ω and established the existence of such solutions, for the general case, we define a class of *admissible solutions*, make the necessary *a priori* estimates of such solutions, and then employ these estimates to prove the existence of solutions in this class. Our motivation for the definition of admissible solutions is from the following properties of supersonic regular reflection solutions φ for the wedge angles close to $\frac{\pi}{2}$; or more generally, for the supersonic regular reflection solutions φ satisfying that $\|\varphi - \varphi_2^{(\theta_{\rm w})}\|_{C^1(\Omega)}$ is small: If (4.8) is strictly elliptic for φ in $\overline{\Omega}\backslash\overline{\Gamma_{\text{sonic}}}$, then φ satisfies (4.22) and the monotonicity properties:

$$\partial_{\xi_2}(\varphi_1 - \varphi) \leqslant 0, \quad D(\varphi_1 - \varphi)\cdot \mathbf{e}_{\mathcal{S}_1} \leqslant 0 \qquad \text{in } \Omega \ \text{ for } \mathbf{e}_{\mathcal{S}_1} = \tfrac{P_0P_1}{|P_0P_1|}. \tag{4.43}$$

We now present the outline of the proof of Theorem 4.1 in the following four steps:

1. Motivated by the discussion above, for the general case, we define the admissible solutions as the solutions of Problem 4.3 (thus the solutions with weak regular reflection-diffraction configuration of either supersonic or subsonic type) satisfying the following properties:

Definition 4.8. *Let* $\theta_{\rm w} \in (\theta_{\rm w}^{\rm d}, \frac{\pi}{2})$. *A function* $\varphi \in C^{0,1}(\overline{\Lambda})$ *is an admissible solution of the regular reflection problem if* φ *is a solution of* Problem 4.3 *extended to* Λ *by* (4.21) (*where* $P_0P_1P_4$ *is a point in the subsonic and sonic cases*) *and satisfies the following properties*:

(i) *The structure of solutions*:
   - *If* $|D\varphi_2(P_0)| > c_2$, *then* φ *is of the* supersonic *regular shock reflection-diffraction configuration shown on Fig. 4.1 and satisfies that the curved part of reflected-diffracted shock* Γ_{shock} *is* C^2 *in its relative interior*; *curves* Γ_{shock}, Γ_{sonic}, Γ_{wedge}, *and* Γ_{sym} *do not have common points except their endpoints*; $\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda\backslash(\mathcal{S}_0 \cup \overline{P_0P_1P_2}))$ *and* $\varphi \in C^1(\overline{\Omega}) \cap C^3(\overline{\Omega}\backslash(\overline{\Gamma_{\text{sonic}}} \cup \{P_2, P_3\}))$.
   - *If* $|D\varphi_2(P_0)| \leqslant c_2$, *then* φ *is of the* subsonic *regular shock reflection-diffraction configuration shown on Fig. 4.2 and satisfies that the reflected-diffracted shock* Γ_{shock} *is* C^2 *in its relative interior*; *curves* Γ_{shock}, Γ_{wedge}, *and* Γ_{sym} *do not have common points except their endpoints*; $\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda\backslash(\mathcal{S}_0 \cup \overline{\Gamma_{\text{shock}}}))$ *and* $\varphi \in C^1(\overline{\Omega}) \cap C^3(\overline{\Omega}\backslash\{P_0, P_3\})$.

Moreover, in both the supersonic and subsonic cases, the extended curve $\Gamma^{\text{ext}}_{\text{shock}} := \Gamma_{\text{shock}} \cup \{P_0\} \cup \Gamma^{-}_{\text{shock}}$ *is* C^1 *in its relative interior, where* $\Gamma^{-}_{\text{shock}}$ *is the reflection of* Γ_{shock} *with respect to the* ξ_1*–axis.*

(ii) *Equation* (4.8) *is strictly elliptic in* $\overline{\Omega}\backslash\overline{\Gamma_{\text{sonic}}}$*, i.e.,* $|D\varphi| < c(|D\varphi|^2, \varphi)$ *in* $\overline{\Omega}\backslash\overline{\Gamma_{\text{sonic}}}$*.*

(iii) $\partial_{\boldsymbol{\nu}}\varphi_1 > \partial_{\boldsymbol{\nu}}\varphi > 0$ *on* Γ_{shock}*, where* $\boldsymbol{\nu}$ *is the normal to* Γ_{shock}*, pointing to the interior of* Ω*.*

(iv) *Inequalities hold*:

$$\varphi_2 \leqslant \varphi \leqslant \varphi_1 \qquad \text{in } \Omega. \tag{4.44}$$

(v) (4.43) *is satisfied, where vector* $\mathbf{e}_{\mathcal{S}_1}$ *is defined as the unit vector parallel to* $\mathcal{S}_1$ *and pointing into* Λ *at* P_0 *for the general case.*

Note that (4.43) implies that

$$D(\varphi_1 - \varphi)\cdot \mathbf{e} \leqslant 0 \qquad \text{in } \overline{\Omega} \;\text{ for all } \mathbf{e} \in \overline{Cone(\mathbf{e}_{\xi_2}, \mathbf{e}_{\mathcal{S}_1})}, \tag{4.45}$$

where $Cone(\mathbf{e}_{\xi_2}, \mathbf{e}_{\mathcal{S}_1}) = \{a\,\mathbf{e}_{\xi_2} + b\,\mathbf{e}_{\mathcal{S}_1} \;:\; a, b > 0\}$ with $\mathbf{e}_{\xi_2} = (0,1)$. Notice that $\mathbf{e}_{\xi_2}$ and $\mathbf{e}_{\mathcal{S}_1}$ are not parallel if $\theta_{\text{w}} \neq \frac{\pi}{2}$.

2. To prove the existence of admissible solutions for each wedge angle in Theorem 4.1, we derive uniform *a priori* estimates for admissible solutions with any wedge angle $\theta_{\text{w}} \in [\theta^{\text{d}}_{\text{w}} + \sigma, \frac{\pi}{2}]$ for each small $\sigma > 0$, show the compactness of this subset of admissible solutions in the appropriate norm, and then apply the degree theory to establish the existence of admissible solutions for each $\theta_{\text{w}} \in [\theta^{\text{d}}_{\text{w}} + \sigma, \frac{\pi}{2}]$, starting from the unique normal reflection solution for $\theta_{\text{w}} = \frac{\pi}{2}$. To derive the *a priori* estimates for admissible solutions, we first obtain the required estimates related to the geometry of shock Γ_{shock} and domain Ω, as well as the basic estimates of solution φ. We prove:

(a) The inequality in (4.45) is strict for any $\mathbf{e} \in Cone(\mathbf{e}_{\xi_2}, \mathbf{e}_{\mathcal{S}_1})$. Combined with the first inequality in (4.44) and the fact that $\varphi = \varphi_1$ on Γ_{shock}, this implies that Γ_{shock} is a Lipschitz graph with a uniform Lipschitz estimate for all admissible solutions.

