ON ASYMPTOTIC RIGIDITY AND CONTINUITY PROBLEMS IN NONLINEAR ELASTICITY ON MANIFOLDS AND HYPERSURFACES

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ABSTRACT. Intrinsic nonlinear elasticity deals with the deformations of elastic bodies as isometric immersions of Riemannian manifolds into the Euclidean spaces (see Ciarlet [9, 10]). In this paper, we study the rigidity and continuity properties of elastic bodies for the intrinsic approach to nonlinear elasticity. We first establish a geometric rigidity estimate for mappings from Riemannian manifolds to spheres (in the spirit of Friesecke–James–Müller [23]), which is the first result of this type for the non-Euclidean case as far as we know. Then we prove the asymptotic rigidity of elastic membranes under suitable geometric conditions. Finally, we provide a simplified geometric proof of the continuous dependence of deformations of elastic bodies on the Cauchy–Green tensors and second fundamental forms, which extends the Ciarlet–Mardare theorem in [18] to arbitrary dimensions and co-dimensions.

Résumé

L'élasticité non-linéaire intrinsèque considère les déformations de corps élastiques comme des immersions isométriques de variétés Riemanniennes dans l'espace Euclidien (voir Ciarlet [9,10]). Dans cet article, en suivant l'approche intrinsèque de l'élasticité non-linéaire, nous étudions les propriétés de rigidité et de continuité de corps élastiques. Premièrement, nous prouvons une estimée de rigidité géomètrique pour les applications de variétés Riemanniennes aux sphères (dans l'esprit de Friesecke–James–Müller [23]), qui est à notre connaissance le premier résultat de ce type dans le cas non-Euclidien. Ensuite, nous prouvons la rigidité asymptotique de membranes élastiques sous des hypothèses géométriques appropriées. Enfin, nous donnons une preuve géométrique simplifiée de la dépendence continue des déformations de corps élastiques par rapport à leur tenseur de Cauchy–Green et leur seconde forme fondamentale, ce qui étend le théorème de Ciarlet–Mardare [18] en dimensions et codimensions arbitraires.

1. INTRODUCTION

Nonlinear elasticity has long been an important subject in mathematics, physics, and engineering. One of the major objectives of elasticity theory is to determine the *deformation* undergone by the elastic bodies in response to external forces and boundary conditions (*cf.* [9, 16]). For an elastic body modelled as a 2- or 3-dimensional isometrically immersed submanifold of the Euclidean space $\mathbb{R}^3$, its deformation Φ is an isometric immersion. The *intrinsic approach* to nonlinear elasticity recasts the problems concerning deformation Φ to those concerning the *Cauchy–Green tensor*, *i.e.*, the Riemannian metric determined by Φ (see Antman [2], Ciarlet [9, 10], Ciarlet–Mardare [15, 17], and the references cited therein).

The intrinsic approach to nonlinear elasticity is based on the *fundamental theorem of surface theory*. It says that, under suitable topological and regularity conditions, deformation Φ of a surface M isometrically immersed in $\mathbb{R}^3$ can be recovered from the Riemannian metric g (*i.e.*, the Cauchy–Green tensor) and the *second fundamental form B*. Tensor B is determined by the

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manner in which M is immersed in the ambient space $\mathbb{R}^3$. In differential geometry, one says that g determines the *intrinsic geometry* of M, and B the *extrinsic geometry.* The literature on this topic is abundant: we refer to [9–12, 14] for the fundamental theorem of surface theory, to [30–32] for generalizations to surfaces with lower regularity (*i.e.*, $W^{2,p}$), and to [4, 38, 39] for generalizations to arbitrary dimensions and co-dimensions, among others.

Throughout this paper, *elastic bodies* are modelled by Riemannian manifolds isometrically immersed in the Euclidean spaces, which are said to have lower regularity if the associated isometric immersions have only $W^{2,p}$–regularity (*cf.* [4, 18, 31]). An *elastic membrane* is an elastic body with co-dimension 1, *i.e.*, a d-dimensional hypersurface isometrically immersed in $\mathbb{R}^{d+1}$. Case $d = 2$ is the situation of physical relevance, though we investigate for the general dimension case $d \geq 2$ mathematically. More precisely, we address three problems concerning the rigidity and continuity properties of elastic bodies.

We first investigate the Riemannian analogues of the geometric rigidity estimate due to Friesecke–James–Müller [23]: If the gradient of a vector field on a Euclidean Lipschitz domain is close to a Euclidean rigid motion on average, then it is close to a specific rigid motion; see Proposition 3.1 below for the precise statement. It is a quantitative version of the rigidity theorem due to Rešetnjak [35,36]. It remains an open question whether the analogous results for mappings between Riemannian manifolds remain valid; *cf.* Kupferman–Maor–Shachar [25]. In this paper, as a first step towards the above open question, we establish a geometric rigidity estimate from d-dimensional Riemannian manifolds to spheres of dimension $d \geq 2$. To achieve this, we exploit the Riemannian Piola identity established in [26] and several ideas for dealing with harmonic maps to tackle the nonlinearities, with crucial use of the special geometry of spheres. We remark in passing that other rigidity results (*e.g.*, infinitesimal rigidity and dimension reduction) have been established in the literature; see [24, 27–29] and the references cited therein.

We then consider a family of elastic membranes $\{(M^\varepsilon, g^\varepsilon)\}_{\varepsilon>0}$ in $\mathbb{R}^{d+1}$ via $W^{2,p}$–isometric immersions $\{\Phi^\varepsilon\}_{\varepsilon>0}$ with bi-Lipschitz homeomorphisms $F^\varepsilon : (M, g) \to (M^\varepsilon, g^\varepsilon)$ for a fixed Riemannian manifold (M, g). The *asymptotic problem* is whether it is possible to extract a subsequence of $\{\Phi^\varepsilon \circ F^\varepsilon\}_{\varepsilon>0}$ that converges weakly to an isometric immersion of (M, g) with the corresponding extrinsic geometries, *i.e.*, second fundamental forms, provided that F^ε are "asymptotically isometric" and Φ^ε are uniformly bounded in $W^{2,p}$. Our second result addresses the asymptotic problem by establishing the convergence of extrinsic geometries under natural geometric assumptions; also see the recent work by Alpern–Kupferman–Maor [1]. Our approach to the asymptotic problem is based on the weak continuity of the *Gauss–Codazzi equations* for isometric immersions. The Gauss–Codazzi equations are a first-order nonlinear system of partial differential equations (PDEs) in terms of the second fundamental form. Its weak continuity has been established in [4–7] by first observing its intrinsic div-curl structure and developing a global *compensated compactness* approach. Based on these developments, we answer the asymptotic rigidity problem in the affirmative, under the assumption that $F^\varepsilon_* g - g^\varepsilon$ converges to zero in suitable Sobolev norms where F^ε_* denotes the pushforward under F^ε.

Finally, we analyze the question of continuous dependence of the isometric immersion Φ in the $W^{2,p}$–norm with respect to the $W^{1,p}$–norm of metric g and the L^p–norm of the second fundamental form B. In Ciarlet–Mardare [18], the continuous dependence result for M as a simply-connected, open, bounded subsets of $\mathbb{R}^2$ with Lipschitz boundary was obtained. This has been further generalized to other Fréchet topologies in [13]. We provide here a simplified geometric proof, which also applies to isometric immersions of simply-connected Riemannian manifolds with arbitrary dimensions and co-dimensions. The proof is based on the arguments in our proof of the realisation theorem [4, Theorem 5.1], as well as the *Cartan structural equations* for isometric immersions (see [3]) and the analytic lemmas due to Mardare [30–32].

The rest of the paper is organized as follows: In §2, we introduce some notations and present some basics of differential geometry and non-Euclidean elasticity needed for subsequent developments. In §3, we establish the geometric rigidity estimate theorem, Theorem 3.2, for mappings from d-dimensional Riemannian manifolds to spheres of dimension $d \geq 2$, which extends the results of Friesecke–James–Müller in the Euclidean spaces in [23]. In §4, we establish the asymptotic rigidity of elastic bodies, Theorem 4.1. In §5, we provide a simplified proof of the continuous dependence of the deformations on the Cauchy–Green tensor and extrinsic geometry, Theorem 5.2.

2. Geometry of Elastic Bodies and Curvatures

In this section, we introduce some notations and present some basics of differential geometry and non-Euclidean elasticity needed for subsequent development.

2.1. **Riemannian Submanifold Theory.** Let (M, g) and $\{(M^\varepsilon, g^\varepsilon)\}_{\varepsilon>0}$ be Riemannian manifolds, *a.k.a.* elastic bodies. Let $F^\varepsilon : M \to M^\varepsilon$ be a bi-Lipschitz homeomorphism for each $\varepsilon > 0$. The pushforward $F^\varepsilon_* g$ defines another metric on M^ε:

$$F^\varepsilon_* g(X_\varepsilon, Y_\varepsilon) := g((F^\varepsilon)^* X_\varepsilon, (F^\varepsilon)^* Y_\varepsilon) \qquad \text{for } X_\varepsilon, Y_\varepsilon \in \Gamma(TM^\varepsilon),$$

where $(F^\varepsilon)^* X_\varepsilon := \mathrm{d}(F^\varepsilon)^{-1} X_\varepsilon$ is the pullback vector field of X_ε which is well-defined as F^ε is a bi-Lipschitz homeomorphism (see also §2.2 below), TM^ε is the tangent bundle of M^ε, and $\Gamma(TM^\varepsilon)$ is the space of tangential vector fields to M^ε (similar for M).

Let ∇ and ∇^ε be the Levi-Civita connections on (M, g) and $(M^\varepsilon, g^\varepsilon)$, respectively. Then $F^\varepsilon_* \nabla$ defines another affine connection on M^ε, known as the pushforward connection:

$$[F^\varepsilon_* \nabla]_{X_\varepsilon} Y_\varepsilon := \nabla_{(F^\varepsilon)^* X_\varepsilon}\left((F^\varepsilon)^* Y_\varepsilon\right) \qquad \text{for } X_\varepsilon, Y_\varepsilon \in \Gamma(TM^\varepsilon).$$

Unless F^ε is an isometry, $F^\varepsilon_* \nabla$ and g^ε are unrelated in general.

Consider an isometric immersion $\Phi^\varepsilon : M^\varepsilon \to \mathbb{R}^D$, where $\dim M^\varepsilon = \dim M = d$, and $\mathbb{R}^D$ is equipped with the Euclidean metric $\mathfrak{e}$. By definition, differential $\mathrm{d}\Phi^\varepsilon$ is everywhere injective, and $(\Phi^\varepsilon)^* \mathfrak{e} = g^\varepsilon$. It defines the second fundamental form:

$$B^\varepsilon : \Gamma(TM^\varepsilon) \times \Gamma(TM^\varepsilon) \to \Gamma((TM^\varepsilon)^\perp),$$

where $(TM^\varepsilon)^\perp$ is the normal bundle of Φ^ε:

$$(TM^\varepsilon)^\perp := T\mathbb{R}^D / T(\Phi^\varepsilon(M^\varepsilon)).$$

That is, $(TM^\varepsilon)^\perp$ is the quotient bundle of two tangent bundles — for each $x \in M^\varepsilon$, its fiber $(T_x M^\varepsilon)^\perp$ is the quotient vector space $T_{\Phi^\varepsilon(x)}\mathbb{R}^D / T_{\Phi^\varepsilon(x)}(\Phi^\varepsilon(M^\varepsilon))$. More precisely, if $\widetilde{\nabla}$ denotes the Levi-Civita connection on $\mathbb{R}^D$, then

$$B^\varepsilon(X_\varepsilon, Y_\varepsilon) := \widetilde{\nabla}_{\mathrm{d}\Phi^\varepsilon(X_\varepsilon)} \mathrm{d}\Phi^\varepsilon(Y_\varepsilon) - \nabla^\varepsilon_{X_\varepsilon} Y_\varepsilon \qquad \text{for } X_\varepsilon, Y_\varepsilon \in \Gamma(TM^\varepsilon). \tag{2.1}$$

The right-hand side of (2.1) can be understood as follows: we can locally extend $\mathrm{d}\Phi^\varepsilon(X_\varepsilon)$ and $\mathrm{d}\Phi^\varepsilon(Y_\varepsilon)$, which are tangential vector fields on $\Phi^\varepsilon \circ F^\varepsilon(M) \subset \mathbb{R}^D$, to vector fields X'_ε and Y'_ε on

$\mathbb{R}^D$, respectively, and then set

$$B^\varepsilon(X_\varepsilon, Y_\varepsilon) := \widetilde{\nabla}_{X'_\varepsilon} Y'_\varepsilon - \nabla^\varepsilon_{X'_\varepsilon} Y'_\varepsilon.$$

This is independent of the choice of extensions X'_ε and Y'_ε; see do Carmo [19, pp.126–127, §6].

A compatibility condition for the isometric immersion $\Phi^\varepsilon : M^\varepsilon \to \mathbb{R}^D$ is the Gauss–Codazzi equations (GCE). They arise from the compatibility of curvatures: the flat curvature of $\mathbb{R}^D$ decomposes along TM^ε and $(TM^\varepsilon)^\perp$. Denote by $R^\varepsilon : \Gamma((TM^\varepsilon)^{\otimes 4}) \to \mathbb{R}$ the Riemann curvature tensor of $(M^\varepsilon, g^\varepsilon, \nabla^\varepsilon)$. The Gauss equation reads

$$\mathfrak{e}(B^\varepsilon(X_\varepsilon, Z_\varepsilon), B^\varepsilon(Y_\varepsilon, W_\varepsilon)) - \mathfrak{e}(B^\varepsilon(X_\varepsilon, W_\varepsilon), B^\varepsilon(Y_\varepsilon, Z_\varepsilon)) = R^\varepsilon(X_\varepsilon, Y_\varepsilon, Z_\varepsilon, W_\varepsilon), \qquad (2.2)$$

and the Codazzi equation reads

$$\widetilde{\nabla}_{\mathrm{d}\Phi^\varepsilon(X_\varepsilon)} B^\varepsilon(Y_\varepsilon, Z_\varepsilon) = \widetilde{\nabla}_{\mathrm{d}\Phi^\varepsilon(Y_\varepsilon)} B^\varepsilon(X_\varepsilon, Z_\varepsilon), \qquad (2.3)$$

for $X_\varepsilon, Y_\varepsilon, Z_\varepsilon, W_\varepsilon \in \Gamma(TM^\varepsilon)$.