(b) The uniform bounds on $\text{diam}(\Omega)$, $\|\varphi\|_{C^{0,1}(\Omega)}$, and the directional monotonicity of $\varphi - \varphi_2$ near the sonic arc for a cone of directions.

(c) The uniform positive lower bound for the distance between the shock and the wedge, and the uniform separation of the shock and the symmetry line (that is, Γ_{shock} is away from a uniform conical neighborhood of Γ_{sym} with vertex at their common endpoint P_2);

(d) The uniform positive lower bound for the distance between the shock and the sonic circle $B_{c_1}((u_1, 0))$ of state (1), by using the properties described in Remark 4.7. This allows us to estimate the ellipticity of (4.8) for φ in Ω (depending on the distance to the sonic arc P_1P_4 for the supersonic regular shock reflection-diffraction configuration and to P_0 for the subsonic regular shock reflection-diffraction configuration).

(e) Estimate (4.29) holds in the supersonic case, by using the monotonicity of $\psi = \varphi - \varphi_2$ near the sonic arc in a cone of directions shown in (b) and the conditions on Γ_{sonic} in Problem 4.3.

The results of (a)–(c) are obtained via the maximum principle, by considering equation (4.12) as a linear elliptic equation for ϕ and using the boundary conditions on Γ_{shock}, Γ_{sonic}, Γ_{wedge}, and Γ_{sym} in Problem 4.3 and (4.44)–(4.45). The results of (c), combined with (a), show the structure of Ω which allows us to perform the uniform local elliptic estimates in various parts of Ω: the interior, near a point P in a relative interior of Γ_{shock}, Γ_{wedge}, and Γ_{sym}, and locally near corners P_2 and P_3.

Based on estimates (a)–(d), we show the uniform regularity estimates for the solution and the free boundary in the weighted and scaled $C^{2,\alpha}$ norms away from the sonic arc in the supersonic case and away from P_0 in the subsonic case, *i.e.*, in $\Omega\backslash\Omega_\varepsilon$, for any small $\varepsilon > 0$, for some $\alpha \in (0,1)$. The equation is uniformly elliptic in this region, with the ellipticity constant depending on ε. Thus, the estimates depend on ε.

3. Below we discuss the estimates near Γ_{sonic} (resp. near P_0 in the subsonic/sonic case), *i.e.*, the estimates in $\Omega_{2\varepsilon}$ for some ε, independent of $\theta_{\text{w}} \in [\theta^{\text{d}}_{\text{w}} + \sigma, \frac{\pi}{2}]$, which allows us to complete the uniform

a priori estimates for admissible solutions with wedge angles $\theta_{\rm w} \in [\theta_{\rm w}^{\rm d} + \sigma, \frac{\pi}{2}]$. We obtain the estimates near $\Gamma_{\rm sonic}$ (or P_0 for the subsonic reflection), *i.e.*, in $\Omega_{2\varepsilon}$, in scaled and weighted $C^{2,\alpha}$ for φ and the free boundary $\Gamma_{\rm shock} \cap \partial\Omega_{2\varepsilon}$, by considering separately four cases depending on $\frac{|D\varphi_2|}{c_2}$ at P_0:

(i) Supersonic: $\frac{|D\varphi_2|}{c_2} \geqslant 1 + \delta$,

(ii) Supersonic (almost sonic): $1 < \frac{|D\varphi_2|}{c_2} < 1 + \delta$,

(iii) Subsonic (almost sonic, including sonic): $1 - \delta \leqslant \frac{|D\varphi_2|}{c_2} \leqslant 1$,

(iv) Subsonic: $\frac{|D\varphi_2|}{c_2} \leqslant 1 - \delta$,

for small $\delta > 0$ chosen so that the estimates can be obtained. The choice of δ determines ε.

For cases (i)–(ii), equation (4.8) is degenerate elliptic in Ω near P_1P_4 on Fig. 4.1. For case (iii), except the sonic case $\frac{|D\varphi_2(P_0)|}{c_2} = 1$, the equation is uniformly elliptic in $\overline{\Omega}$, but the ellipticity constant is small and tends to zero near P_0 on Fig. 4.2 as $\frac{|D\varphi_2^{(\theta_{\rm w})}(P_0)|}{c_2} \to 1^-$, *i.e.*, as the subsonic angles $\theta_{\rm w}$ tend to the sonic angle. Thus, for cases (i)–(iii), we exploit the local elliptic degeneracy, which allows us to find a comparison function in each case, to show the appropriately fast decay of $\varphi - \varphi_2$ near P_1P_4 for cases (i)–(ii) and near P_0 for case (iii); furthermore, combining with appropriate local non-isotropic rescaling to obtain the uniform ellipticity, we obtain the *a priori* estimates in the weighted and scaled $C^{2,\alpha}$–norms. In cases (i)–(ii), the norms are (4.38). For case (iii), we use the different norms to obtain the estimates that imply the standard $C^{2,\alpha}$–estimates. To obtain these estimates, for case (i), we use the argument developed in Chen-Feldman [33] and described above (see Remark 4.6), where the ellipticity estimate (4.31) follows from the estimates described in (d) above and (4.29) obtained in (e). These estimates hold in Ω_ε with $\varepsilon \lesssim (\text{length}(\Gamma_{\rm sonic}))^2$ because the *rectangles* $R_{(x_0,y_0)}$ defined by (4.39) do not fit into Ω for larger x_0, which means, for example, that $R_{(x_0,y_0)} \cap \Gamma_{\rm wedge} \neq \varnothing$ for $(x_0, y_0) \in \Gamma_{\rm shock} \cap \partial\Omega_\varepsilon$ with $x_0 \geqslant C(\text{length}(\Gamma_{\rm sonic}))^2$ if C is fixed and length$(\Gamma_{\rm sonic})$ is small, because the length of the y-side of $R_{(x_0,y_0)}$ is $\frac{\sqrt{x_0}}{10}$, and $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$ are smooth curves that intersect $\Gamma_{\rm sonic}$ transversally. However, length$(\Gamma_{\rm sonic})$ tends to zero, as $\dfrac{|D\varphi_2^{(\theta_{\rm w})}(P_0)|}{c_2} \to 1^+$, *i.e.*, when the supersonic wedge angle tends to the sonic angle. Thus, a different argument, involving an appropriate scaling, is employed for case (ii) in order to keep ε uniform for all $\theta_{\rm w} \in [\theta_{\rm w}^{\rm d} + \delta, \frac{\pi}{2}]$. Another version of that argument (with a different scaling) is applied for case (iii). For both cases (ii)–(iii), we need to use smaller rectangles than those for case (i), but this requires stronger growth estimates than (4.36) to obtain a bound in $C^{1,1}$ from the corresponding weighted and scaled estimates. We obtain such growth estimates by using the conditions of cases (ii)–(iii) for sufficiently small δ. For case (iv), the equation is uniformly elliptic in Ω for the admissible solution, where the ellipticity constant is not small, and the estimates are more technically challenging than those for cases (i)–(iii). This can be seen as follows: For all cases (i)–(iv), the free boundary has a lower *a priori* regularity in the sense that only the Lipschitz estimate of $\Gamma_{\rm shock}$ is obtained in (a) above; however, for case (iv), the uniform ellipticity combined with oblique boundary conditions does not allow a comparison function that leads to the fast decay of $|\varphi - \varphi_2|$ near P_0. Thus, we prove the C^α–estimates of $D(\varphi - \varphi_2)$ near P_0, by deriving the equations and boundary conditions for two directional derivatives of $\varphi - \varphi_2$ near P_0, and performing the hodograph transform to flatten the free boundary.