If $M^\varepsilon \hookrightarrow \mathbb{R}^D$ are not hypersurfaces, *i.e.*, the normal bundle $(TM^\varepsilon)^\perp$ has rank greater than 1, then there is an additional compatibility equation named after Ricci:

$$g^\varepsilon([S^\varepsilon_{\xi_\varepsilon}, S^\varepsilon_{\eta_\varepsilon}]X_\varepsilon, Y_\varepsilon) = R^{\varepsilon,\perp}(X_\varepsilon, Y_\varepsilon, \eta_\varepsilon, \xi_\varepsilon). \qquad (2.4)$$

Here S^ε is the shape operator of B^ε (which is equivalent to B^ε) and $R^{\varepsilon,\perp}$ is the Riemann curvature of bundle $(TM^\varepsilon)^\perp$ with respect to $(g^\varepsilon, \nabla^\varepsilon)$:

$$\begin{aligned} & R^{\varepsilon,\perp}(X_\varepsilon, Y_\varepsilon, \eta_\varepsilon, \xi_\varepsilon) \\ & := \mathfrak{e}(\nabla^{\varepsilon,\perp}_{\mathrm{d}\Phi^\varepsilon(X_\varepsilon)} \nabla^{\varepsilon,\perp}_{\mathrm{d}\Phi^\varepsilon(Y_\varepsilon)} \eta_\varepsilon - \nabla^{\varepsilon,\perp}_{\mathrm{d}\Phi^\varepsilon(Y_\varepsilon)} \nabla^{\varepsilon,\perp}_{\mathrm{d}\Phi^\varepsilon(X_\varepsilon)} \eta_\varepsilon + \nabla^{\varepsilon,\perp}_{[\mathrm{d}\Phi^\varepsilon(X_\varepsilon), \mathrm{d}\Phi^\varepsilon(Y_\varepsilon)]} \eta_\varepsilon,\, \xi_\varepsilon). \end{aligned}$$

Connection $\nabla^{\varepsilon,\perp}$ is the orthogonal projection of $\widetilde{\nabla}$ onto $(TM^\varepsilon)^\perp$. The vector fields $X_\varepsilon, Y_\varepsilon \in \Gamma(TM^\varepsilon)$ and $\eta_\varepsilon, \xi_\varepsilon \in \Gamma((TM^\varepsilon)^\perp)$ are arbitrary. In the weak regularity case that $\Phi^\varepsilon \in W^{2,p}$, solutions of (2.2)–(2.4) are understood in the sense of distributions, and the injectivity of $\mathrm{d}\Phi^\varepsilon$ is defined for a continuous representative in the Sobolev class.

Recognizing the underlying "div-curl structure" and developing the geometric compensated compactness argument, we have established a weak continuity theorem of the Gauss–Codazzi–Ricci equations (GCRE) (2.2)–(2.4) in [4] (also see [7]):

Proposition 2.1 ([4, Theorem 4.1]). *Let M be a Riemannian manifold with $W^{1,p} \cap L^\infty$–metric g for $p > 2$. Assume that the second fundamental forms and normal connections $\{(B^\varepsilon, \nabla^{\varepsilon,\perp})\}_{\varepsilon>0}$ are solutions of the Gauss–Codazzi–Ricci equations* (2.2)–(2.4) *and have a uniform L^p_{loc}–bound on the Riemannian manifold (M, g). Then, after passing to a subsequence if necessary, $(B^\varepsilon, \nabla^{\varepsilon,\perp})$ converges weakly in L^p_{loc} to a solution of the Gauss–Codazzi–Ricci equations* (2.2)–(2.4).

Its geometric analogue is the weak rigidity theorem of isometric immersions below. Throughout this paper, an immersion Φ is understood in the sense that $D\Phi$ has nonzero determinant *a.e.*. Note that $D\Phi$ is well-defined *a.e.* for $\Phi \in W^{2,p}$.

Proposition 2.2 ([4, Corollary 5.2]). *Let M be a d-dimensional simply-connected Riemannian manifold with $W^{1,p}$–metric g for $p > d$. Assume that $\{\Phi^\varepsilon\}_{\varepsilon>0}$ is a family of isometric immersions of (M, g) into a Euclidean space, uniformly bounded in $W^{2,p}_{\mathrm{loc}}$, whose second fundamental forms and normal connections are $(B^\varepsilon, \nabla^{\varepsilon,\perp})$. Then, after passing to a subsequence if necessary, Φ^ε converges weakly in $W^{2,p}_{\mathrm{loc}}$ to an isometric immersion Φ of (M, g) whose second fundamental form* *and normal connection are the weak $L^p_{\rm loc}$–limits of $(B^\varepsilon, \nabla^{\varepsilon,\perp})$, obeying the Gauss–Codazzi–Ricci equations* (2.2)–(2.4).

Propositions 2.1–2.2 serve as a starting point for §4–§5. In the proof of the asymptotic rigidity theorem, Theorem 4.1 below, we employ a variant of Proposition 2.1. Moreover, the continuous dependence theorem, Theorem 5.2, enables us to obtain a stronger version of Proposition 2.2.

We also introduce two notational conventions. First, the Einstein summation convention is used throughout: the repeated lower and upper indices are understood to be summed over. Second, for two tensor fields S and S' on the same manifold, $S \star S'$ denotes a generic linear combination of quadratic expressions for the components of S and S'.

2.2. **Vector Bundles.** Let $f : (M, g) \to (\widetilde{M}, \tilde{g})$ be a mapping between two Riemannian manifolds. Then its differential $\mathrm{d}f : TM \to T\widetilde{M}$ can be viewed as a section of the vector bundle $T^*M \otimes f^*T\widetilde{M} \cong \mathrm{Hom}(TM; f^*T\widetilde{M})$. Here and hereafter, for a vector bundle E over manifold $\widetilde{M}$, f^*E is the *pull-back bundle* over M.

In the above setting, we define as in [25, §1.1, p.369]:

$$\begin{aligned}\mathrm{SO}(g,\tilde{g}) := \Big\{\xi\,:TM \to T\widetilde{M} \text{ such that, for each } x \in M,\\ \xi_x : T_xM \to T_{\xi'(x)}\widetilde{M} \text{ is an orientation-preserving}\\ \text{isometry with respect to } (g,\tilde{g})\Big\},\end{aligned} \tag{2.5}$$

where $\xi' : M \to \widetilde{M}$ is the map between the base points associated to ξ. That is, for $\xi(F_1) = F_2$,

$$\xi'(\pi_M(F_1)) := \pi_{\widetilde{M}}(F_2), \tag{2.6}$$

where $F_1 \in TM$ and $F_2 \in T\widetilde{M}$ are fibers, and $\pi_M : TM \to M$ and $\pi_{\widetilde{M}} : T\widetilde{M} \to \widetilde{M}$ are natural projections onto the base points.

As a side remark, given $f : M \to \widetilde{M}$, $\mathrm{d}f \in \mathrm{SO}(g,\tilde{g})$ is systematically written as $\mathrm{d}f \in \mathrm{SO}(g; f^*\tilde{g})$ in [25]. However, we adhere to the notation, $\mathrm{SO}(g,\tilde{g})$, which will be more convenient for our purpose (in particular, for the formulation of Theorem 3.2).

In §3 below, we are interested in the distance between a given matrix field over M and $\mathrm{SO}(g,\tilde{g})$. More precisely, for $Q \in \Gamma(T^*M \otimes T\widetilde{M})$ (that is, for each $x \in M$, $Q(x)$ is a linear homomorphism, *i.e.*, a $(d \times d)$–matrix, from T_xM to $T_{Q'(x)}\widetilde{M}$), we consider the map on M:

$$\mathrm{dist}(Q, \mathrm{SO}(g,\tilde{g}))\,:\;\; x \longmapsto \mathrm{dist}(Q'(x), \mathrm{SO}(g,\tilde{g})|_{x;Q}), \tag{2.7}$$

where Q' is as in (2.6) and

$$\begin{aligned}\mathrm{SO}(g,\tilde{g})|_{x;Q} := \big\{S : T_xM \to T_{Q'(x)}\widetilde{M} \text{ is orientation-preserving}\\ \text{with } S^*(\tilde{g}|_{Q'(x)}) = g|_x\big\}.\end{aligned}$$

We need some more notations: Let E be a vector bundle over (M, g) with the bundle metric g^E. The natural metric on $\Gamma(T^*M \otimes E) \equiv \Omega^1(M, E)$ is the product metric of g and g^E, denoted by $g \otimes g^E$. For notational convenience, we sometimes write $[g \otimes g^E](\bullet,\bullet) \equiv \langle \bullet,\bullet\rangle_{g\otimes g^E}$, and similarly for the other metrics. Moreover, $\Gamma_0(E)$ denotes the space of sections $\sigma : M \to E$ such that $\sigma|_{\partial M} = 0$, and $\Gamma_{\rm c}(E)$ denotes the space of compactly supported sections. For a manifold M, we reserve symbol ∇^M for the Levi-Civita connection on M.

3. Geometric Rigidity for Mappings from Riemannian Manifolds to Spheres

A geometric rigidity estimate was first established in Friesecke–James–Müller [23, §4] as stated below. Roughly speaking, if the gradient of a vector field v is close to a Euclidean rigid motion on average, then it is indeed close to a specific rigid motion.

Proposition 3.1 ([23], Theorem 3.1 and the ensuing comment). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with $d \geq 2$, and let $1 < p < \infty$. Then there is $C = C(\Omega, p) > 0$ so that, for each $v \in W^{1,p}(\Omega, \mathbb{R}^d)$, there exists $\mathcal{R} \in \mathrm{SO}(d)$ such that*

$$\|\nabla v - \mathcal{R}\|_{L^p(\Omega)} \leq C \, \|\mathrm{dist}(\nabla v, \mathrm{SO}(d))\|_{L^p(\Omega)}, \tag{3.1}$$

where $\mathrm{SO}(d)$ is the group of all rotations about the origin in $\mathbb{R}^d$ under the operation of composition.

The above result can be viewed as a quantitative version of the Rešetnjak rigidity theorem in nonlinear elasticity in [35, 36]. In the recent work by Kupferman–Maor–Shachar [25], the Rešetnjak rigidity theorem was generalized to Lipschitz maps $f : M \to \widetilde{M}$ between Riemannian manifolds. It is raised as an open question in [25] whether the quantitative inequality (3.1) admits generalizations to mappings (not necessarily Lipschitz) between Riemannian manifolds.

The difficulty of the above question lies in the nonlinearity induced by the geometry of Riemannian manifolds. Indeed, the proof of Proposition 3.1 by Friesecke–James–Müller [23] relies essentially on the flat geometry of $\mathbb{R}^d$, especially the interior *a priori* estimates for harmonic functions on $\mathbb{R}^d$. In contrast, in the Riemannian setting, the role of harmonic functions is, roughly speaking, played by *harmonic maps*. The PDEs for harmonic maps are a quasilinear elliptic system, for which the desired regularity theory and *a priori* estimates are largely missing.

In this section, we present a variant of Proposition 3.1 for mappings $f : M \to \widetilde{M}$ under the restrictions both that $p = 2$ and that f satisfies the higher regularity assumptions. Throughout this section, M is a Riemannian manifold, and $\mathbb{S}^d = \{x = (x^1, \cdots, x^{d+1}) \in \mathbb{R}^{d+1} : |x| = 1\}$ is the unit d-sphere equipped with the *round metric can* parameterized by d-angles $(\phi_1, \cdots, \phi_d)$:

$$\mathrm{can} = \mathrm{d}\phi_1 \otimes \mathrm{d}\phi_1 + \sum_{i=2}^{d} \big(\prod_{j=1}^{i-1} \sin^2 \phi_j\big) \mathrm{d}\phi_i \otimes \mathrm{d}\phi_i,$$

so that $x \in \mathbb{S}^d$ is represented by

$$x^1 = \cos \phi_1, \quad x^j = \cos \phi_j \prod_{i=1}^{j-1} \sin \phi_i \ \text{ for } 2 \leq j \leq d, \quad x^{d+1} = \prod_{i=1}^{d} \sin \phi_i.$$

The key ingredients of the proof include the specific structures of $\mathbb{S}^d$-valued harmonic map equations (see [20, 22]), as well as the Riemannian Piola identity established by Kupferman–Maor–Shachar (see [25, Theorem 2] and [26]).

Here and hereafter in this section, an element $\mathcal{R}$ in $\mathrm{SO}(g, \mathrm{can})$ is said to be a rigid motion if it equals to a constant orthogonal $(d+1) \times (d+1)$-matrix restricted onto $\mathbb{S}^d \subset \mathbb{R}^{d+1}$.

Theorem 3.2. *Let $d \in \mathbb{Z}_{\geq 2}$ (i.e., $d \geq 2$ integers), and let $\mathbf{n}(d)$ be any number strictly greater than $\frac{d}{2}$. Let (M, g) be a Riemannian manifold that can be $C^{\mathbf{n}(d)+2}$-isometrically immersed into the round sphere $(\mathbb{S}^d, can)$, with possibly non-empty boundary ∂M. Then, for each $f \in W_0^{\mathbf{n}(d)+1,\infty}(M, g; \mathbb{S}^d, \mathrm{can})$, there exists a rigid motion $\mathcal{R}$ such that*

$$\|\mathrm{d}f - \mathcal{R}\|_{L^2(M,g;\, \mathbb{S}^d, \mathrm{can})} \leq C \|\mathrm{dist}(\mathrm{d}f, SO(g, \mathrm{can}))\|_{L^2(M,g)}, \tag{3.2}$$

where $C>0$ depends only on $\|f\|_{W^{\mathbf{n}(d)+1,\infty}(M,g;\,\mathbb{S}^d,\mathrm{can})}$ and the $C^{\mathbf{n}(d)+2}$-geometry of (M,g).

The assumption that $\|f\|_{W^{\mathbf{n}(d)+1,\infty}(M,g;\,\mathbb{S}^d,\mathrm{can})}<\infty$ is not sharp; in fact, our goal here is not to find the sharp conditions on f. For example, the assumption that $\|f\|_{W^{2,\infty}\cap W^{3,2}(M,g;\,\mathbb{S}^d,\mathrm{can})}<\infty$ will suffice when $d\in\{2,3\}$.

By the dilation: $can\mapsto\lambda^2 can$, $\lambda\neq 0$, the theorem holds when the target manifold is replaced by any round d-sphere, with constant C depending additionally on λ. The theorem is also invariant under the transform: $\mathrm{d}f\mapsto O\circ\mathrm{d}f$ for a rigid motion O, and inequality (3.2) holds with the same constant C.

In contrast to the Euclidean case (Proposition 3.1), we impose the higher integrability assumption on f and obtain estimate (3.2) for the non-Euclidean case in Theorem 3.2. It would be interesting to see whether the result can be improved by finding a uniform constant C (depending only on the geometry of M and $\mathbb{S}^d$).

To understand Theorem 3.2, we now present the following variant of an asymptotic rigidity result that was established for the more general case of nearly conformal maps by Rešetnjak [35]; see also [23, Corollary 3.3].