4. In order to prove the existence of solutions, we perform an iteration, which is an extension of the iteration process used in Chen-Feldman [33]. First, given an admissible solution φ for the wedge angle $\theta_{\rm w}$, we map its elliptic domain $\Omega(\varphi, \theta_{\rm w})$ to a unit square $Q = (0,1)^2$ so that, for the supersonic case, the boundary parts $\Gamma_{\rm shock}$, $\Gamma_{\rm sonic}$, $\Gamma_{\rm wedge}$, and $\Gamma_{\rm sym}$ are mapped to the respective sides of Q, and the other properties of this map are satisfied. For the subsonic case, the map is discontinuous at $P_0 = \overline{\Gamma_{\rm sonic}}$ (mapping the triangular domain to a square). Moreover, we define a function u on Q by expressing $u := \varphi - \tilde{\varphi}_2^{(\theta_{\rm w})}$ in the coordinates on Q, where $\tilde{\varphi}_2^{(\theta_{\rm w})}$ is a function determined by $\theta_{\rm w}$ and equals to

φ_2 near $\overline{\Gamma_{\rm sonic}}$; we skip the complete technical definition here. For appropriate functions u on Q and the wedge angle $\theta_{\rm w}$, this map can be inverted, *i.e.*, the elliptic domain $\Omega(u,\theta_{\rm w})$ and the iteration free boundary $\Gamma_{\rm shock}(u,\theta_{\rm w})$ can be determined, and a function $\varphi^{(u,\theta_{\rm w})}$ on $\Omega(u,\theta_{\rm w})$ is defined by expressing u in the coordinates on $\Omega(u,\theta_{\rm w})$ and adding $\tilde{\varphi}_2^{(\theta_{\rm w})}$ so that, if u is obtained from the admissible solution φ with the elliptic domain Ω as described above, then $\Omega(u,\theta_{\rm w})=\Omega$ and $\varphi^{(u,\theta_{\rm w})}=\varphi$ in Ω. Moreover, the map: $\Omega(u,\theta_{\rm w})\to Q$ and its inverse satisfy certain continuity properties with respect to $(u,\theta_{\rm w})$. The iteration is performed in terms of the functions defined on Q. The iteration set consists of pairs $(u,\theta_{\rm w})$, where u is in a weighted and scaled $C^{2,\alpha}$ space on Q, denoted as $C^{2,\alpha}_{**}$ (its definition is technical, so we skip it here), and satisfy

(i) $\|u\|_{C^{2,\alpha}_{**}}\leqslant M(\theta_{\rm w})$, where $M(\theta_{\rm w})$ is defined explicitly, based on the *a priori* estimates discussed above;
(ii) $\Omega(u,\theta_{\rm w})$, $\Gamma_{\rm shock}(u,\theta_{\rm w})$, and $\varphi^{(u,\theta_{\rm w})}$ on $\Omega(u,\theta_{\rm w})$ satisfy some geometric and analytical properties.

The iteration map: $(\hat{u},\theta_{\rm w})\to(u,\theta_{\rm w})$ is defined by solving the iteration problem on $\Omega(u,\theta_{\rm w})$ and then mapping its solution φ to a function u on Q. This mapping includes additional steps, compared to the one described above. Specifically, we modify the iteration free boundary by using the solution φ of the iteration problem so that, in the mapping: $(\varphi,\theta_{\rm w})\to u$, the resulting function u on Q keeps the regularity obtained from solving the iteration problem. This yields the compactness of the iteration map. We prove that, for a fixed point $(u,\theta_{\rm w})$ of the iteration map, $\varphi^{(u,\theta_{\rm w})}$ on $\Omega(u,\theta_{\rm w})$ is an admissible solution. We use the degree theory to establish the existence of admissible solutions as fixed points of the iteration map for each $\theta_{\rm w}\in[\theta_{\rm w}^{\rm d}+\delta,\frac{\pi}{2}]$, starting from the unique normal reflection solution for $\theta_{\rm w}=\frac{\pi}{2}$. The compactness of the iteration map described above is necessary for that. The *a priori* estimates of admissible solutions discussed above are used in the degree theory argument in order to define the iteration set such that a fixed point of the iteration map (*i.e.*, admissible solution) cannot occur on the boundary of the iteration set, since that would contradict the *a priori* estimates. With all of these arguments, we complete the proof of Theorem 4.1. This provides a solution to the von Neumann's conjectures.

More details can be found in Chen-Feldman [35]; also see [33].

4.2. **The Prandtl-Meyer Problem for Unsteady Supersonic Flow onto Solid Wedges.** As we discussed in §2–§3, steady shocks appear when a steady supersonic flow hits a straight wedge; see Figure 3.1. Since both weak and strong steady shock solutions are stable in the steady regime, the static stability analysis alone is not able to single out one of them in this sense, unless an additional condition is posed on the speed of the downstream flow at infinity. Then the dynamic stability analysis becomes more significant to understand the non-uniqueness issue of the steady oblique shock solutions. However, the problem for the dynamic stability of the steady shock solutions for supersonic flow past solid wedges involves several additional difficulties. The recent efforts have been focused on the construction of the global Prandtl-Meyer reflection configurations in the self-similar coordinates for potential flow.

As we discussed earlier, if a supersonic flow with a constant density $\rho_0>0$ and a velocity $\mathbf{u}_0=(u_{10},0)$, $u_{10}>c_0:=c(\rho_0)$, impinges toward wedge W in (3.11), and if $\theta_{\rm w}$ is less than the detachment angle $\theta_{\rm w}^{\rm d}$, then the well-known *shock polar analysis* shows that there are two different steady weak solutions: *the steady weak shock solution* $\bar{\Phi}$ and *the steady strong shock solution*, both of which satisfy the entropy condition and the slip boundary condition (see Fig. 3.1).

Then the dynamic stability of the weak transonic shock solution for potential flow can be formulated as the following problem:

Problem 4.4 (Initial-Boundary Value Problem)**.** *Given $\gamma>1$, fix (ρ_0,u_{10}) with $u_{10}>c_0$. For a fixed $\theta_{\rm w}\in(0,\theta_{\rm w}^{\rm d})$, let W be given by* (3.11)*. Seek a global entropy solution $\Phi\in W^{1,\infty}_{\rm loc}(\mathbb{R}_+\times(\mathbb{R}^2\backslash W))$ of Eq.* (4.5) *with ρ determined by* (4.4) *and $B=\frac{u_{10}^2}{2}+h(\rho_0)$ so that Φ satisfies the initial condition at $t=0$:*

$$(\rho,\Phi)|_{t=0}=(\rho_0,u_{10}x_1)\qquad \text{for } \mathbf{x}\in\mathbb{R}^2\backslash W, \tag{4.46}$$

and the slip boundary condition along the wedge boundary ∂W:

$$\nabla_{\mathbf{x}}\Phi \cdot \boldsymbol{\nu}_{\mathrm{w}}|_{\partial W} = 0, \tag{4.47}$$

where $\boldsymbol{\nu}_{\mathrm{w}}$ *is the exterior unit normal to* ∂W.

In particular, we seek a solution $\Phi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+ \times (\mathbb{R}^2\backslash W))$ *that converges to the steady weak oblique shock solution* $\bar{\Phi}$ *corresponding to the fixed parameters* $(\rho_0, u_{10}, \gamma, \theta_{\mathrm{w}})$ *with* $\bar{\rho} = h^{-1}(B - \frac{1}{2}|\nabla\bar{\Phi}|^2)$, *when* $t \to \infty$, *in the following sense*: *For any* $R > 0$, Φ *satisfies*

$$\lim_{t\to\infty} \|(\nabla_{\mathbf{x}}\Phi(t,\cdot) - \nabla_{\mathbf{x}}\bar{\Phi}, \rho(t,\cdot) - \bar{\rho})\|_{L^1(B_R(\mathbf{0})\backslash W)} = 0 \tag{4.48}$$

for $\rho(t,\mathbf{x})$ *given by* (4.4).

Since the initial data functions in (4.46) do not satisfy the boundary condition (4.47), a boundary layer is generated along the wedge boundary starting at $t = 0$, which forms the Prandtl-Meyer reflection configurations; see Bae-Chen-Feldman [6] and the references cited therein.

Notice that the initial-boundary value problem, Problem 4.4, is invariant under scaling (4.6). Thus, we study the existence of self-similar solutions determined by equation (4.8) with (4.9) through (4.7).

As the upstream flow has the constant velocity $(u_{10}, 0)$, noting the choice of B in Problem 4.4, the corresponding pseudo-potential φ_0 has the expression of

$$\varphi_0 = -\frac{1}{2}|\boldsymbol{\xi}|^2 + u_{10}\xi_1 \tag{4.49}$$

in self-similar coordinates $\boldsymbol{\xi} = \frac{\mathbf{x}}{t}$, as shown directly from (4.14). Notice also the symmetry of the domain and the upstream flow in Problem 4.4 with respect to the x_1–axis. Problem 4.4 can then be reformulated as the following boundary value problem in the domain:

$$\Lambda := \mathbb{R}^2_+\backslash\{\boldsymbol{\xi} : \xi_2 \leqslant \xi_1 \tan\theta_{\mathrm{w}},\, \xi_1 \geqslant 0\}$$

in the self-similar coordinates $\boldsymbol{\xi}$, which corresponds to domain $\{(t,\mathbf{x}) : \mathbf{x} \in \mathbb{R}^2_+\backslash W,\, t > 0\}$ in the $(t,\mathbf{x})$–coordinates, where $\mathbb{R}^2_+ = \{\boldsymbol{\xi} : \xi_2 > 0\}$.

Problem 4.5 (Boundary Value Problem)**.** *Seek a solution* φ *of equation* (4.8) *in the self-similar domain* Λ *with the slip boundary condition*:

$$D\varphi \cdot \boldsymbol{\nu}|_{\partial\Lambda} = 0 \tag{4.50}$$

and the asymptotic boundary condition:

$$\varphi - \varphi_0 \longrightarrow 0 \tag{4.51}$$

along each ray $R_\theta := \{\xi_1 = \xi_2 \cot\theta, \xi_2 > 0\}$ *with* $\theta \in (\theta_{\mathrm{w}}, \pi)$ *as* $\xi_2 \to \infty$ *in the sense that*

$$\lim_{r\to\infty} \|\varphi - \varphi_0\|_{C(R_\theta\backslash B_r(0))} = 0. \tag{4.52}$$

In particular, we seek a global entropy solution of Problem 4.5 with two types of Prandtl-Meyer reflection configurations whose occurrence is determined by the wedge angle θ_{w} for the two different cases: One contains a straight weak oblique shock $\mathcal{S}_0$ attached to the wedge vertex O and connected to a normal shock $\mathcal{S}_1$ through a curved shock Γ_{shock} when $\theta_{\mathrm{w}} < \theta^{\mathrm{s}}_{\mathrm{w}}$, as shown in Fig. 4.4; the other contains a curved shock Γ_{shock} attached to the wedge vertex and connected to a normal shock $\mathcal{S}_1$ when $\theta^{\mathrm{s}}_{\mathrm{w}} \leqslant \theta_{\mathrm{w}} < \theta^{\mathrm{d}}_{\mathrm{w}}$, as shown in Fig. 4.5, in which the curved shock Γ_{shock} is tangential to the straight weak oblique shock $\mathcal{S}_0$ at the wedge vertex.

To seek a global entropy solution of Problem 4.5 with the structure of Fig. 4.4 or Fig. 4.5, one needs to compute the pseudo-potential function φ below $\mathcal{S}_0$.