Corollary 3.3. *Let $d\in\mathbb{Z}_{\geq 2}$, and let $\mathbf{n}(d)$ be any number strictly greater than $\frac{d}{2}$. Let (M,g) be a Riemannian manifold that can be $C^{\mathbf{n}(d)+2}$-isometrically immersed in $(\mathbb{S}^d,\mathrm{can})$. Let $\{f_\varepsilon\}$ be a family of mappings from M to $\mathbb{S}^d$ with a uniform bound in $W_0^{\mathbf{n}(d)+1,\infty}(M,g;\,\mathbb{S}^d,\mathrm{can})$. If the L^2-norm of the distance between $\mathrm{d}f_\varepsilon$ and the group of orientation-preserving isometries from (M,g) to $(\mathbb{S}^d,can)$ shrinks to zero at a rate of $\mathcal{O}(\varepsilon)$, then there exists a particular rigid motion whose L^2-distance to $\mathrm{d}f_\varepsilon$ shrinks to zero also at a rate of $\mathcal{O}(\varepsilon)$, after passing to subsequences.*

PROOF. By assumption, $\{\mathrm{d}f_\varepsilon\}$ is uniformly bounded in $W^{\mathbf{n}(d),\infty}(M,g;\,\mathbb{S}^d,\mathrm{can})$. Hence, thanks to a standard compactness argument, it contains a subsequence $\{\mathrm{d}f_{\varepsilon_j}\}$ that converges to some $F\in W^{\mathbf{n}(d),\infty}(M,g;\,\mathbb{S}^d,\mathrm{can})$ strongly in L^2. Since $\|\mathrm{dist}(\mathrm{d}f_\varepsilon,\mathrm{SO}(g,\mathrm{can}))\|_{L^2}\to 0$ at rate $\mathcal{O}(\varepsilon)$, it follows that $F\in\mathrm{SO}(g,\mathrm{can})$ *a.e.*. On the other hand, by Theorem 3.2, for each j, there is a rigid motion $\mathcal{R}_{\varepsilon_j}$ with $\|\mathrm{d}f_{\varepsilon_j}-\mathcal{R}_{\varepsilon_j}\|_{L^2}\to 0$ at rate $\mathcal{O}(\varepsilon_j)$. Since the group of rigid motions is compact, there is a further subsequence $\{\varepsilon_{j_k}\}\subset\{\varepsilon_j\}$ such that $\|\mathcal{R}_{\varepsilon_{j_k}}-\mathcal{R}_0\|_{L^2}\to 0$ for a constant rigid motion $\mathcal{R}_0$. Then it follows from the uniqueness of limits that $F=\mathcal{R}_0$ *a.e.*.

To prove the geometric rigidity theorem, Theorem 3.2, we need the Piola identity in terms of the extrinsic geometry.

3.1. **The Riemannian Piola Identity.** Recall that, for a Euclidean mapping $f:\Omega\subset\mathbb{R}^d\to\mathbb{R}^d$, the classical Piola identity reads as

$$\mathrm{div}[\mathrm{cof}(\mathrm{d}f)]=0,$$

where $\mathrm{cof}(\mathrm{d}f)$ is the cofactor matrix of the $(d\times d)$-matrix $\mathrm{d}f$. Kupferman–Maor–Shachar [25, Theorems 1–2] generalized it to the mappings between Riemannian manifolds (also see [26]). Here we only collect some results in [25] related to our subsequent development.

Consider a smooth mapping $f:(M,g)\to(\widetilde{M},\tilde{g})$ between the Riemannian manifolds. We first derive some identities for general manifolds $\widetilde{M}$ and then specialize to $\widetilde{M}=\mathbb{S}^d$. As in §2.2, we can view $\mathrm{d}f\in\Gamma(T^*M\otimes f^*T\widetilde{M})$; equivalently, $\mathrm{d}f\in\Omega^1(M,f^*T\widetilde{M})$, that is, $\mathrm{d}f$ is a $f^*T\widetilde{M}$-valued differential 1-form over M. By a standard application of multi-linear algebra (see [25, §2.1]), $\mathrm{cof}(\mathrm{d}f)\in\Omega^1(M,f^*T\widetilde{M})$ may be defined intrinsically. Furthermore, define the co-derivative:

$$\delta_{\nabla^{f^*T\widetilde{M}}}:\Omega^1(M,f^*T\widetilde{M})\to\Omega^0(M,f^*T\widetilde{M}) \tag{3.3}$$

as the formal L^2–adjoint operator of the differential between $f^*T\widetilde{M}$-valued differential forms. Then

$$\delta_{\nabla^{f^*T\widetilde{M}}}\big(\mathrm{cof}(\mathrm{d}f)\big)=0, \tag{3.4}$$

which is known as the *Riemannian Piola identity.* In the geometric literature, it is more common to denote δ_∇ by ∇^* or $\nabla^\dagger$.

For our purpose, we need to express the Piola identity (3.4) in terms of the extrinsic geometry. More precisely, assume that $\iota:(\widetilde{M},\tilde{g})\to(\mathbb{R}^D,\mathfrak{e})$ is an isometric embedding into the Euclidean space. Such ι always exists for large enough D, due to Nash's embedding theorem in [33]. Let B be the second fundamental form of $\widetilde{M}$ with respect to ι, *i.e.*, $B:\Gamma(T\widetilde{M})\times\Gamma(T\widetilde{M})\to\Gamma((T\widetilde{M})^\perp)$, such that, for $u,v\in\Gamma(T\widetilde{M})$ and $\eta\in\Gamma((T\widetilde{M})^\perp)$,

$$\mathfrak{e}(B(u,v),\eta)=\mathfrak{e}(\nabla_v^{\widetilde{M}\times\mathbb{R}^D}\eta,\mathrm{d}\iota\circ u); \tag{3.5}$$

also see the equation below (2.8) in [25].

We remark in passing on the following notations in [25]: $\nabla^{M\times\mathbb{R}^D}$ and $\nabla^{\widetilde{M}\times\mathbb{R}^D}$ should be understood as $(\iota\circ f)^*\nabla^{\mathbb{R}^D}$ and $\iota^*\nabla^{\mathbb{R}^D}$, respectively; that is, they are the pullback affine connections, where $\nabla^{\mathbb{R}^D}$ denotes the Levi-Civita connection on $\mathbb{R}^D$ as before. Indeed, it can be checked that (3.5) is equivalent to the definition in §2.1 above. Correspondingly, in [25], symbols $M\times\mathbb{R}^D$ and $\widetilde{M}\times\mathbb{R}^D$ should be understood respectively as the pullback bundles $(\iota\circ f)^*T\mathbb{R}^D\equiv f^*(\iota^*T\mathbb{R}^D)$ and $\iota^*T\mathbb{R}^D$.

With the above preparations, we can now state the weak formulation of the Riemannian Piola identity as in [25, Theorem 2]. It is derived by splitting the tangent bundle $T\mathbb{R}^D$ into the tangential ($T\widetilde{M}$) and normal ($(T\widetilde{M})^\perp$) directions, and by applying the pullback operations under ι and f suitably:

$$\begin{aligned}&\int_M\big\langle(f^*\mathrm{d}\iota)\circ\mathrm{cof}(\mathrm{d}f),\,\nabla^{M\times\mathbb{R}^D}\zeta\big\rangle_{g\otimes\mathfrak{e}}\,\mathrm{d}V_g\\&=\int_M\big\langle\mathrm{tr}_g[(f^*B)(\mathrm{d}f,\mathrm{cof}(\mathrm{d}f))],\,\zeta\big\rangle_{\mathfrak{e}}\,\mathrm{d}V_g\qquad\text{for each }\zeta\in\Gamma_0(M\times\mathbb{R}^D).\end{aligned} \tag{3.6}$$

We understand the terms in (3.6) as follows: On the left-hand side,

(i) $f^*\mathrm{d}\iota:f^*T\widetilde{M}\to f^*(\iota^*T\mathbb{R}^D)\cong T(M\times\mathbb{R}^D)$,
(ii) $\mathrm{cof}(\mathrm{d}f):TM\to f^*T\widetilde{M}$,
(iii) $\nabla^{M\times\mathbb{R}^D}\zeta:TM\to T(M\times\mathbb{R}^D)$;

while, on the right-hand side,

(i) $f^*B:f^*T\widetilde{M}\times f^*T\widetilde{M}\to f^*((T\widetilde{M})^\perp)\subset T(M\times\mathbb{R}^D)$,
(ii) $f^*B(\mathrm{d}f,\mathrm{cof}(\mathrm{d}f)):TM\times TM\to f^*((T\widetilde{M})^\perp)$,
(iii) $\mathrm{tr}_g[f^*B(\mathrm{d}f,\mathrm{cof}(\mathrm{d}f))]\in\Gamma(f^*((T\widetilde{M})^\perp))\subset\Gamma[T(M\times\mathbb{R}^D)]$.

Thus, both sides of (3.6) are well-defined so that we can integrate (3.6) over M with respect to the volume measure $\mathrm{d}V_g$ induced by g. By the Sobolev embeddings and an elementary approximation argument, (3.6) is valid for any $f\in W^{1,p}(M;\widetilde{M})$ with $p\geq 2(d-1)$ ($p>2$ if $d=2$) and $\zeta\in(W_0^{1,2}\cap L^\infty)(M;\mathbb{R}^D)$.

3.2. **Proof of the Geometric Rigidity Theorem, Theorem 3.2.** Our proof follows the strategies in Friesecke–James–Müller [23]. Nevertheless, it is far from a straightforward adaptation; we have to apply some ideas for dealing with harmonic maps (see *e.g.*, Hélein [21] and Hélein–Wood [22]) to tackle with the nonlinearities originated from the non-Euclidean geometry.

Proof. We divide the proof into seven steps.

1. To start with, we derive an equation for $\mathrm{cof}(\mathrm{d}f)-\mathrm{d}f$. It follows from the weak formulation of the Piola identity (3.6) that, for each $\zeta \in \Gamma_0((\iota\circ f)^*T\mathbb{R}^D)$,

$$\begin{aligned}&\int_M \left\langle (f^*\mathrm{d}\iota)\circ[\mathrm{cof}(\mathrm{d}f)-\mathrm{d}f],\, \nabla^{(\iota\circ f)^*T\mathbb{R}^D}\zeta\right\rangle_{g\otimes\mathfrak{e}}\,\mathrm{d}V_g\\ &=\int_M \left\langle \mathrm{tr}_g[(f^*B)(\mathrm{d}f,\mathrm{cof}(\mathrm{d}f))],\,\zeta\right\rangle_{\mathfrak{e}}\,\mathrm{d}V_g\\ &\quad-\int_M \left\langle (f^*\mathrm{d}\iota)\circ \mathrm{d}f,\, \nabla^{(\iota\circ f)^*T\mathbb{R}^D}\zeta\right\rangle_{g\otimes\mathfrak{e}}\,\mathrm{d}V_g.\end{aligned}\tag{3.7}$$

As before, ∇^M and $\nabla^{\mathbb{R}^D}$ denote the Levi-Civita connections on M and $\mathbb{R}^D$, respectively. In view of the definitions of the co-derivative and $\langle\bullet,\bullet\rangle_{g\otimes\mathfrak{e}}$ and the fact that

$$(f^*\mathrm{d}\iota)\circ\mathrm{d}f=\mathrm{d}(\iota\circ f)\in\Gamma_0((\iota\circ f)^*T\mathbb{R}^D),$$

we have

$$\begin{aligned}&\int_M \left\langle (f^*\mathrm{d}\iota)\circ \mathrm{d}f,\, \nabla^{(\iota\circ f)^*T\mathbb{R}^D}\zeta\right\rangle_{g\otimes\mathfrak{e}}\,\mathrm{d}V_g\\ &=\int_M \left\langle \mathrm{d}(\iota\circ f),\, \nabla^{(\iota\circ f)^*T\mathbb{R}^D}\zeta\right\rangle_{g\otimes\mathfrak{e}}\,\mathrm{d}V_g\\ &=\int_M \left\langle \delta_{\nabla^{(\iota\circ f)^*T\mathbb{R}^D}}\circ\nabla^{(\iota\circ f)^*T\mathbb{R}^D}(\iota\circ f),\,\zeta\right\rangle_{g\otimes\mathfrak{e}}\,\mathrm{d}V_g\\ &=\int_M \mathrm{tr}_g\left\{\left\langle \delta_{\nabla^{(\iota\circ f)^*T\mathbb{R}^D}}\circ\nabla^{(\iota\circ f)^*T\mathbb{R}^D}(\iota\circ f),\,\zeta\right\rangle_{\mathfrak{e}}\right\}\mathrm{d}V_g.\end{aligned}$$

Recall that, for a vector bundle E over M, the Bochner Laplacian

$$\mathrm{tr}_g(\delta_{\nabla^E}\circ\nabla^E)\qquad\text{over }\Gamma(E)\equiv\Omega^0(M,E)$$

coincides with the Hodge Laplacian, *i.e.*, the negative of the Laplace–Beltrami operator Δ_g. Then, identifying $(\iota\circ f)^*T\mathbb{R}^D$ with the trivial bundle $M\times\mathbb{R}^D$ and hence viewing both $\iota\circ f$ and ζ as mappings from M to $\mathbb{R}^D$, we may infer from (3.7) that

$$\begin{aligned}&\int_M \left\langle (f^*\mathrm{d}\iota)\circ[\mathrm{cof}(\mathrm{d}f)-\mathrm{d}f],\, \nabla^{M\times\mathbb{R}^D}\zeta\right\rangle_{g\otimes\mathfrak{e}}\,\mathrm{d}V_g\\ &=\int_M \left\langle \mathrm{tr}_g[(f^*B)(\mathrm{d}f,\mathrm{cof}(\mathrm{d}f))],\,\zeta\right\rangle_{\mathfrak{e}}\,\mathrm{d}V_g+\int_M\left\langle\Delta_g(\iota\circ f),\,\zeta\right\rangle_{\mathfrak{e}}\,\mathrm{d}V_g.\end{aligned}\tag{3.8}$$

Now, using the identity

$$\delta_{\nabla^{M\times\mathbb{R}^D}}\circ(f^*\mathrm{d}\iota)=\delta_{\nabla^{f^*T\widetilde{M}}}$$

and the arbitrariness of the test differential form $\zeta\in\Gamma_0((\iota\circ f)^*T\mathbb{R}^D)$, we have

$$\begin{aligned}&\delta_{\nabla^{f^*T\widetilde{M}}}[\mathrm{cof}\,(\mathrm{d}f)-\mathrm{d}f]\\ &=\Delta_g(\iota\circ f)+\mathrm{tr}_g[(f^*B)(\mathrm{d}f,\mathrm{d}f)]+\mathrm{tr}_g[(f^*B)(\mathrm{d}f,\mathrm{cof}\,(\mathrm{d}f)-\mathrm{d}f)].\end{aligned}\tag{3.9}$$

We will carry out the estimates for f based on this equation.

2. We decompose $f = w + z$ with

$$\begin{cases} \Delta_g(\iota\circ z) = \delta_{\nabla^{f^*T\widetilde{M}}}[\mathrm{cof}\,(\mathrm{d}f) - \mathrm{d}f] \\ \qquad\qquad\quad -\mathrm{tr}_g[(f^*B)(\mathrm{d}f, \mathrm{cof}\,(\mathrm{d}f) - \mathrm{d}f)] \qquad \text{on } M, \\ z|_{\partial M} = 0 \end{cases} \tag{3.10}$$

and

$$\begin{cases} \Delta_g(\iota\circ w) = -\mathrm{tr}_g[(f^*B)(\mathrm{d}f, \mathrm{d}f)] \qquad \text{on } M, \\ w|_{\partial M} = 0. \end{cases} \tag{3.11}$$

This is possible, since $f|_{\partial M} = 0$ by assumption. We set

$$\epsilon := \|\mathrm{dist}(\mathrm{d}f, \mathrm{SO}(g,\tilde{g}))\|_{L^2(M,g)}$$

and assume $\epsilon \leq 1$ without loss of generality. Since $\mathrm{d}f \in \mathrm{SO}(g,\tilde{g})$ implies that $\mathrm{cof}\,(\mathrm{d}f) = \mathrm{d}f$ *a.e.* (*cf.* [25, Corollary 4]), there is a uniform constant $C = C(M,g)$ such that

$$|\mathrm{cof}\,(\mathrm{d}f) - \mathrm{d}f|^2 \leq C\,\mathrm{dist}^2(\mathrm{d}f, \mathrm{SO}(g,\tilde{g})) \qquad a.e..$$

It follows that

$$\|\mathrm{cof}\,(\mathrm{d}f) - \mathrm{d}f\|_{L^2(M,g;\widetilde{M},\tilde{g})} \leq C\,\epsilon. \tag{3.12}$$

Notice that, if $B \equiv 0$, *i.e.*, $\widetilde{M}$ is Euclidean, then w is taken to be a harmonic function. This agrees with the case in Friesecke–James–Müller [23].

3. We first derive the estimate for z. Multiplying $\iota\circ z$ to (3.10) and recalling that ι is an isometric embedding, we obtain

$$\begin{aligned} \int_M |\mathrm{d}z|^2\,\mathrm{d}V_g \leq{}& \int_M \big|\langle \nabla^{f^*T\widetilde{M}}(\iota\circ z),\, \mathrm{cof}(\mathrm{d}f) - \mathrm{d}f\rangle\big|\,\mathrm{d}V_g \\ &+ \int_M \big|z\,\mathrm{tr}_g[(f^*B)(\mathrm{d}f, \mathrm{cof}\,(\mathrm{d}f) - \mathrm{d}f)]\big|\,\mathrm{d}V_g, \end{aligned} \tag{3.13}$$

where the norm of $\mathrm{d}z$ is taken with respect to both metrics g and $\tilde{g}$:

$$|\mathrm{d}z| := \sqrt{\langle \mathrm{d}z, \mathrm{d}z\rangle_{g\otimes\tilde{g}}}. \tag{3.14}$$

From now on, we focus on the case: $(\widetilde{M},\tilde{g}) = (\mathbb{S}^d, \mathrm{can})$. In this case, we have

$$(f^*B)(\mathrm{d}f, \mathrm{cof}(\mathrm{d}f) - \mathrm{d}f) = f\langle \mathrm{d}f, \mathrm{cof}(\mathrm{d}f) - \mathrm{d}f\rangle_{g\otimes\mathrm{can}}; \tag{3.15}$$

see [22, (26)]. Also, $|f| = 1$. Thus, for some $C = C(M, g, \|f\|_{W^{1,\infty}(M,g)})$,

$$\int_M |\mathrm{d}z|^2\,\mathrm{d}V_g \leq C\Big\{\int_M |\mathrm{d}z|\big|\mathrm{cof}\,(\mathrm{d}f) - \mathrm{d}f\big|\,\mathrm{d}V_g + \int_M |z|\big|\mathrm{cof}\,(\mathrm{d}f) - \mathrm{d}f\big|\,\mathrm{d}V_g\Big\}.$$

Here and hereafter, for notational convenience, we write

$$\|f\|_{W^{1,\infty}(M,g)} \equiv \|f\|_{W^{1,\infty}(M,g;\mathbb{S}^d,\mathrm{can})} \equiv \|f\|_{W^{1,\infty}(M,g;\mathbb{R}^{d+1},\mathfrak{e})}.$$

The latter equality holds since $(\mathbb{S}^d, \mathrm{can})$ is isometrically embedded in $(\mathbb{R}^{d+1}, \mathfrak{e})$. Since M is bounded and $z|_{\partial M} = 0$, we deduce from the Cauchy–Schwarz inequality, $|z| = |\iota\circ z|_{g\otimes\mathfrak{e}}$, and the Poincaré inequality (for functions on manifold M, *i.e.*, 0-forms):

$$\int_M |\phi|^2\,\mathrm{d}V_g \leq c_0^2 \int_M |\mathrm{d}\phi|^2\,\mathrm{d}V_g \qquad \text{for each } \phi \in W_0^{1,2}(M;\mathbb{R}^D)$$

with $c_0 = c_0(M,g)$ that

$$\int_M |\mathrm{d}z|^2 \,\mathrm{d}V_g \leq C(M, g, \|f\|_{W^{1,\infty}(M,g)})\,\epsilon^2. \tag{3.16}$$

4. Next, we bound w from (3.11). Using $f = w+z$, $f \in \widetilde{M} = \mathbb{S}^d$ *a.e.* on M, and the specific geometric properties of $\mathbb{S}^d$ (namely identity (3.15)), we have

$$\int_M \left(|\mathrm{d}w|^2 + |w|^2|\mathrm{d}f|^2\right) \mathrm{d}V_g = -\int_M wz|\mathrm{d}f|^2 \,\mathrm{d}V_g.$$

Since $\|f\|_{W^{1,\infty}(M,g)} \leq C_1$, using the Poincaré inequality, we can estimate

$$\begin{aligned}\int_M \left(|\mathrm{d}w|^2 + |w|^2|\mathrm{d}f|^2\right) \mathrm{d}V_g &\leq C_1^2 \int_M |w||z| \,\mathrm{d}V_g \\ &\leq \frac{1}{2}\int_M |\mathrm{d}w|^2 \,\mathrm{d}V_g + 8C_1^4 c_0^4 \int_M |\mathrm{d}z|^2 \,\mathrm{d}V_g.\end{aligned}$$

Thus, together with estimate (3.13) for $\mathrm{d}z$, we conclude

$$\int_M |\mathrm{d}w|^2 \,\mathrm{d}V_g \leq C\big(M, g, \|f\|_{W^{1,\infty}(M,g)}\big)\,\epsilon^2. \tag{3.17}$$

In view of the Poincaré inequality again, (3.16)–(3.17) can be summarized as

$$\|(w,z)\|_{W^{1,2}(M,g)} \leq C(M,g)\,\epsilon. \tag{3.18}$$

The $W^{2,2}$-estimates for w can be derived directly from (3.11):

$$\begin{aligned}\|w\|_{W^{2,2}(M,g;\,\widetilde{M},\tilde{g})} &\leq C_1\big(\|\Delta_g(\iota\circ w)\|_{L^2(M,g)} + \|w\|_{L^2(M,g)}\big) \\ &\leq C_1\big(\big\|f|\mathrm{d}f|^2\big\|_{L^2(M,g)} + \|w\|_{L^2(M,g)}\big) \\ &\leq C(M, g, \|f\|_{W^{1,\infty}(M,g;\,\widetilde{M},\tilde{g})})\,\epsilon. \end{aligned} \tag{3.19}$$

In the first inequality above, we have used the Calderón–Zygmund estimates on M (see, *e.g.* Wang [40]) so that C_1 depends only on the C^3–geometry of (M,g). The second inequality follows from (3.11). The final inequality holds by the assumption that $\|f\|_{W^{1,\infty}(M,g)} \leq C$ and (3.18).

5. To proceed, let us estimate up to the $W^{4,2}$-norm of w. We *claim* that

$$\|w\|_{W^{4,2}(M,g)} \leq C(M, g, \|f\|_{W^{2,\infty}\cap W^{3,2}(M,g)})\,\epsilon. \tag{3.20}$$

To see this, taking two derivatives to the right-hand side of (3.11) and then expressing it in local coordinates for the sake of clarity, we have

$$\int_M |\nabla^2\Delta_g w|^2 \,\mathrm{d}V_g = \int_M \left|g^{k\ell}\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^\ell}\Big(fg^{pq}\mathrm{can}_{ij}\frac{\partial f^i}{\partial x^p}\frac{\partial f^j}{\partial x^q}\Big)\right|^2 \mathrm{d}V_g.$$

By the chain rule, we obtain

$$\begin{aligned}&\int_M |\nabla^2\Delta_g w|^2 \,\mathrm{d}V_g \\ &\leq C\int_M \Big(|\nabla^2 f|^2|\nabla f|^4 + |f|^2|\nabla^2 f|^4 + |f|^2|\nabla^3 f|^2|\nabla f|^2\Big)\,\mathrm{d}V_g,\end{aligned} \tag{3.21}$$

where C depends only on the C^3-geometry of M. A similar computation yields

$$\int_M |\nabla\Delta_g w|^2 \,\mathrm{d}V_g \leq C\int_M \left(|\nabla f|^3 + |f||\nabla f||\nabla^2 f|\right)^2 \mathrm{d}V_g. \tag{3.22}$$

By the $W^{1,2}$-estimate (3.18) for f and the assumption that $\|f\|_{W^{2,\infty}\cap W^{3,2}(M,g)} < \infty$, the right-hand sides of (3.21)–(3.22) are both controlled by ϵ^2. Therefore, we may conclude the *claim* from the Calderón–Zygmund estimates on M.

6. We can now conclude the proof for $d \in \{2,3\}$. Indeed, utilizing the Sobolev embedding: $W^{4,2} \hookrightarrow W^{2,\infty}$, it follows from (3.21)–(3.22) and (3.19) that

$$\|w\|_{W^{2,\infty}(M,g)} \leq C\,\epsilon,$$

where $C = C(M, g, \|f\|_{W^{2,\infty}\cap W^{3,2}(M,g)})$. Thus, by the fundamental theorem of calculus, there is a $d \times d$ matrix $\mathcal{R}$ such that

$$\sup_{x\in M} |\mathrm{d}w(x) - \mathcal{R}| \leq C\epsilon, \tag{3.23}$$

where the norm $|\bullet|$ is defined as in (3.14) above, with $\mathrm{d}w - \mathcal{R}$ in lieu of $\mathrm{d}z$. On the other hand, by the definition of ϵ and (3.16), we have

$$\begin{aligned}&\left\|\mathrm{dist}(\mathrm{d}w, \mathrm{SO}(g, \mathrm{can}))\right\|_{L^2(M,g)}\\ &\leq \left\|\mathrm{dist}(\mathrm{d}f, \mathrm{SO}(g, \mathrm{can}))\right\|_{L^2(M,g)} + \left\|\mathrm{dist}(\mathrm{d}z, \mathrm{SO}(g, \mathrm{can}))\right\|_{L^2(M,g)} \leq C\epsilon.\end{aligned}$$

Furthermore, the left-hand side of (3.23) is invariant under the actions by isometries of $\mathbb{R}^{d+1}$ so that we can take $\mathcal{R} \in \mathrm{SO}(g, \mathrm{can})$ in (3.23); that is, $\mathcal{R}$ is a rigid motion. Thanks to (3.16) and (3.23), the proof is now complete for $d \in \{2,3\}$.

7. Finally, we now explain how to modify the arguments in Steps 5–6 above to deal with the general case: $d = \dim M$. Observe first that Step 6 remains valid, once we can estimate w in some norm that is stronger than $\|\bullet\|_{W^{2,\infty}(M,g)}$. It suffices to obtain a bound of form $\int_M |\nabla^{\mathbf{n}(d)}\Delta_g w|^2 \,\mathrm{d}V_g < \infty$, which is equivalent to estimating $\|w\|_{W^{\mathbf{n}(d)+2,2}(M,g)}$ by the Calderón–Zygmund estimates, since the following Sobolev continuous embedding holds when $\mathbf{n}(d) > \frac{d}{2}$:

$$W^{\mathbf{n}(d)+2,2}(M,g) \hookrightarrow W^{2,\infty}(M,g).$$

Now we are left to bound $\int_M |\nabla^{\mathbf{n}(d)}\Delta_g w|^2 \,\mathrm{d}V_g$. Notice that

$$\int_M |\nabla^{\mathbf{n}(d)}\Delta_g w|^2 \,\mathrm{d}V_g = \int_M \left|\nabla^{\mathbf{n}(d)}\left(f g^{pq}\mathrm{can}_{ij}\frac{\partial f^i}{\partial x^p}\frac{\partial f^j}{\partial x^q}\right)\right|^2 \mathrm{d}V_g.$$

The integrand on the right-hand side contains, by the Leibniz rule, at most $\mathbf{n}(d)+1$ derivatives in both f and the metric. Since (M,g) has finite volume as a submanifold of $(\mathbb{S}^d, \mathrm{can})$, the right-hand side is controlled by $\|f\|_{W^{\mathbf{n}(d)+1,\infty}(M,g)}$ and the $C^{\mathbf{n}(d)+2}$-geometry of (M,g). This completes the proof.

3.3. **Remarks.** Concerning the geometric rigidity theorem (Theorem 3.2) and its proof, we have the following remarks in order:

1. Starting from Step 3 in the above proof of Theorem 3.2, our argument deviates from Step 2 in the proof of [23, Proposition 3.4]. In [23], map w is harmonic, so an interior bound for $\|\nabla^2 w\|_{L^2}$ follows easily; see (3.16) therein. However, in our case, $\Delta_g(\iota\circ w) = -\mathrm{tr}_g[(f^*B)(\mathrm{d}f, \mathrm{d}f)]$ in M; see (3.11). We proceed as in (3.19) by the estimate: $\|w\|_{W^{2,2}} \lesssim \|\Delta_g(\iota\circ w)\|_{L^2(M,g)} + \|w\|_{L^2(M,g)}$. This is where the additional regularity assumptions on f come into play.

2. We have imposed an extra condition $f = 0$ (hence $w = 0$) on ∂M in Theorem 3.2, while no boundary condition for f is needed in Proposition 3.1. This is due to the fact that w is not a *harmonic function* in our case. Instead, PDE (3.11) for w is a "perturbed" harmonic map equation: when $w = f$, it is exactly the harmonic map equation. Therefore, the weighted Hessian estimate as in [23, (3.26)] is no longer valid in general, thus preventing us from adapting the arguments in [23] to derive the estimates up to the boundary, unless $f = 0$ on ∂M.