Given $M_0 > 1$, ρ_1 and $\mathbf{u}_1$ are determined by using the shock polar as in Fig. 3.1 for steady potential flow (note that the shock polar is now different from the one for the full Euler system but has the same shape as in Fig. 3.1). Similar to those in §3.1, in the potential flow case, for any wedge angle $\theta_{\mathrm{w}} \in (0, \theta^{\mathrm{s}}_{\mathrm{w}})$, line $u_2 = u_1 \tan\theta_{\mathrm{w}}$ and the shock polar intersect at a point $\mathbf{u}_1$ with $|\mathbf{u}_1| > c_1$ and $u_{11} < u_{10}$; while, for any wedge angle $\theta_{\mathrm{w}} \in [\theta^{\mathrm{s}}_{\mathrm{w}}, \theta^{\mathrm{d}}_{\mathrm{w}})$, they intersect at a point $\mathbf{u}_1$ with $u_{11} > u_{1\mathrm{d}}$ and $|\mathbf{u}_1| < c_1$, where $u_{1\mathrm{d}}$ is the u_1–component of the unique detachment state $\mathbf{u}_{\mathrm{d}}$ when $\theta_{\mathrm{w}} = \theta^{\mathrm{d}}_{\mathrm{w}}$. The intersection

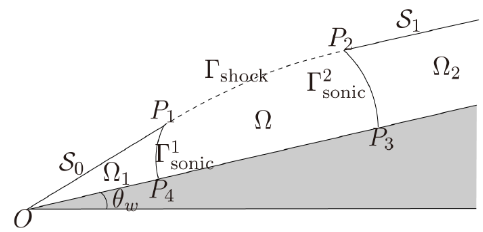

FIGURE 4.4. Self-similar solutions for $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm s})$ in the self-similar coordinates $\boldsymbol{\xi}$ (*cf.* [6])

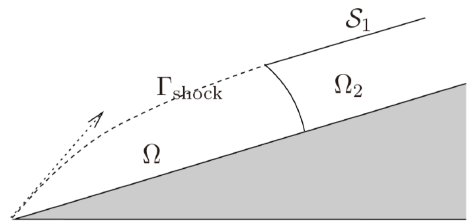

FIGURE 4.5. Self-similar solutions for $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$ in the self-similar coordinates $\boldsymbol{\xi}$ (*cf.* [6])

state $\mathbf{u}_1$ is the velocity for steady potential flow behind an oblique shock $\mathcal{S}_0$ attached to the wedge vertex with angle $\theta_{\rm w}$. The strength of shock $\mathcal{S}_0$ is relatively weak compared to the other shock given by the other intersection point on the shock polar, hence we call $\mathcal{S}_0$ a *weak oblique shock*, and the corresponding state $\mathbf{u}_1$ is a *weak state*.

We also note that states $\mathbf{u}_1$ depend smoothly on u_{10} and $\theta_{\rm w}$, and such states are supersonic when $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm s})$ and subsonic when $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$.

Once $\mathbf{u}_1$ is determined, by (4.17) and (4.49), the pseudo-potential φ_1 below the weak oblique shock $\mathcal{S}_0$ is

$$\varphi_1 = -\frac{1}{2}|\boldsymbol{\xi}|^2 + \mathbf{u}_1 \cdot \boldsymbol{\xi}. \tag{4.53}$$

Similarly, by (4.16)–(4.17) and (4.49)–(4.50), the pseudo-potential φ_2 below the normal shock $\mathcal{S}_1$ is of the form:

$$\varphi_2 = -\frac{1}{2}|\boldsymbol{\xi}|^2 + \mathbf{u}_2 \cdot \boldsymbol{\xi} + k_2 \tag{4.54}$$

for constant state $\mathbf{u}_2$ and constant k_2; see (4.14). Then it follows from (4.9) and (4.53)–(4.54) that the corresponding densities ρ_1 and ρ_2 are constants, respectively. In particular, we have

$$\rho_k^{\gamma-1} = \rho_0^{\gamma-1} + \frac{\gamma-1}{2}\big(u_{10}^2 - |\mathbf{u}_k|^2\big) \qquad \text{for } k = 1, 2. \tag{4.55}$$

Denote $\Gamma_{\rm wedge} := \partial W \cap \partial\Lambda$. Next, we define the sonic arcs $\Gamma^1_{\rm sonic} = P_1P_4$ on Fig. 4.4 and $\Gamma^2_{\rm sonic} = P_2P_3$ on Figs. 4.4–4.5. The sonic circle $\partial B_{c_1}(\mathbf{u}_1)$ of the uniform state φ_1 intersects line $\mathcal{S}_0$, where $c_1 = \rho_1^{\frac{\gamma-1}{2}}$ by (4.11). For the supersonic case $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm s})$, there are two arcs of this sonic circle between $\mathcal{S}_0$ and $\Gamma_{\rm wedge}$ in Λ. We denote by $\Gamma^1_{\rm sonic}$ the *lower* arc (*i.e.*, located to the left from another arc) in the orientation on Fig. 4.4. Note that $\Gamma^1_{\rm sonic}$ tends to point O as $\theta_{\rm w} \nearrow \theta_{\rm w}^{\rm s}$ and is outside of Λ for the subsonic case $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$. Similarly, the sonic circle $\partial B_{c_2}(\mathbf{u}_2)$ of the uniform state φ_2 intersects line $\mathcal{S}_1$, where $c_2 = \rho_2^{\frac{\gamma-1}{2}}$. There are two arcs of this circle between $\mathcal{S}_1$ and the line containing $\Gamma_{\rm wedge}$. For all $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, the *upper* arc (*i.e.*, located to the right of the other arc) in the orientation on Figs. 4.4–4.5 is within Λ, which is denoted as $\Gamma^2_{\rm sonic}$.

Then Problem 4.5 can be reformulated into the following free boundary problem:

Problem 4.6 (Free Boundary Problem). *For $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, find a free boundary (curved shock) $\Gamma_{\rm shock}$ and a function φ defined in domain Ω, as shown in Figs.* 4.4–4.5, *such that φ satisfies*

(i) *Equation* (4.8) *in Ω,*
(ii) $\varphi = \varphi_0$ *and* $\rho D\varphi \cdot \boldsymbol{\nu}_{\rm s} = \rho_0 D\varphi_0 \cdot \boldsymbol{\nu}_{\rm s}$ *on* $\Gamma_{\rm shock}$,
(iii) $\varphi = \hat{\varphi}$ *and* $D\varphi = D\hat{\varphi}$ *on* $\Gamma^1_{\rm sonic} \cup \Gamma^2_{\rm sonic}$ *when* $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm s})$ *and on* $\Gamma^2_{\rm sonic} \cup \{O\}$ *when* $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$ *for* $\hat{\varphi} := \max(\varphi_1, \varphi_2)$,
(iv) $D\varphi \cdot \boldsymbol{\nu}_{\rm w} = 0$ *on* $\Gamma_{\rm wedge}$,

where $\boldsymbol{\nu}_{\rm s}$ and $\boldsymbol{\nu}_{\rm w}$ are the unit normals to $\Gamma_{\rm shock}$ and $\Gamma_{\rm wedge}$ pointing to the interior of Ω, respectively.

Remark 4.9. *It can be shown that $\varphi_1 > \varphi_2$ on $\Gamma^1_{\rm sonic}$ and the opposite inequality holds on $\Gamma^2_{\rm sonic}$. This justifies the requirements in* Problem 4.6(iii) *above.*

Remark 4.10. *Similar to* Problem 4.3*, the conditions in* Problem 4.6(ii)–(iii) *are the Rankine-Hugoniot conditions* (4.16)–(4.17) *on $\Gamma_{\rm shock}$ and $\Gamma^1_{\rm sonic} \cup \Gamma^2_{\rm sonic}$ or $\Gamma^2_{\rm sonic} \cup \{O\}$, respectively; see the discussions right after* Problem 4.3.