3. It would be interesting to further generalize Theorem 3.2 to general ambient manifolds $(\widetilde{M}, \tilde{g})$. In this case, it is natural to define that $\mathcal{R} \in \mathrm{SO}(g, \tilde{g})$ is a rigid motion if and only if $\mathcal{R}$ is an isometry of the tangent bundle of the ambient manifold with respect to $\tilde{g}$. We may view $\mathcal{R} \in \mathrm{SO}(g, \tilde{g})$ by identifying $\mathcal{R}$ with its restriction on TM. Such a definition entails that, for most of the ambient manifolds $(\widetilde{M}, \tilde{g})$, $\mathrm{SO}(g, \tilde{g}) = \{\mathrm{Id}_M\}$, whence (3.2) trivially holds with $C = 1$. The nontrivial case that $\mathrm{SO}(g, \tilde{g})$ has more than one element occurs when, roughly speaking, $(T\widetilde{M}, \tilde{g})$ has a large degree of symmetries, *e.g.*, when $(\widetilde{M}, \tilde{g})$ is a space form (*i.e.*, a Riemannian manifold with constant sectional curvature).

4. In the case of a general ambient manifold $(\widetilde{M}, \tilde{g})$, the main difficulty lies in obtaining a smallness estimate for $\|\mathrm{d}w\|_{L^2(M,g;\widetilde{M},\tilde{g})}$. Indeed, when multiplying w to both sides of Eq. (3.11) and integrating over M, we obtain

$$\int_M |\mathrm{d}w|^2 \,\mathrm{d}V_g = -\int_M \left\{ g_{\gamma\delta} g^{ij} w^\delta \big(\widetilde{\Gamma}^\gamma_{\alpha\beta} \circ f\big)(\mathrm{d}f)^\alpha_i (\mathrm{d}f)^\beta_j \right\} \mathrm{d}V_g$$

in local coordinates with Einstein's summation convention, where $\widetilde{\Gamma}^\gamma_{\alpha\beta} \in C^2(\widetilde{M}, \tilde{g})$ are the Christoffel symbols on $\widetilde{M}$ of the Levi-Civita connection for $\tilde{g}$; see [20, 22]. There is no apparent structure for the integrand on the right-hand side that leads to $\|\mathrm{d}w\|_{L^2(M,g;\widetilde{M},\tilde{g})} \lesssim \epsilon$ as in (3.17).

5. In the recent preprint [1], Alpern–Kupferman–Maor further extended their asymptotic rigidity theorem in [25] to hypersurfaces in space forms. As remarked in [1], a quantitative result in the form of a geometric rigidity theorem may help to extend the asymptotic rigidity theorem therein to arbitrary ambient Riemannian manifolds. It would be interesting to further investigate such extensions.

4. Asymptotic Rigidity of Elastic Membranes

In this section, we formulate and prove an asymptotic rigidity theorem for elastic membranes, *i.e.*, immersed hypersurfaces M^ε in the Euclidean space $(\mathbb{R}^{d+1}, \mathfrak{e})$, in which $\mathfrak{e}$ is the Euclidean metric as before. The theorem addresses the convergence of both deformations and extrinsic geometries.

Theorem 4.1. *Let M^ε be a sequence of d-dimensional Riemannian manifolds and $p > d \geq 2$. Let $\Phi^\varepsilon : (M^\varepsilon, g^\varepsilon) \hookrightarrow (\mathbb{R}^{d+1}, \mathfrak{e})$ be $W^{2,p}_{\mathrm{loc}}$-isometric immersions. Let M be a Riemannian manifold with $W^{1,p}_{\mathrm{loc}}$-metric g so that there are bi-Lipschitz homeomorphisms $F^\varepsilon : M \to M^\varepsilon$ whose bi-Lipschitz constants are uniformly bounded on compact sets such that*

$$(F^\varepsilon)^*[g^\varepsilon] - g \longrightarrow 0 \qquad \text{in } W^{1,p'}_{\mathrm{loc}}(M) \quad \text{for } p' = \frac{p}{p-1} \in [1, 2), \tag{4.1}$$

and that $\Phi^\varepsilon \circ F^\varepsilon$ are uniformly bounded in $W^{2,p}_{\mathrm{loc}}(M, g; \mathbb{R}^{d+1}, \mathfrak{e})$. Then, after passing to a subsequence if necessary, $\Phi^\varepsilon \circ F^\varepsilon$ converges weakly in $W^{2,p}_{\mathrm{loc}}$ to an isometric immersion $\overline{\Phi} : (M, g) \to (\mathbb{R}^{d+1}, \mathfrak{e})$ so that its second fundamental form is a weak L^p_{loc}-limit of the second fundamental forms of Φ^ε, obeying the Gauss–Codazzi equations (2.2)–(2.3).

PROOF. Throughout the proof, unless otherwise specified, all the Sobolev spaces $\mathbf{X} = W^{k,p}, W^{-k,p}, L^p, \ldots$ are understood as $\mathbf{X}(M, g; \mathbb{R}^{d+1}, \mathfrak{e})$. The proof is divided into six steps.

1. We first fix some notations. Denote by ∇ the Levi-Civita connection on (M, g) and by ∇^ε the Levi-Civita connection on $(M^\varepsilon, g^\varepsilon)$. By assumption, $\Phi^\varepsilon : (M^\varepsilon, g^\varepsilon) \to (\mathbb{R}^{d+1}, \mathfrak{e})$ are isometric immersions and F^ε are Lipschitz homeomorphisms. Then

$$\Phi^\varepsilon \circ F^\varepsilon : \big(M, (F^\varepsilon)^* g^\varepsilon\big) \longrightarrow (\mathbb{R}^{d+1}, \mathfrak{e})$$

are also isometric immersions. We write $\underline{\nu^\varepsilon}$ for the outward unit normal vector field of $\Phi^\varepsilon \circ F^\varepsilon$; in the proof below, we view it as defined on $\mathbb{R}^{d+1}$.

We also set

$$\widehat{g^\varepsilon} := (F^\varepsilon)^* g^\varepsilon, \quad \widehat{\nabla^\varepsilon} := (F^\varepsilon)^* \nabla^\varepsilon \qquad \text{on } M, \tag{4.2}$$

and

$$\widehat{B^\varepsilon}(X, Y) := -\mathfrak{e}(\widetilde{\nabla}_{(\Phi^\varepsilon \circ F^\varepsilon)_*(X)} \underline{\nu^\varepsilon}, (\Phi^\varepsilon \circ F^\varepsilon)_*(Y)) \quad \text{for } X, Y \in \Gamma(TM), \tag{4.3}$$

which are well-defined since F^ε are bi-Lipschitz homeomorphisms. For notational convenience, write

$$X_\varepsilon := (F^\varepsilon)_* X = \mathrm{d}F^\varepsilon(X) \in \Gamma(TM^\varepsilon) \qquad \text{for each } X \in \Gamma(TM).$$

That is, X and X_ε are F^ε*-related.* The same convention applies to $Y_\varepsilon, Z_\varepsilon, W_\varepsilon \ldots$. Also, by the locality of the theorem, without loss of generality, we may take M to be compact, so that the subscripts "$_{\rm loc}$" are dropped from now on.

The tensor $\widehat{B^\varepsilon} : \Gamma(TM) \times \Gamma(TM) \to \mathbb{R}$ defined in (4.3) is the second fundamental form of the isometric immersion $\Phi^\varepsilon \circ F^\varepsilon : (M, \widehat{g^\varepsilon}) \to (\mathbb{R}^{d+1}, \mathfrak{e})$. Note that our convention here for the second fundamental form is slightly different from that in §2: we view $\widehat{B^\varepsilon}$ as a $\mathbb{R}$-valued function, which is more convenient for the case of codimension one.

The definition in (4.3) is motivated by the following observations:

- If $\Phi^\varepsilon \circ F^\varepsilon$ is smooth for each ε, then, by definition (see §2),
$$\widehat{B^\varepsilon}(X, Y) = \mathfrak{e}(\widetilde{\nabla}_{(\Phi^\varepsilon \circ F^\varepsilon)_*(X)} (\Phi^\varepsilon \circ F^\varepsilon)_*(Y), \underline{\nu^\varepsilon}). \tag{4.4}$$
The right-hand side of (4.4) can be understood as follows: we can locally extend $(\Phi^\varepsilon \circ F^\varepsilon)_* X$ and $(\Phi^\varepsilon \circ F^\varepsilon)_* Y$, which are vector fields on $\Phi^\varepsilon \circ F^\varepsilon(M) \subset \mathbb{R}^{d+1}$, to vector fields X'_ε and Y'_ε on $\mathbb{R}^{d+1}$, respectively, and then set $\widehat{B^\varepsilon}(X, Y) = \mathfrak{e}\big(\widetilde{\nabla}_{X'_\varepsilon} Y'_\varepsilon, \underline{\nu^\varepsilon}\big)$. This is independent of the choice of extensions X'_ε and Y'_ε; see do Carmo [19, pp.126–127, §6].

- Since $\Phi^\varepsilon \circ F^\varepsilon$ is uniformly bounded in $W^{2,p}$, the right-hand side of (4.4) is a product of two $W^{1,p}$-terms and one L^p-term. Since $p > d = \dim M$, we see that $\widehat{B^\varepsilon}$ is uniformly bounded in L^p (see Step 5 below for details).

- The above remarks justify the computation below:
$$\begin{aligned}\widehat{B^\varepsilon}(X, Y) = &(\Phi^\varepsilon \circ F^\varepsilon)_*(X)\{\mathfrak{e}((\Phi^\varepsilon \circ F^\varepsilon)_*(Y), \underline{\nu^\varepsilon})\}\\ &- \mathfrak{e}(\widetilde{\nabla}_{(\Phi^\varepsilon \circ F^\varepsilon)_*(X)} \underline{\nu^\varepsilon}, (\Phi^\varepsilon \circ F^\varepsilon)_*(Y)).\end{aligned}$$
Since $(\Phi^\varepsilon \circ F^\varepsilon)_*(Y)$ is tangential and $\underline{\nu^\varepsilon}$ is vertical, both with respect to the immersed hypersurface $\Phi^\varepsilon \circ F^\varepsilon(M)$, the first term on the right-hand side vanishes. Therefore, we arrive at (4.4).

In addition, by passing to subsequences if necessary, we have

$$\Phi^\varepsilon \circ F^\varepsilon \rightharpoonup \overline{\Phi} \qquad \text{in } W^{2,p}.$$

Note that $D^\top \overline{\Phi} \cdot D\overline{\Phi} \in W^{1,p}$ for $p > d$, thanks to the Sobolev–Morrey embedding. Since $\widehat{g^\varepsilon} := (F^\varepsilon)^* g^\varepsilon = (\Phi^\varepsilon \circ F^\varepsilon)^* \mathfrak{e}$ converges in $W^{1,p'}$ to g on M, by a compactness argument and the uniqueness of limits, we conclude that $g = D^\top \overline{\Phi} \cdot D\overline{\Phi}$ as $W^{1,p}$-tensor fields. Thus, $\overline{\Phi}$ is an isometry. Moreover, $|\det(D\overline{\Phi})| = \sqrt{\det g} > 0$ in the *a.e.* sense, so $\overline{\Phi}$ is also an immersion.

2. In order to prove the weak convergence of second fundamental forms, we will prove that $\widehat{B^\varepsilon}$ is an *approximate solution* for the Gauss–Codazzi equation on (M, g), which will be made precise in Lemma 4.2 below. In what follows, we let $X, Y, Z, W \in \Gamma(TM)$ be arbitrary, and consider F^ε-related vector fields $X_\varepsilon, Y_\varepsilon, Z_\varepsilon, W_\varepsilon \in \Gamma(TM^\varepsilon)$. We identify them (without relabelling) with extensions X', Y', Z', W', $X'_\varepsilon, Y'_\varepsilon, Z'_\varepsilon, W'_\varepsilon \in \Gamma(T\mathbb{R}^{d+1})$ as in Step 1.

As before, $\Phi^\varepsilon \circ F^\varepsilon : (M, \widehat{g^\varepsilon}) \longrightarrow (\mathbb{R}^{d+1}, \mathfrak{e})$ are isometric immersions for $\widehat{g^\varepsilon} \equiv (F^\varepsilon)^* g^\varepsilon$. Then $\widehat{B^\varepsilon}$ defined in (4.3) satisfies the Gauss equation:

$$\widehat{B^\varepsilon}(X, Z)\widehat{B^\varepsilon}(Y, W) - \widehat{B^\varepsilon}(X, W)\widehat{B^\varepsilon}(Y, Z) = \widehat{R^\varepsilon}(X, Y, Z, W),$$

where $\widehat{R^\varepsilon}$ is the Riemann curvature tensor of $(M, \widehat{g^\varepsilon}, \widehat{\nabla^\varepsilon})$ and we have used the fact that $\mathfrak{e}(\underline{\nu^\varepsilon}, \underline{\nu^\varepsilon}) = 1$.

Denoting by R^ε the Riemann curvature tensor of $(M^\varepsilon, g^\varepsilon, \nabla^\varepsilon)$, we have

$$\widehat{R^\varepsilon} = (F^\varepsilon)^* R^\varepsilon.$$

This follows from the tensorial property of the Riemann curvature: For each $P \in M$,

$$\begin{aligned}
&[(F^\varepsilon)^* R^\varepsilon](X, Y, Z, W)|_P \\
&= R^\varepsilon(X_\varepsilon, Y_\varepsilon, Z_\varepsilon, W_\varepsilon)|_{F^\varepsilon(P)} \\
&= g^\varepsilon(\nabla^\varepsilon_{X_\varepsilon}\nabla^\varepsilon_{Y_\varepsilon} Z_\varepsilon - \nabla^\varepsilon_{Y_\varepsilon}\nabla^\varepsilon_{X_\varepsilon} Z_\varepsilon - \nabla^\varepsilon_{[X_\varepsilon, Y_\varepsilon]} Z_\varepsilon,\, W_\varepsilon)\big|_{F^\varepsilon(P)} \\
&= [(F^\varepsilon)^* g^\varepsilon](F^{\varepsilon *}\{\nabla^\varepsilon_{X_\varepsilon}\nabla^\varepsilon_{Y_\varepsilon} Z_\varepsilon - \nabla^\varepsilon_{Y_\varepsilon}\nabla^\varepsilon_{X_\varepsilon} Z_\varepsilon - \nabla^\varepsilon_{[X_\varepsilon, Y_\varepsilon]} Z_\varepsilon\},\, (F^\varepsilon)^* W_\varepsilon)\big|_P \\
&= \widehat{g^\varepsilon}(\widehat{\nabla^\varepsilon}_X\{(F^\varepsilon)^*(\nabla^\varepsilon_{Y_\varepsilon} Z_\varepsilon)\} - \widehat{\nabla^\varepsilon}_Y\{(F^\varepsilon)^*(\nabla^\varepsilon_{X_\varepsilon} Z_\varepsilon)\} - \widehat{\nabla^\varepsilon}_{F^{\varepsilon *}[X_\varepsilon, Y_\varepsilon]}(F^{\varepsilon *} Z_\varepsilon),\, W)\big|_P \\
&= \widehat{g^\varepsilon}(\widehat{\nabla^\varepsilon}_X \widehat{\nabla^\varepsilon}_Y Z - \widehat{\nabla^\varepsilon}_Y \widehat{\nabla^\varepsilon}_X Z - \widehat{\nabla^\varepsilon}_{[X,Y]} Z,\, W)\big|_P \\
&=: \widehat{R^\varepsilon}(X, Y, Z, W)|_P.
\end{aligned}$$

For the penultimate equality, we have used the Lie bracket identity $f_*([X, Y]) = [f_* X, f_* Y]$.

From the computations above, we infer that

$$\widehat{B^\varepsilon}(X, Z)\widehat{B^\varepsilon}(Y, W) - \widehat{B^\varepsilon}(X, W)\widehat{B^\varepsilon}(Y, Z) = R(X, Y, Z, W) + [\text{Error}]_1, \tag{4.5}$$

where R denotes the Riemann curvature tensor of (M, g, ∇), and

$$[\text{Error}]_1 := \{\widehat{R^\varepsilon} - R\}(X, Y, Z, W).$$

On the other hand, the Codazzi equation for $\widehat{B^\varepsilon}$ reads as

$$\widetilde{\nabla}_X\big(\widehat{B^\varepsilon}(Y, Z)\underline{\nu^\varepsilon}\big) - \widetilde{\nabla}_Y\big(\widehat{B^\varepsilon}(X, Z)\underline{\nu^\varepsilon}\big) = 0. \tag{4.6}$$

3. To proceed, we invoke the weak continuity of the Gauss–Codazzi equations (2.2)–(2.3) established in [4]. We employ a variant of Proposition 2.1, which deals with the weak continuity of "approximate solutions" for the Gauss–Codazzi equations (2.2)–(2.3) (see [4, Remark 4.1]).

Lemma 4.2. *Let (M, g) be a d-dimensional Riemannian manifold with $W^{1,p}_{\mathrm{loc}} \cap L^{\infty}_{\mathrm{loc}}$-metric for $p > 2$. Suppose that the tensor fields*

$$\mathrm{II}^{\varepsilon} : \Gamma(TM^{\varepsilon}) \times \Gamma(TM^{\varepsilon}) \to \Gamma((TM^{\varepsilon})^{\perp})$$

both have a uniform bound in L^p_{loc} and are "approximate solutions" of the Gauss–Codazzi equations (2.2)–(2.3) *in the following sense:*

$$\mathfrak{e}(\mathrm{II}^{\varepsilon}(u,w), \mathrm{II}^{\varepsilon}(v,z)) - \mathfrak{e}(\mathrm{II}^{\varepsilon}(u,z), \mathrm{II}^{\varepsilon}(v,w)) - R(u,v,w,z) = \mathcal{O}^{(1)}_{\varepsilon}, \tag{4.7}$$

$$\widetilde{\nabla}_v \mathrm{II}^{\varepsilon}(u,w) - \widetilde{\nabla}_u \mathrm{II}^{\varepsilon}(v,w) = \mathcal{O}^{(2)}_{\varepsilon}, \tag{4.8}$$

for arbitrary fixed $u, v, w, z \in \Gamma(TM)$, where $\mathcal{O}^{(1)}_{\varepsilon}, \mathcal{O}^{(2)}_{\varepsilon} \to 0$ in $W^{-1,r}_{\mathrm{loc}}$ as $\varepsilon \to 0$ for some $r > 1$, and R is the Riemann curvature tensor of g and the Levi-Civita connection ∇ on M. Then $\{\mathrm{II}^{\varepsilon}\}$ converges weakly in L^p_{loc} to a weak solution of the Gauss–Codazzi equations (2.2)–(2.3).

We will apply this lemma to $\mathrm{II}^{\varepsilon} := \widehat{B^{\varepsilon}}\underline{\nu^{\varepsilon}}$, with (u, v, w, z) replaced by (X, Y, Z, W). By (4.5) and (4.6), we see that $[\mathrm{Error}]_1 = \mathcal{O}^{(1)}_{\varepsilon}$ and $\mathcal{O}^{(2)}_{\varepsilon} \equiv 0$ here. It suffices to show that $\mathcal{O}^{(1)}_{\varepsilon} \to 0$ in $W^{-1,r}$ for some $r > 1$ as $\varepsilon \to 0$ and that $\mathrm{II}^{\varepsilon}$ is uniformly bounded in L^p.

4. Using the definition of the Riemann curvature tensor, $\mathcal{O}^{(1)}_{\varepsilon} = \{\widehat{R^{\varepsilon}} - R\}(X, Y, Z, W)$ can be expressed as

$$\begin{aligned}
\mathcal{O}^{(1)}_{\varepsilon} &= (\widehat{g^{\varepsilon}} - g)(\widehat{\nabla^{\varepsilon}}_X \widehat{\nabla^{\varepsilon}}_Y Z,\, W) + g(\{\widehat{\nabla^{\varepsilon}}_X - \nabla_X\}\nabla_Y Z,\, W) \\
&\quad + g(\nabla_X\{\widehat{\nabla^{\varepsilon}}_Y - \nabla_Y\}Z,\, W) - (\widehat{g^{\varepsilon}} - g)(\widehat{\nabla^{\varepsilon}}_Y \widehat{\nabla^{\varepsilon}}_X Z,\, W) \\
&\quad - g(\{\widehat{\nabla^{\varepsilon}}_Y - \nabla_Y\}\nabla_X Z,\, W) - g(\nabla_Y\{\widehat{\nabla^{\varepsilon}}_X - \nabla_X\}Z,\, W) \\
&\quad - (\widehat{g^{\varepsilon}} - g)(\widehat{\nabla^{\varepsilon}}_{[X,Y]} Z,\, W) - g(\{\widehat{\nabla^{\varepsilon}}_{[X,Y]} - \nabla_{[X,Y]}\}Z,\, W) \\
&=: \sum_{\ell=1}^{8} J_{\ell}.
\end{aligned} \tag{4.9}$$

We first observe that

$$\widehat{\Gamma^{\varepsilon}} \quad \text{is uniformly bounded in } L^p. \tag{4.10}$$

Indeed, since $\Phi^{\varepsilon} \circ F^{\varepsilon}$ is uniformly bounded in $W^{2,p}$, it follows that $\widehat{g^{\varepsilon}}$ is uniformly bounded in $W^{1,p}$. This can be seen from explicit computations: in the local orthonormal coordinates $\{e_{\alpha}\}$ on (M, g),

$$\widehat{g^{\varepsilon}}_{\alpha\beta} = D(\Phi^{\varepsilon} \circ F^{\varepsilon})^{\alpha}_{\gamma} D(\Phi^{\varepsilon} \circ F^{\varepsilon})^{\gamma}_{\beta}.$$

The Sobolev–Morrey embedding then implies the uniform boundedness of $\{\widehat{g^{\varepsilon}}\}$ in C^0. Thus, (4.10) follows from the continuity of the matrix inverse and the formula:

$$\widehat{\Gamma^{\varepsilon}}^{\alpha}_{\beta\gamma} = \frac{1}{2}(g^{\varepsilon})^{\alpha\delta}\{\partial_{\beta} g^{\varepsilon}_{\delta\gamma} + \partial_{\gamma} g^{\varepsilon}_{\delta\beta} - \partial_{\delta} g^{\varepsilon}_{\gamma\beta}\}.$$

In what follows, we justify the convergence $J_{\ell} \to 0$ term by term.

<u>J_1 and J_4.</u> A direct computation yields that

$$\begin{aligned}
\widehat{\nabla^{\varepsilon}}_X \widehat{\nabla^{\varepsilon}}_Y Z &= X^{\delta}\partial_{\delta}\big(Y^{\alpha}\partial_{\alpha}Z^{\beta}\big)\partial_{\beta} + X^{\delta}Y^{\alpha}(\partial_{\alpha}Z^{\beta})\widehat{\Gamma^{\varepsilon}}^{\gamma}_{\delta\beta}\partial_{\gamma} \\
&\quad + X^{\delta}Y^{\alpha}Z^{\beta}\widehat{\Gamma^{\varepsilon}}^{\gamma}_{\alpha\beta}\widehat{\Gamma^{\varepsilon}}^{\kappa}_{\delta\gamma}\partial_{\kappa} + X^{\delta}\partial_{\delta}\big(Y^{\alpha}Z^{\beta}\widehat{\Gamma^{\varepsilon}}^{\gamma}_{\alpha\beta}\big)\partial_{\gamma},
\end{aligned}$$

where the last term of the right-hand side is most singular. We deduce from this equality that $\widehat{\nabla^{\varepsilon}}_X \widehat{\nabla^{\varepsilon}}_Y Z$ is uniformly bounded in $W^{-1,p}$ for any given $X, Y, Z \in \Gamma(TM)$. Thanks to the assumption of the strong convergence $\widehat{g^\varepsilon} \to g$ in $W^{1,p'}$, we then obtain that

$$J_1 \to 0 \qquad \text{in } W^{-1,r} \text{ as } \varepsilon \to 0$$

for any $r < d' = \frac{d}{d-1}$. The argument for J_4 is analogous.

$\underline{J_2 \text{ and } J_5.}$ This is more direct. Since $\nabla_Y Z \in L^p$,

$$(\widehat{\nabla^\varepsilon}_X - \nabla_X)\nabla_Y Z = X^i (\nabla_Y Z)^j (\widehat{\Gamma^\varepsilon} - \Gamma)^k_{ij}\partial_k \rightharpoonup 0 \qquad \text{in } L^{p/2}.$$

Using $g \in W^{1,p} \hookrightarrow C^0$, the Rellich lemma, and the Sobolev embedding,

$$J_2 \to 0 \qquad \text{in } W^{-1,r} \text{ as } \varepsilon \to 0$$

for all $r \in (1,\infty)$ if $d \leq \frac{p}{2}$, and for $r \in (1, \frac{dp}{2d-p})$ if $d > \frac{p}{2}$. The argument for J_5 is analogous.

$\underline{J_3 \text{ and } J_6.}$ It suffices to argue for J_3. As before, after passing to subsequences, $\widehat{\nabla^\varepsilon}_Y Z - \nabla_Y Z \rightharpoonup 0$ in L^p so that $\nabla_X(\widehat{\nabla^\varepsilon}_Y Z - \nabla_Y Z) \rightharpoonup 0$ in $W^{-1,p}$. Since $g \in W^{1,p}$, it follows from the Sobolev embedding that

$$J_3 \to 0 \qquad \text{in } W^{-1,r} \text{ as } \varepsilon \to 0 \text{ for any } r < d' = \frac{d}{d-1}.$$

$\underline{J_8.}$ Note that $[X,Y] = \nabla_X Y - \nabla_Y X$ as ∇ is the Levi-Civita connection on (M,g), hence $[X,Y] \in L^p$. It follows that

$$\{\widehat{\nabla^\varepsilon}_{[X,Y]} - \nabla_{[X,Y]}\}Z = [X,Y]^i Z^j (\widehat{\Gamma^\varepsilon} - \Gamma)^k_{ij}\partial_k \rightharpoonup 0 \qquad \text{in } L^{p/2}.$$

The rest of the argument is similar to that for J_2 and J_5 above.

$\underline{J_7.}$ Finally, note as before that $\widehat{\nabla^\varepsilon}_{[X,Y]}Z$ is uniformly bounded in $L^{p/2}$. In J_7, this term is paired with $\widehat{g^\varepsilon} - g$, which converges strongly to zero in $W^{1,p'}$. By Sobolev embedding, for $p > d$, we have the compact embedding: $W^{1,p'} \hookrightarrow L^{\frac{dp'}{d-p'}} = L^{\frac{dp}{d(p-1)-p}}$, which is greater than or equal to the Hölder conjugate $\frac{p}{p-2} = (\frac{p}{2})'$. Thus, $J_7 \to 0$ in L^1. By Sobolev embedding again,

$$J_7 \to 0 \qquad \text{in } W^{-1,r} \text{ as } \varepsilon \to 0 \text{ for any } r < d' = \frac{d}{d-1}.$$

To summarize, by the arguments above, we conclude that $\mathcal{O}^{(1)}_\varepsilon \to 0$ in $W^{-1,r}$ for some $r > 1$, where $\mathcal{O}^{(1)}_\varepsilon$ is defined in (4.9).

5. Now we show that $\mathrm{II}^\varepsilon = \widehat{B^\varepsilon}\underline{\nu^\varepsilon}$ is uniformly bounded in L^p. Recall from (4.4) that

$$\mathrm{II}^\varepsilon(X,Y) = -\mathfrak{e}(\widetilde{\nabla}_{(\Phi^\varepsilon\circ F^\varepsilon)_* X}\underline{\nu^\varepsilon},\, (\Phi^\varepsilon \circ F^\varepsilon)_* Y)\underline{\nu^\varepsilon}$$

for any vector fields $X, Y \in \Gamma(TM)$. For ease of notations, we write here $\underline{X_\varepsilon} := (\Phi^\varepsilon \circ F^\varepsilon)_* X$ and $\underline{Y_\varepsilon} := (\Phi^\varepsilon \circ F^\varepsilon)_* Y$. By assumptions, they are uniformly bounded in $W^{1,p}$. Thus, for the Euclidean coordinate frame $\{\partial_i\}$ on $\mathbb{R}^{d+1}$, we may express

$$\mathrm{II}^\varepsilon(X,Y) = -(\underline{X_\varepsilon})^i \big[\partial_i(\underline{\nu^\varepsilon})^j\big](\underline{Y_\varepsilon})^j \underline{\nu^\varepsilon}. \tag{4.11}$$

Assume for the moment that $\underline{\nu^\varepsilon}$ is uniformly bounded in L^p. Then the right-hand side of (4.11) is a product of three terms uniformly bounded in $W^{1,p}$ and another term uniformly bounded in L^p. By assumption $p > d = \dim M$, the Sobolev–Morrey embedding shows that $W^{1,p} \hookrightarrow C^0$. Hence, II^ε is uniformly bounded in L^p.

To justify the above claim, we make use of the following expression for $\underline{\nu^\varepsilon}$:

$$\underline{\nu^\varepsilon} = \frac{\mathrm{d}(\Phi^\varepsilon \circ F^\varepsilon)^1 \wedge \cdots \wedge \mathrm{d}(\Phi^\varepsilon \circ F^\varepsilon)^d}{\|\mathrm{d}(\Phi^\varepsilon \circ F^\varepsilon)^1 \wedge \cdots \wedge \mathrm{d}(\Phi^\varepsilon \circ F^\varepsilon)^d\|} \llcorner \mathrm{d}V_{\mathfrak{e}}, \tag{4.12}$$

where $dV_{\mathfrak{e}}$ is the Euclidean volume form on $(\mathbb{R}^{d+1}, \mathfrak{e})$, $\llcorner$ is the interior multiplication, and $\| \bullet \|$ is the mass norm for vector fields. This equality is understood modulo obvious isomorphisms between the tangent and cotangent bundles. Note that the denominator $\|d(\Phi^\varepsilon \circ F^\varepsilon)^1 \wedge \cdots \wedge d(\Phi^\varepsilon \circ F^\varepsilon)^d\| \geq c_0$ for a uniform constant $c_0 > 0$, since Φ^ε are isometric immersions (hence non-degenerate) and F^ε have uniformly bounded bi-Lipschitz constants. Moreover, $d(\Phi^\varepsilon \circ F^\varepsilon)^1 \wedge \cdots \wedge d(\Phi^\varepsilon \circ F^\varepsilon)^d$ is a wedge product of d differential 1-forms uniformly bounded in $W^{1,p}$. Again, since $W^{1,p} \hookrightarrow C^0$ for $p > d$, we deduce that $\underline{\nu^\varepsilon}$ is uniformly bounded in L^p.