Let φ be a solution of Problem 4.6 such that $\Gamma_{\rm shock}$ is a C^1–curve up to its endpoints and $\varphi \in C^1(\overline{\Omega})$. To obtain a solution of Problem 4.5 from φ, we consider two cases:

For the supersonic case $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm s})$, we divide region Λ into four separate regions; see Fig. 4.4. We denote by $\mathcal{S}_{0,{\rm seg}}$ the line segment $OP_1 \subset \mathcal{S}_0$, and by $\mathcal{S}_{1,{\rm seg}}$ the portion (half-line) of $\mathcal{S}_1$ with left endpoint P_2 so that $\mathcal{S}_{1,{\rm seg}} \subset \Lambda$. Let $\Omega_{\mathcal{S}}$ be the unbounded domain below curve $\overline{\mathcal{S}_{0,{\rm seg}} \cup \Gamma_{\rm shock} \cup \mathcal{S}_{1,{\rm seg}}}$ and above $\Gamma_{\rm wedge}$ (see Fig. 4.4). In $\Omega_{\mathcal{S}}$, let Ω_1 be the bounded domain enclosed by $\mathcal{S}_0, \Gamma^1_{\rm sonic}$, and $\Gamma_{\rm wedge}$. Set $\Omega_2 := \Omega_{\mathcal{S}} \backslash \overline{\Omega_1 \cup \Omega}$. Define a function φ_* in Λ by

$$\varphi_* = \begin{cases} \varphi_0 & \text{in}\, \Lambda \backslash \Omega_{\mathcal{S}}, \\ \varphi_1 & \text{in}\, \Omega_1, \\ \varphi & \text{in}\, \Gamma^1_{\rm sonic} \cup \Omega \cup \Gamma^2_{\rm sonic}, \\ \varphi_2 & \text{in}\, \Omega_2. \end{cases} \tag{4.56}$$

By Problem 4.6(ii)–(iii), φ_* is continuous in $\Lambda \backslash \Omega_{\mathcal{S}}$ and C^1 in $\overline{\Omega_{\mathcal{S}}}$. In particular, φ_* is C^1 across $\Gamma^1_{\rm sonic} \cup \Gamma^2_{\rm sonic}$. Moreover, using Problem 4.6(i)–(iii), we obtain that φ_* is a global entropy solution of equation (4.8) in Λ.

For the subsonic case $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$, region $\Omega_1 \cup \Gamma^1_{\rm sonic}$ in φ_* reduces to one point $\{O\}$; see Fig. 4.5. The corresponding function φ_* is a global entropy solution of equation (4.8) in Λ.

The first unsteady analysis of the steady supersonic weak shock solution as the long-time behavior of an unsteady flow is due to Elling-Liu [56], in which they succeeded in establishing a stability theorem for an important class of physical parameters determined by certain assumptions for the wedge angle $\theta_{\rm w}$ less than the sonic angle $\theta_{\rm w}^{\rm s} \in (0, \theta_{\rm w}^{\rm d})$ for potential flow.

Recently, in Bae-Chen-Feldman [6], we have removed the assumptions [56] and established the stability theorem for the steady (supersonic or transonic) weak shock solutions as the long-time asymptotics of the global Prandtl-Meyer reflection configurations for unsteady potential flow for all the admissible physical parameters even up to the detachment angle $\theta_{\rm w}^{\rm d}$ (beyond the sonic angle $\theta_{\rm w}^{\rm s} < \theta_{\rm w}^{\rm d}$).

To achieve this, we solve the free boundary problem (Problem 4.6), involving transonic shocks, for all the wedge angles $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$ by employing the techniques developed in Chen-Feldman [35], described in §4.1 above. Similar to Definition 4.8, we define admissible solutions in the present case:

Definition 4.11. *Let $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$. A function $\varphi \in C^{0,1}(\overline{\Lambda})$ is an admissible solution of* Problem 4.6 *if φ is a solution of* Problem 4.6 *extended to Λ by* (4.56) *and satisfies the following properties*:

(i) *The structure of solutions is as follows*:
- *If $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm s})$, then φ has the configuration shown on Fig.* 4.4 *such that $\Gamma_{\rm shock}$ is C^2 in its relative interior, $\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda \backslash (\overline{\mathcal{S}_{0,{\rm seg}}} \cup \overline{\Gamma_{\rm shock}} \cup \overline{\mathcal{S}_{1,{\rm seg}}}))$, and $\varphi \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega} \backslash (\overline{\Gamma^1_{\rm sonic}} \cup \overline{\Gamma^2_{\rm sonic}})) \cap C^3(\Omega)$.*

- *If* $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$, *then* φ *has the configuration shown on Fig.* 4.5 *such that* $\Gamma_{\rm shock}$ *is* C^2 *in its relative interior,* $\varphi \in C^{0,1}(\Lambda) \cap C^1(\Lambda\backslash(\Gamma_{\rm shock} \cup \overline{\mathcal{S}_{1,\rm seg}}))$, *and* $\varphi \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega}\backslash(\{O\} \cup \overline{\Gamma_{\rm sonic}^2})) \cap C^3(\Omega)$.

(ii) *Equation* (4.8) *is strictly elliptic in* $\overline{\Omega}\backslash(\overline{\Gamma_{\rm sonic}^1} \cup \overline{\Gamma_{\rm sonic}^2})$, *i.e.,* $|D\varphi| < c(|D\varphi|^2, \varphi)$ *in* $\overline{\Omega}\backslash(\overline{\Gamma_{\rm sonic}^1} \cup \overline{\Gamma_{\rm sonic}^2})$.

(iii) $\partial_{\boldsymbol\nu}\varphi_0 > \partial_{\boldsymbol\nu}\varphi > 0$ *on* $\Gamma_{\rm shock}$, *where* $\boldsymbol\nu$ *is the normal to* $\Gamma_{\rm shock}$, *pointing to the interior of* Ω.