6. Finally, by Lemma 4.2 and Steps 4–5, we conclude that $\widehat{B^\varepsilon} \underline{\nu^\varepsilon}$ converge weakly in L^p to a weak solution of the Gauss–Codazzi equations (2.2)–(2.3). This completes the proof.

5. Continuous Dependence of the Deformation on the Cauchy–Green Tensor and Second Fundamental Form

In Ciarlet-Mardare [18], the following continuous dependence of the deformation was established:

Theorem 5.1 ([18, Theorem 6.1]). *Let Ω be a simply-connected, open, bounded subset of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Assume that Ω lies locally on the same side of $\partial\Omega$, and $p > 2$. Define the spaces:*

$$\mathbb{T}(\Omega) := \left\{ (g, B) \, : \, \begin{array}{l} g \in W^{1,p}(\Omega; \mathrm{Sym}^+_{2\times 2} T^*\Omega), B \in L^p(\Omega; \mathrm{Sym}_{2\times 2} T^*\Omega) \\ (g, B) \text{ satisfies GCE } (2.2)\text{–}(2.3) \end{array} \right\}$$

and

$$\mathbf{V}(\Omega) := W^{2,p}(\Omega; \mathbb{R}^3)/\mathrm{Isom}_+(\mathbb{R}^3),$$

which are equipped with the natural topologies inherited from the corresponding Sobolev spaces $W^{k,p}$ for $k \in \{0, 1, 2\}$. Let $\Phi : \mathbb{T}(\Omega) \to \mathbf{V}(\Omega)$ map (g, B) to the immersion f such that f is an isometric immersion from (Ω, g) to $(\mathbb{R}^3, \mathfrak{e})$ with the second fundamental form B. Then Φ is locally Lipschitz continuous.

In Theorem 5.1 above, $\mathrm{Isom}_+(\mathbb{R}^3)$ denotes the group of orientation-preserving isometries in the 3-D Euclidean space, *i.e.*, $\mathrm{Isom}_+(\mathbb{R}^3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$, and $\mathrm{Sym}_{d\times d}$ is the space of symmetric $d \times d$ matrices with superscript "+" to designate the positive definiteness.

In nonlinear elasticity, Theorem 5.1 states the continuous dependence of the deformation of elastic bodies on the Cauchy–Green tensor (*i.e.,* the metric) and the extrinsic geometry, for 2-D elastic bodies with lower regularity. Its proof is based on a programme developed by P. G. Ciarlet, C. Mardare, and S. Mardare; see [14,17,18] and the references cited therein. The goal of the programme (as summarized in [11]) is to extend the "fundamental theorem of surface theory" (that is, a simply-connected surface immersed in $\mathbb{R}^3$ can be uniquely recovered from its metric g and second fundamental form B modulo $\mathrm{Isom}_+(\mathbb{R}^3)$–actions) to the case of surfaces with lower regularity (*i.e.*, when $g \in W^{1,p}$ and $B \in L^p$ for $p > 2$). Also see Szopos [38] for the higher dimensional case.

The proof of Theorem 5.1 in [14, 17, 18] may be outlined as follows: First, the Gauss–Codazzi equations (2.2)–(2.3) are transformed into two types of first-order, nonlinear, matrix-valued PDEs, known as the *Pfaff* and *Poincaré* systems. Then, applying the analytic results due to Mardare [30–32], the Pfaff and Poincaré systems are solved, and the continuous dependence of solutions in suitable Sobolev spaces is proved. On the other hand, the transformation from the

Gauss–Codazzi equations (2.2)–(2.3) to the Pfaff–Poincaré systems in [14,17,18] appears highly intricate, which involves many different types of geometric quantities (*e.g.*, metrics, connections, curvatures, ...) in local coordinates as the entries of the same matrices of enormous size.

In [4], we provided a simpler, more direct approach of proving the existence of $W^{2,p}$–isometric immersions with respect to the prescribed $W^{1,p}$–metrics and L^p–second fundamental forms by using the Cartan formalism of exterior calculus on manifolds. In what follows, we explain how Theorem 5.1 can be recovered by utilizing the method in [4].

In fact, we establish an analogue of Theorem 5.1 on a simply-connected Riemannian manifold (M, g) in arbitrary dimensions and co-dimensions (but neglecting the effects of boundary). For this, let E be a $\mathbb{R}^k$-vector bundle over d-dimensional Riemannian manifold M with bundle metric g^E and compatible bundle connection ∇^E. We use Latin letters $X, Y, Z, W, \ldots$ to denote the tangential vector fields in $\Gamma(TM)$, and Greek letters $\xi, \eta \in \Gamma(f(TM)^\perp)$ or $\Gamma(E)$ to denote the vector fields on the normal bundle $f(TM)^\perp$ or on some given vector bundle E. Also, S_η is the shape operator determined by the second fundamental form B via $S_\eta(X,Y) = \langle \eta, B(X,Y)\rangle$, $\widetilde{\nabla}$ is the Levi-Civita connection on the ambient space $\mathbb{R}^{d+k}$, R denotes the Riemann curvature tensor on M, and R^E denotes the Riemann curvature tensor on E:

$$R^E(X,Y) = [\nabla^E_X, \nabla^E_Y] - \nabla^E_{[X,Y]}.$$

Then the Gauss, Codazzi, and Ricci equations on E are as follows in order:

$$\langle B(Y,W), B(X,Z)\rangle - \langle B(X,W), B(Y,Z)\rangle = R(X,Y,Z,W), \tag{5.1}$$

$$\widetilde{\nabla}_Y B(X,Z) = \widetilde{\nabla}_X B(Y,Z), \tag{5.2}$$

$$\langle [S_\eta, S_\xi]X, Y\rangle = R^E(X,Y,\eta,\xi). \tag{5.3}$$

In [4, Theorem 5.2], we established the equivalence of the following three clauses for $p > d$:

(i) the existence of a weak solution $(g, B, \nabla^E) \in W^{1,p}_{\rm loc} \times L^p_{\rm loc} \times L^p_{\rm loc}$ of the Gauss–Codazzi–Ricci equations (5.1)–(5.3) as above;

(ii) the Cartan formalism (see [4, §5.2] for details);

(iii) the existence of a $W^{2,p}_{\rm loc}$–isometric immersion whose metric, second fundamental form, and normal connection are the weak solution (g, B, ∇^E) in (i) above.

Based on these, we have

Theorem 5.2. *Let M be a d-dimensional simply-connected differentiable Riemannian manifold. Let E be a $\mathbb{R}^k$-vector bundle over M with bundle metric g^E and compatible bundle connection ∇^E. Denote by $\mathcal{A}(E)$ the space of affine connections on E over M. For $p > d$, define the spaces*:

$$\mathbb{T}(M) := \left\{ (g, B, \nabla^E) : \begin{array}{l} g \in W^{1,p}(M; {\rm Sym}^+_{d\times d}T^*M), \nabla^E \in L^p\big(M; \mathcal{A}(E)\big) \\ B \in L^p(M; {\rm Sym}_{d\times d}T^*M \otimes TM^\perp) \\ (g, B, \nabla^E) \textit{ satisfies GCRE } (5.1)\text{–}(5.3) \textit{ on } E \end{array} \right\}$$

and

$$\mathbf{V}(M) := W^{2,p}(M; \mathbb{R}^n)/{\rm Isom}_+(\mathbb{R}^{d+k}). \tag{5.4}$$

Let $\Phi : \mathbb{T}(M) \to \mathbf{V}(M)$ map (g, B, ∇^E) to an isometric immersion $f : (M, g) \to (\mathbb{R}^{d+k}, \mathfrak{e})$ with second fundamental form B and normal connection ∇^E. Then Φ is locally Lipschitz continuous.

In the above, ${\rm Sym}_{d\times d}T^*M$ denotes the space of symmetric 2–forms on the cotangent bundle T^*M (which can be expressed as $d \times d$ symmetric matrices), and ${\rm Sym}^+_{d\times d}T^*M$ consists of positive definite elements in $\mathrm{Sym}_{d\times d}T^*M$. Also, it is classical (see [34, p.56, Exercise 8(2)]) that $\mathrm{Isom}_+(\mathbb{R}^{d+k}) = \mathbb{R}^{d+k} \rtimes \mathrm{SO}(d+k)$; that is, orientation-preserving Euclidean rigid motions on $\mathbb{R}^{d+k}$ consist of translations and orientation-preserving rotations.

Proof. In this proof, denote by $\|\bullet\|_{\mathbb{T}}$ and $\|\bullet\|_{\mathbf{V}}$ the natural norms induced from the suitable product or quotient topologies on $\mathbb{T}(M)$ and $\mathbf{V}(M)$, respectively. More precisely, for any $h \in W^{1,p}(M;\mathfrak{gl}(d;\mathbb{R}))$, $D \in L^p(M;\mathfrak{gl}(d;\mathbb{R}))$, and $\Lambda \in L^p(M;\mathcal{A}(E))$, we write

$$\|(h,D,\Lambda)\|_{\mathbb{T}} := \|h\|_{W^{1,p}} + \|D\|_{L^p} + \|\Lambda\|_{L^p};$$

and, for any $\vartheta \in \mathbf{V}(M)$, we write

$$\|\vartheta\|_{\mathbf{V}} := \inf_{\mathcal{I}\in \mathrm{Isom}_+(\mathbb{R}^{d+k})} \|\vartheta\circ\mathcal{I}\|_{W^{2,p}}.$$

In general, arguments h and D in $\|(h,D,\Lambda)\|_{\mathbb{T}}$ above are not required to be tensorial. Furthermore, in the above, the domains over which the Sobolev norms are taken are suitable subsets of M, which will be clear from the context.

We first use [4, Theorem 5.2] to conclude that mapping $\Phi : \mathbb{T}(M) \to \mathbf{V}(M)$ is well-defined. Then it suffices to establish

$$\|\Phi(g,B,\nabla^E) - \Phi(g',B',\nabla^{E'})\|_{\mathbf{V}} \le C\|(g,B,\nabla^E) - (g',B',\nabla^{E'})\|_{\mathbb{T}} \tag{5.5}$$

for some constant C, provided that the right-hand side is sufficiently small. The arguments are divided into three steps.

1. The crucial step is to translate the Gauss–Codazzi–Ricci equations (5.1)–(5.3) into the Pfaff–Poincaré system. This is achieved via the Cartan structural equations (*cf.* [8] for details). Let $U \subset M$ be a local chart on which E is trivialized. For a local orthonormal frame of vector fields $\{\partial_i\}_{i=1}^d \subset \Gamma(TU)$, let $\{\omega^i\}_{i=1}^d$ be the dual co-frame of differential 1-forms. In addition, let $\{\eta_\alpha\}_{\alpha=d+1}^{d+k}$ be an orthonormal frame on the typical fibre of E. Then, for indices $1 \le i,j \le d$, $d+1 \le \alpha,\beta \le d+k$, and $1 \le a,b,c \le d+k$, define differential 1-forms $\{\omega^a_b\}$ by

$$\omega^i_j(\partial_k) := \langle \nabla_{\partial_k}\partial_j, \partial_i\rangle, \tag{5.6}$$

$$\omega^i_\alpha(\partial_j) = -\omega^\alpha_i(\partial_j) := \langle B(\partial_i,\partial_j), \eta_\alpha\rangle, \tag{5.7}$$

$$\omega^\alpha_\beta(\partial_j) := \langle \nabla^E_{\partial_j}\eta_\alpha, \eta_\beta\rangle. \tag{5.8}$$

We write $\mathbf{W} := \{\omega^a_b\}$ as the $\mathfrak{so}(d+k)$-valued differential 1-forms on U, or equivalently, the 1-form-valued matrix field, where $\mathfrak{so}(d+k)$ is the space of anti-symmetric $(d+k)\times(d+k)$ matrices. Schematically, we may write

$$\mathbf{W} = \begin{bmatrix} \nabla & B \\ -B & \nabla^E \end{bmatrix},$$

where $\nabla = \nabla^g$ is the Levi-Civita connection on M corresponding to metric g. On the other hand, we augment the 1-forms $\{\omega^i\}$ by setting

$$w := \big(\omega^1, \omega^2, \cdots, \omega^d, \underbrace{0, \cdots, 0}_{k \text{ times}}\big)^\top. \tag{5.9}$$

Thus, w is a 1-form-valued $(d+k)$-vector, or a $\mathbb{R}^{d+k}$-valued 1-form.

It can be checked that the Gauss–Codazzi–Ricci equations (5.1)–(5.3) are equivalent to the following two *structural equations* (see [4, §5.3, Step 4]):

$$\mathrm{d}w = w \wedge \mathbf{W}, \qquad \mathrm{d}\mathbf{W} + \mathbf{W} \wedge \mathbf{W} = 0, \tag{5.10}$$

even for the lower regularity under consideration. Moreover, the desired isometric immersion $f \in W^{2,p}(M; \mathbb{R}^{d+k})$ can be solved from the systems of first-order nonlinear PDEs as below:

$$\mathbf{W} = \mathrm{d}A \cdot A^\top, \qquad A(x_0) = A_0, \tag{5.11}$$

and

$$\mathrm{d}f = w \cdot A, \qquad f(x_0) = f_0. \tag{5.12}$$

In the above, d is the exterior derivative, $\wedge$ denotes the intertwining of wedge product on differential forms and the matrix product $(\cdot)$, A is a field of orthogonal $d+k$ matrices to be solved in some neighbourhood $V \subset U \subset M$ containing point x_0, A_0 and f_0 are arbitrary initial data, (5.11) is known as the *Pfaff* system, and (5.12) as the *Poincaré* system. We also note that $\mathbf{W}$ is in L^p.

2. We now prove (5.5) for the case that $g = g'$. With the above preparation, the analytic lemmas due to Mardare [31] may be directly applied. By [31, Theorem 7], the Pfaff system (5.11) has a unique solution $A \in W^{1,p}(V; \mathfrak{so}(d+k))$ if and only if a compatibility condition of involutiveness holds in the sense of distributions; see [4, Eq. (5.12)]. Also, by [32, Theorem 6.5] and a result by Schwartz [37], the Poincaré system (5.12) has a unique solution $f \in W^{2,p}(V; \mathbb{R}^{d+k})$ if and only if a compatibility condition of exactness holds in the sense of distributions; see [4, Eq. (5.14)]. Furthermore, the solutions for the Pfaff and Poincaré systems depend continuously on the source term: if A and A' are two solutions for (5.11) with the same initial data associated with source terms $\mathbf{W}$ and $\mathbf{W}'$ respectively, then

$$\|A - A'\|_{W^{1,p}(V)} \leq C\|\mathbf{W} - \mathbf{W}'\|_{L^p(V)}. \tag{5.13}$$

If f and f' are two solutions to (5.12) with the same initial data, then

$$\|f - f'\|_{W^{2,p}(V)} \leq C\|w \cdot A - w' \cdot A'\|_{W^{1,p}(V)}, \tag{5.14}$$

where $C = C(d, k, p, V)$, and V is a sufficiently small neighbourhood on M, *i.e.*, these estimates are local. In this case, $g = g'$ so that $w = w'$. Then we have

$$\|f - f'\|_{W^{2,p}(V)} \leq C\|A - A'\|_{W^{1,p}(V)}.$$

However, by well-known computations in differential geometry (see [4, Steps 4–5, §5.3] for details), the structural equations (5.10) are equivalent to the aforementioned compatibility conditions for the Pfaff and Poincaré systems, respectively.

As indicated earlier, $\Phi : \mathbb{T}(M) \to \mathbf{V}(M)$ is well-defined. In (5.13)–(5.14), we have proved that

$$\|f - f'\|_{W^{2,p}(V)} \leq C\|\mathbf{W} - \mathbf{W}'\|_{L^p(V)},$$

provided that the initial data coincide. Without loss of generality, we can always assume the same initial data — this can be achieved by applying a Euclidean isometry in $\mathrm{Isom}_+(\mathbb{R}^{d+k})$, which is negligible by the quotient construction of $\mathbf{V}(M)$. By the definition of $\mathbf{W}$, it is clear that

$$\|\mathbf{W} - \mathbf{W}'\|_{L^p(V)} = \|B - B'\|_{L^p(V)} + \|\nabla^E - \nabla^{E'}\|_{L^p(V)}.$$

Thus, the proof is complete on a local chart V in the case that $g = g'$. The same holds if V is replaced by M, thanks to the assumption that M is simply-connected and a monodromy argument. See Mardare [32, Theorems 6.4 and 6.5] for detailed arguments on simply-connected Euclidean domains, which can be adapted to manifolds; also see Step 7 in [4, §5.3].

3. It remains to prove (5.5) for the general case $g \neq g'$. It follows from the definition of $\mathbf{W}$ given in Step 1 that

$$\|\mathbf{W} - \mathbf{W}'\|_{L^p(V)} \leq C \left\{ \|\nabla^g - \nabla^{g'}\|_{L^p(V)} + \|B - B'\|_{L^p(V)} + \|\nabla^E - \nabla^{E'}\|_{L^p(V)} \right\},$$

where C is purely dimensional.

Note that $\|\nabla^g - \nabla^{g'}\|_{L^p(V)}$ is in turn controlled by $\|g - g'\|_{W^{1,p}(V)}$. Indeed, the coordinate-wise components of the Levi-Civita connection is given by the Christoffel symbols $\Gamma^k_{ij} = \frac{1}{2} g^{kl} \{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\}$, where $g^{ij} = (g_{ij})^{-1}$. As in §4, recall that $\nabla = g^{-1} \star \partial g$, where ∂ denotes the (Euclidean) derivative. Then

$$\begin{aligned}
\|\nabla^g - \nabla^{g'}\|_{L^p(V)} &= \big\| \big(g^{-1} - (g')^{-1}\big) \star \partial g + (g')^{-1} \star \partial(g - g') \big\|_{L^p(V)} \\
&\leq \|g^{-1}(g - g')(g')^{-1}\|_{L^\infty(V)} \|\partial g\|_{L^p(V)} \\
&\quad + \|(g')^{-1}\|_{L^\infty(V)} \|\partial(g - g')\|_{L^p(V)},
\end{aligned}$$

which is bounded by $C\|g - g'\|_{L^p(V)}$, thanks to the Sobolev–Morrey embedding: $W^{1,p}_{\mathrm{loc}}(M) \hookrightarrow L^\infty_{\mathrm{loc}}(M)$ for $p > d$, where C depends on the $W^{1,p}$-norm of g and g' (which is allowed as the theorem concerns only the local continuity of Φ). Thus, in view of Eq. (5.13), we have

$$\begin{aligned}
\|A - A'\|_{W^{1,p}(V)} &\leq C\|\mathbf{W} - \mathbf{W}'\|_{L^p(V)} \\
&\leq C\|(g, B, \nabla^E) - (g', B', \nabla^{E'})\|_{\mathbb{T}}.
\end{aligned} \tag{5.15}$$

In particular, $A \in W^{1,p}(V; \mathfrak{gl}(n, \mathbb{R}))$. Then we consider the Poincaré system (as before, with the same initial data) associated to g and g':

$$\mathrm{d}f = w \cdot A, \qquad \mathrm{d}f' = w' \cdot A',$$

where w and w' are defined as those in (5.9) with respect to g and g', respectively. Since w and w' consist of the dual co-frames of the same orthonormal frame on M, then

$$\|w - w'\|_{W^{1,p}(V)} \leq C\|g - g'\|_{W^{1,p}(V)}, \tag{5.16}$$

where C is a dimensional constant. Thus, taking the difference of the two Poincaré systems, we see that

$$\mathrm{d}(f - f') = (w - w')A + w(A - A'). \tag{5.17}$$

Then we have the following estimates:

$$\begin{aligned}
&\|f - f'\|_{W^{2,p}(V)} \\
&\leq \|(w - w')A + w(A - A')\|_{W^{1,p}(V)} \\
&\leq C\big\{\|A\|_{L^\infty(V)}\|w - w'\|_{W^{1,p}(V)} + \|A\|_{W^{1,p}(V)}\|w - w'\|_{L^\infty(V)} \\
&\qquad + \|w\|_{L^\infty(V)}\|A - A'\|_{W^{1,p}(V)} + \|w\|_{W^{1,p}(V)}\|A - A'\|_{L^\infty(V)}\big\} \\
&\leq C\big\{\|A\|_{W^{1,p}(V)}\|w - w'\|_{W^{1,p}(V)} + \|w\|_{W^{1,p}(V)}\|A - A'\|_{W^{1,p}(V)}\big\} \\
&\leq C'\big\{\|w - w'\|_{W^{1,p}(V)} + \|A - A'\|_{W^{1,p}(V)}\big\},
\end{aligned} \tag{5.18}$$

where we have used the estimates for the Poincaré system (*cf.* Mardare [32] and Schwartz [37]) in the first inequality, the usual interpolation inequality in the second inequality, and the Morrey–Sobolev embedding in the third inequality, and constant C' depends on $\|A\|_{W^{1,p}(V)}$ and $\|w\|_{W^{1,p}(V)}$, which again is allowed due to the locality of Φ.

Therefore, combining Eqs. (5.15)–(5.16) with (5.18) together, we complete the proof.

Finally, let us revisit the weak rigidity theorem of isometric immersions established in [4] (*i.e.*, Proposition 2.2). A slightly stronger version can be deduced — In view of Theorem 5.2, we can drop the uniform $W^{2,p}_{\rm loc}$–boundedness of isometric immersions in the hypothesis:

Corollary 5.3. *Let M be a d-dimensional simply-connected Riemannian manifold with $W^{1,p}$-metric g for $p > d$. Let $\{\Phi^\varepsilon\}_{\varepsilon>0}$ be a family of isometric immersions of (M, g) into the Euclidean space $\mathbb{R}^D$, whose second fundamental forms and normal connections are B^ε and $\nabla^{\varepsilon,\perp}$. Assume that B^ε and $\nabla^{\varepsilon,\perp}$ are uniformly bounded in $L^p_{\rm loc}$. Then, modulo translations and rotations, Φ^ε converges weakly in $W^{2,p}_{\rm loc}$ to an isometric immersion Φ of (M, g) and, more importantly, the second fundamental form and normal connection of Φ are weak $L^p_{\rm loc}$-limits of B^ε and $\nabla^{\varepsilon,\perp}$, obeying the Gauss–Codazzi–Ricci equations* (5.1)–(5.3).

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References

[1] I. Alpern, R. Kupferman, C. Maor, Asymptotic rigidity for shells in non-Euclidean elasticity, arXiv preprint, arXiv: 2012.12075, 2021.

[2] S. S. Antman, Ordinary differential equations of nonlinear elasticity I: Foundations of the theories of non-linearly elastic rods and shells, Arch. Ration. Mech. Anal. 61 (1976) 307–351.

[3] H. Cartan, Formes Différentielles, Paris, Hermann, 1967.

[4] G.-Q. Chen, S. Li, Global weak rigidity of the Gauss–Codazzi–Ricci equations and isometric immersions of Riemannian manifolds with lower regularity, J. Geom. Anal. 28 (2018) 1957–2007.

[5] G.-Q. Chen, S. Li, Weak continuity of Cartan's structural equation on semi-Riemannian manifolds with lower regularity, Arch. Ration. Mech. Anal. 241 (2021) 579–641.

[6] G.-Q. Chen, M. Slemrod, D. Wang, Isometric immersions and compensated compactness, Commun. Math. Phys. 294 (2010) 411–437.

[7] G.-Q. Chen, M. Slemrod, D. Wang, Weak continuity of the Gauss–Codazzi–Ricci system for isometric embedding, Proc. Amer. Math. Soc. 138 (2010) 1843–1852.

[8] S. S. Chern, W. H. Chen, K. S. Lam, Lectures on Differential Geometry, Vol. 1 of World Scientific Publishing Company, 1999.

[9] P. G. Ciarlet, Mathematical Elasticity, Vol.1 of Three–Dimensional Elasticity, North-Holland, Amsterdam, 1988.
[10] P. G. Ciarlet, An Introduction to Differential Geometry with Applications to Elasticity, Springer, Dordrecht, 2005.
[11] P. G. Ciarlet, L. Gratie, C. Mardare, A new approach to the fundamental theorem of surface theory, Arch. Ration. Mech. Anal. 188 (2008) 457–473.
[12] P. G. Ciarlet, F. Laurent, Continuity of a deformation as a function of its Cauchy–Green tensor, Arch. Ration. Mech. Anal. 167 (2003) 255–269.
[13] P. G. Ciarlet, M. Malin, C. Mardare, Continuity in Fréchet topologies of a surface as a function of its fundamental forms, J. Math. Pures Appl. 142 (2020) 243–265.
[14] P. G. Ciarlet, C. Mardare, Recovery of a surface with boundary and its continuity as a function of its two fundamental forms, Anal. Appl. 3 (2005) 99–117.
[15] P. G. Ciarlet, C. Mardare, Existence theorems in intrinsic nonlinear elasticity, J. Math. Pures Appl. 94 (2010) 229–243.
[16] P. G. Ciarlet, C. Mardare, Boundary conditions in intrinsic nonlinear elasticity, J. Math. Pures Appl. 101 (2014) 458–472.
[17] P. G. Ciarlet and S. Mardare, Nonlinear Korn inequalities in $\mathbb{R}^n$ and immersions in $W^{2,p}$, $p > n$, considered as functions of their metric tensors in $W^{1,p}$, J. Math. Pures Appl. 105 (2016) 873–906.
[18] P. G. Ciarlet, C. Mardare, A surface in $W^{2,p}$ is a locally Lipschitz-continuous function of its fundamental forms in $W^{1,p}$ and L^p, $p > 2$, J. Math. Pures Appl. 124 (2019) 300–318.
[19] M. P. do Carmo, Riemannian Geometry, translated from the second Portuguese edition by Francis Flaherty, Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992.
[20] J. Eells, Jr., J. H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964) 109–160.
[21] F. Hélein, Harmonic Maps, Conservation Laws and Moving Frames, Second Edition, Vol. 150 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2002.
[22] F. Hélein, J. C. Wood, Harmonic Maps, in: Handbook of Global Analysis, pp. 417–491, 1213, Elsevier Sci. B. V., Amsterdam, 2008.
[23] G. Friesecke, R. D. James, S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity, Comm. Pure Appl. Math. 55 (2002) 1461–1506.
[24] P. Hornung, M. Lewicka, M. R. Pakzad, Infinitesimal isometries on developable surfaces and asymptotic theories for thin developable shells, J. Elasticity 111 (2013) 1–19.
[25] R. Kupferman, C. Maor, A. Shachar, Reshetnyak rigidity of Riemannian manifolds, Arch. Ration. Mech. Anal. 231 (2019) 367–408.
[26] R. Kupferman, A. Shachar, A geometric perspective on the Piola identity in Riemannian settings, J. Geom. Mech. 11 (2019) 59–76.
[27] M. Lewicka, Quantitative immersability of Riemann metrics and the infinite hierarchy of prestrained shell models, Arch. Ration. Mech. Anal. 236 (2020) 1677–1707.
[28] M. Lewicka, L. Mahadevan, M. R. Pakzad, The Monge–Ampère constraint: matching of isometries, density and regularity, and elastic theories of shallow shells, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017) 45–67.
[29] M. Lewicka, M. G. Mora, M. R. Pakzad, The matching property of infinitesimal isometries on elliptic surfaces and elasticity of thin shells, Arch. Ration. Mech. Anal. 200 (2011) 1023–1050.
[30] S. Mardare, The fundamental theorem of surface theory with little regularity, J. Elasticity 73 (2003) 251–290.
[31] S. Mardare, On Pfaff systems with L^p coefficients and their applications in differential geometry, J. Math. Pures Appl. 84 (2005) 1659–1692.
[32] S. Mardare, On systems of first order linear partial differential equations with L^p coefficients, Adv. Diff. Eq. 12 (2007) 301–360.
[33] J. F. Nash Jr., The imbedding problem for Riemannian manifolds, Ann. of Math. 63 (1956) 20–63.
[34] P. Petersen, Riemannian Geometry, Second Edition, Vol. 171 of Graduate Texts in Mathematics, Springer (2006).
[35] Yu. G. Rešetnjak, Liouville's conformal mapping theorem under minimal regularity hypotheses, Sib. Mat. Zhurnal 8 (1967) 835–840.

[36] Yu. G. Rešetnjak, On the stability of conformal mappings in multidimensional spaces, Sib. Mat. Zhurnal 8 (1967) 91–114.
[37] L. Schwartz, Théorie des Distributions, Second Edition, Hermann Press, 1966.
[38] M. Szopos, An existence and uniqueness result for isometric immersions with little regularity, Rev. Roumaine. Math. Pures Appl. 53 (2008) 555–565.
[39] K. Tenenblat, On isometric immersions of Riemannian manifolds, Bull. Brazilian Math. Soc. 2 (1971) 23–36.
[40] C. Wang, The Calderon–Zygmund inequality on a compact Riemannian manifold, Pacific J. Math. 217 (2004) 181–200.
[41] F. W. Warner, Foundations of Differentiable Manifolds and Lie Groups, Scott, Foresman, and Co., Glenview, 1971.
[42] A. Yavari, A. Goriely, Riemann–Cartan geometry of nonlinear dislocation mechanics, Arch. Ration. Mech. Anal. 205 (2012) 59–118.

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