(iv) *The inequalities hold*:
$$\max\{\varphi_1, \varphi_2\} \leqslant \varphi \leqslant \varphi_0 \qquad in\ \Omega, \tag{4.57}$$

(v) *The monotonicity properties hold:*
$$D(\varphi_0 - \varphi) \cdot \mathbf{e}_{\mathcal{S}_1} \geqslant 0, \quad D(\varphi_0 - \varphi) \cdot \mathbf{e}_{\mathcal{S}_0} \leqslant 0 \qquad in\ \Omega, \tag{4.58}$$
where $\mathbf{e}_{\mathcal{S}_0}$ *and* $\mathbf{e}_{\mathcal{S}_1}$ *are the unit vectors along lines* $\mathcal{S}_0$ *and* $\mathcal{S}_1$ *pointing to the positive* ξ_1*–direction, respectively.*

Similar to (4.45), the monotonicity properties in (4.58) imply that
$$D(\varphi_1 - \varphi) \cdot \mathbf{e} \leqslant 0 \qquad \text{in } \overline{\Omega} \ \text{ for all } \mathbf{e} \in \overline{Cone(-\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\mathcal{S}_0})}, \tag{4.59}$$
where $Cone(-\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\mathcal{S}_0}) = \{-a\,\mathbf{e}_{\mathcal{S}_1} + b\,\mathbf{e}_{\mathcal{S}_0} \ : \ a, b > 0\}$. We note that $\mathbf{e}_{\mathcal{S}_0}$ and $\mathbf{e}_{\mathcal{S}_1}$ are not parallel if $\theta_{\rm w} \neq 0$. Then we establish the following theorem.

Theorem 4.2. *Let* $\gamma > 1$ *and* $u_{10} > c_0$. *For any* $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$, *there exists a global entropy solution* φ *of* Problem 4.6 *such that the following regularity properties are satisfied for some* $\alpha \in (0,1)$:

(i) *If* $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm s})$, *the reflected shock* $\overline{\mathcal{S}_{0,\rm seg}} \cup \Gamma_{\rm shock} \cup \overline{\mathcal{S}_{1,\rm seg}}$ *is* $C^{2,\alpha}$*–smooth, and* $\varphi \in C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\overline{\Omega}\backslash(\overline{\Gamma_{\rm sonic}^1} \cup \overline{\Gamma_{\rm sonic}^2}))$*;*

(ii) *If* $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$, *the reflected shock* $\overline{\Gamma_{\rm shock}} \cup \overline{\mathcal{S}_{1,\rm seg}}$ *is* $C^{1,\alpha}$ *near* O *and* $C^{2,\alpha}$ *away from* O, *and* $\varphi \in C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\overline{\Omega}\backslash(\{O\} \cup \overline{\Gamma_{\rm sonic}^2}))$.

Moreover, in both cases, φ *is* $C^{1,1}$ *across the sonic arcs, and* $\Gamma_{\rm shock}$ *is* C^∞ *in its relative interior.*

Furthermore, φ *is an admissible solution in the sense of Definition* 4.11*, so* φ *satisfies further properties listed in Definition* 4.11*.*

We follow the argument described in §4.1 so that, for any small $\delta > 0$, we obtain the required uniform estimates of admissible solutions with wedge angles $\theta_{\rm w} \in [0, \theta_{\rm w}^{\rm d} - \delta]$. Using these estimates, we apply the Leray-Schauder degree theory to obtain the existence for each $\theta_{\rm w} \in [0, \theta_{\rm w}^{\rm d} - \delta]$ in the class of admissible solutions, starting from the unique normal solution for $\theta_{\rm w} = 0$. Since $\delta > 0$ is arbitrary, the existence of a global entropy solution for any $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$ can be established. More details can be found in Bae-Chen-Feldman [6]; see also Chen-Feldman [35].

The existence results in Bae-Chen-Feldman [6] indicate that the steady weak supersonic/transonic shock solutions are the asymptotic limits of the dynamic self-similar solutions, the Prandtl-Meyer reflection configurations, in the sense of (4.52) in Problem 4.5 for all $\theta_{\rm w} \in (0, \theta_{\rm w}^{\rm d})$ and all $\gamma > 1$.

On the other hand, it is shown in Elling [55] and Bae-Chen-Feldman [6] that, for each $\gamma > 1$, there is no self-similar *strong* Prandtl-Meyer reflection configuration for the unsteady potential flow in the class of admissible solutions. This means that the situation for the dynamic stability of the strong steady oblique shocks is more sensitive.

5. Convexity of Self-Similar Transonic Shocks and Free Boundaries

We now discuss some recent developments in the analysis of geometric properties of transonic shocks as free boundaries in the 2-D self-similar coordinates for compressible fluid flows. In Chen-Feldman-Xiang [36], we have developed a general framework for the analysis of the convexity of transonic shocks as free boundaries. For both applications discussed above, the von Neumann problem for shock reflection-diffraction in §4.1 and the Prandtl-Meyer problem for unsteady supersonic flow onto solid wedges in §4.2, the admissible solutions satisfy the conditions of this abstract framework, as shown in [36]. For simplicity, we present below the results on the convexity properties of transonic shocks for these two problems (without discussion on the abstract framework).

For the regular shock reflection-diffraction configurations, we recall that, for admissible solutions in the sense of Definition 4.8, the inequality in (4.45) is shown to be strict for any $\mathbf{e} \in Cone(\mathbf{e}_{\xi_2}, \mathbf{e}_{\mathcal{S}_1})$. From this, it is proved that, for admissible solutions, the shock is a graph in the coordinate system (S,T) with respect to basis $\{\mathbf{e}, \mathbf{e}^{\perp}\}$ for any unit vector $\mathbf{e} \in Cone(\mathbf{e}_{\xi_2}, \mathbf{e}_{\mathcal{S}_1})$, where $\mathbf{e}^{\perp}$ is the unit vector orthogonal to $\mathbf{e}$ and oriented so that $T_{P_1} > T_{P_2}$, and we have used notation (S_P, T_P) for the coordinates of point P. That is, there exists $f_{\mathbf{e}} \in C^{\infty}((T_{P_2}, T_{P_1})) \cap C^1([T_{P_2}, T_{P_1}])$ such that

$$\Gamma_{\text{shock}} = \{(S,T):\ S = f_{\mathbf{e}}(T),\ T_{P_2} < T < T_{P_1}\}, \qquad \Omega \cap \{T_{P_2} < T < T_{P_1}\} \subset \{S < f_{\mathbf{e}}(T)\}, \tag{5.1}$$

where we have used the notational convention (4.20) in the subsonic/sonic case.

Since the convexity or concavity of a shock as a graph depends on the orientation of the coordinate system and Ω will be shown to be a convex domain (corresponding to the concavity of $f_{\mathbf{e}}$ in (5.1)), we do not distinguish them and instead use the term *convexity* for either case below. Then we have

Theorem 5.1 (Convexity of transonic shocks for the regular shock reflection-diffraction configurations)**.** *If a solution of the von Neumann problem for shock reflection-diffraction is admissible in the sense of Definition* 4.8*, then its domain Ω is convex, and the shock curve Γ_{shock} is a strictly convex graph in the following sense*: *For any $\mathbf{e} \in Cone(\mathbf{e}_{\xi_2}, \mathbf{e}_{\mathcal{S}_1})$, the function $f_{\mathbf{e}}$ in* (5.1) *satisfies*

$$f_{\mathbf{e}}'' < 0 \qquad \text{on } (T_{P_2}, T_{P_1}).$$

That is, Γ_{shock} is uniformly convex on any closed subset of its relative interior.

Moreover, for the solution of Problem 4.3 *extended to Λ by* (4.21)*, with pseudo-potential $\varphi \in C^{0,1}(\Lambda)$ satisfying Definition* 4.8(i)–(iv)*, the shock is strictly convex if and only if Definition* 4.8(v) *holds.*

For the Prandtl-Meyer problem for unsteady supersonic flow onto solid wedges, the results are similar. We first note that, based on (4.59) (which is strict for $\mathbf{e} \in Cone(-\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\mathcal{S}_0})$) and the maximum principle, it is proved that, for admissible solutions in the sense of Definition 4.11, the shock is a graph in the coordinate system (S,T) with respect to basis $\{\mathbf{e}, \mathbf{e}^{\perp}\}$ for any unit vector $\mathbf{e} \in Cone(-\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\mathcal{S}_0})$, *i.e.*, (5.1) holds, with $f_{\mathbf{e}} \in C^{\infty}((T_{P_2}, T_{P_1})) \cap C^1([T_{P_2}, T_{P_1}])$, where we have used the notational convention $P_1 = P_0$ for the subsonic/sonic case $\theta_{\rm w} \in [\theta_{\rm w}^{\rm s}, \theta_{\rm w}^{\rm d})$.

Theorem 5.2 (Convexity of transonic shocks for the Prandtl-Meyer reflection configurations)**.** *If a solution of the Prandtl-Meyer problem is admissible in the sense of Definition* 4.11*, then its domain Ω is convex, and the shock curve Γ_{shock} is a strictly convex graph in the following sense*: *For any $\mathbf{e} \in Cone(-\mathbf{e}_{\mathcal{S}_1}, \mathbf{e}_{\mathcal{S}_0})$, the function $f_{\mathbf{e}}$ in* (5.1) *satisfies*

$$f_{\mathbf{e}}'' < 0 \qquad \text{on } (T_{P_2}, T_{P_1}).$$

That is, Γ_{shock} is uniformly convex on any closed subset of its relative interior.

Moreover, for the solution of Problem 4.6 *extended to Λ by* (4.56) (*with the appropriate modification for the subsonic/sonic case*) *with pseudo-potential $\varphi \in C^{0,1}(\Lambda)$ satisfying Definition* 4.11(i)–(iv)*, the shock is strictly convex if and only if Definition* 4.11(v) *holds.*

Theorems 5.1–5.2 indicate that the curvature of Γ_{shock}:

$$\kappa = -\frac{f_{\mathbf{e}}''(T)}{\big(1 + (f_{\mathbf{e}}'(T))^2\big)^{3/2}}$$

has a positive lower bound on any closed subset of (T_{P_2}, T_{P_1}).

Now we discuss the techniques developed in [36] by giving the main steps in the proofs of Theorems 5.1–5.2. While the argument in [36] is carried out in a more general setting, we focus here on the specific cases of the regular shock reflection-diffraction and Prandtl-Meyer reflection configurations; see [36] for the results in the more general setting and the detailed proofs.

For the von Neumann problem, define

$$\phi := \varphi - \varphi_1.$$

For the Prandtl-Meyer problem, define

$$\phi := \varphi - \varphi_0.$$

Then, in both cases, $\phi = 0$ on Γ_{shock}. From this, using Definition 4.8(iii) for the regular reflection-diffraction case and Definition 4.11(iii) for the Prandtl-Meyer reflection case, it follows that, in both problems, $\phi < 0$ in Ω near Γ_{shock}. Since Γ_{shock} is the zero level set of ϕ, the conclusion of Theorems 5.1–5.2 on the strict convexity of Γ_{shock} is equivalent to the following: $\phi_{\boldsymbol{\tau\tau}} > 0$ along Γ^0_{shock}, where Γ^0_{shock} is the relative interior of Γ_{shock}. Also, denote by Con the cone from (4.45) for the von Neuamnn problem and the cone from (4.59) for the Prandtl-Meyer problem.

First, we establish the relation between the strict convexity/concavity of a portion of the shock and the possibility for $\partial_{\mathbf{e}}\phi$, with $\mathbf{e} \in Con$, to attain its local minimum or maximum with respect to $\overline{\Omega}$ on that portion of the shock. More precisely, on a portion of "wrong" convexity on which $f''_{\mathbf{e}} \geqslant 0$ (equivalently, $\phi_{\boldsymbol{\tau\tau}} \leqslant 0$), $\phi_{\mathbf{e}}$ cannot attain its local minimum relative to $\overline{\Omega}$. Then, assuming that a portion of the free boundary has a "wrong" convexity $f''_{\mathbf{e}} > 0$, we show that $\phi_{\mathbf{e}}$ for $\mathbf{e} \in Con$ attains its local minimum relative to Γ_{shock} on the closure of that portion. As we discussed above, it cannot be a local minimum with respect to $\overline{\Omega}$. Starting from that, through a nonlocal argument, with the use of the maximum principle for equation (4.12), considered as a linear elliptic PDE for ϕ, in Ω, and the boundary conditions on various parts of $\partial\Omega$, we reach a contradiction, which implies that the shock is convex, possibly non-strictly, *i.e.*, $f''_{\mathbf{e}} \leqslant 0$ on (T_{P_2}, T_{P_1}), or equivalently, $\phi_{\boldsymbol{\tau\tau}} \geqslant 0$ on Γ_{shock}. Extending the previous argument with use of the real analyticity of Γ^0_{shock}, we improve the result to the locally uniform convexity as in Theorems 5.1–5.2.

Furthermore, with the convexity of reflected-diffracted transonic shocks, the uniqueness and stability of global regular shock reflection-diffraction configurations have also been established in the class of *admissible solutions*; see Chen-Feldman-Xiang [37] for the details.

The nonlinear method, ideas, techniques, and approaches that we have presented above for solving M-D transonic shocks and free boundary problems should be useful to analyze other longstanding and newly emerging problems. Examples of such problems include the unsolved M-D steady transonic shock problems for the full Euler equations (including steady detached shock problems), the unsolved M-D self-similar transonic shock problems (such as the 2-D Riemann problems and the conic body problems) for potential flow, as well as the longstanding open transonic shock problems for both the isentropic and the full Euler equations; also see Chen-Feldman [35]. Certainly, further new ideas, techniques, and methods are still required to be developed in order to solve these mathematically challenging and fundamentally important problems.

Acknowledgements. The research of Gui-Qiang G. Chen was supported in part by the UK Engineering and Physical Sciences Research Council Awards EP/L015811/1, EP/V008854, and EP/V051121/1, and the Royal Society–Wolfson Research Merit Award WM090014. The research of Mikhail Feldman was supported in part by the National Science Foundation under DMS-1764278 and DMS-2054689.

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