Small scales and singularity formation in fluid mechanics

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The background

Leonardo Da Vinci (circa 1500): "... the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large."

Euler equation 1755: ideal fluid. Set on domain $D \subset \mathbb{R}^n$, $n = 2, 3$.

$$\partial_t u + (u \cdot \nabla)u = \nabla p, \;\; \nabla \cdot u = 0, \;\;\; u \cdot n|_{\partial D} = 0, \;\; u(x,0) = u_0(x)$$

Navier-Stokes equation 1845: adds viscosity.

$$\partial_t u + (u \cdot \nabla)u - \nu \Delta u = \nabla p, \;\;\; u|_{\partial D} = 0$$

Global regularity vs finite time blow up? The story is very different in dimensions two and three.
Key quantity: vorticity $\omega = \mathrm{curl}\, u.$

The Euler equation in vorticity form:

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, \;\; u = K_D * \omega, \;\; \omega(x,0) = \omega_0(x).$$

The vortex stretching term on the right hand side is identically zero in two dimensions.

Solutions to the 2D Euler equation are globally regular (Wolibner 1933, Hölder 1933).

Global regularity vs finite time singularity formation question for solutions to the 3D Euler (and Navier-Stokes!) equations is a major open problem.

Singularities and Turbulence: The Zeroth Law

Feynman: turbulence is the most important unsolved problem of classical physics.

Suppose u^ν are solutions of the Navier-Stokes equation, no boundaries.

$$\partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu - \nu \Delta u^\nu = \nabla p^\nu + F, \;\; \nabla \cdot u^\nu = 0.$$

The energy dissipation rate is given by

$$\partial_t \int |u^\nu|^2 \, dx = -2\nu \int |\nabla u^\nu|^2 \, dx + \int F u^\nu \, dx.$$

The zeroth law of turbulence (G.I. Taylor, Kolmogorov):

$$\lim_{\nu \to 0} \nu \langle \int |\nabla u^\nu|^2 \, dx \rangle \to \epsilon > 0$$

$\langle \cdot \rangle$ denotes suitable time average. Confirmed very well in experiments.

Search for singularities: 3D Euler equation

But as $\nu \to 0$, solutions of the Navier-Stokes equation converge to the solutions of Euler equation if everything stays smooth. And in this case the zeroth law could not hold on finite time scales.

Conjecture, Onsager 1949: As $\nu \to 0$, the validity of the zeroth law is enabled by singularities or near singularities of solutions, which create *extreme dissipation regions.*

The validity of this picture has been shown rigorously for Burgers equation

$$\partial_t u^\nu + u^\nu \partial_x u^\nu - \nu \partial_x^2 u^\nu = F$$

by E, Khanin, Mazel and Sinai 2000. Singularities of the inviscid equation are simple: shocks, and they dissipate energy.

Recall the 3D Euler equation:

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, \;\; u = K_D * \omega, \;\; \omega(x,0) = \omega_0.$$

Local well-posedness in sufficiently regular spaces.

Search for singularities: 3D Euler equation

A variety of regularity criteria. Beale-Kato-Majda '84: at blow up time T, must have

$$\int_0^T \|\omega(\cdot, s)\|_{L^\infty}\, ds = \infty.$$

Infinite energy singularities: Stuart '88, Childress-Ierley-Spiegel-Young '89, Constantin '00.

Numerical simulations looking for singular scenarios: Grauer-Sideris '91, Pumir-Siggia '92, Kerr '93, E-Shu '94, Boratav-Pelz '94, Pelz-Gulak '97, Ohkitani-Gibbon '00, Hou-Li '06, Larios-Petersen-Titi-Wingate '15.

Constantin-Fefferman-Majda '96: conditions on vorticity direction sufficient for regularity.

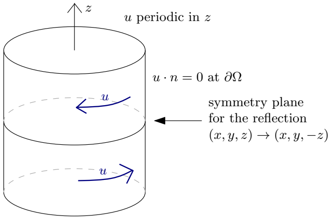
The 3D Euler equation: the Hou-Luo scenario

Luo-Hou '14, numerical experiment: axi-symmetric flow in a cylinder.
Denote $D_t = \partial_t + u^r\partial_r + u^z\partial_z$.
3D Euler equation in cylindrical coordinates:

$$D_t\left(\frac{\omega^\phi}{r}\right) = \frac{\partial_z(ru^\phi)^2}{r^4}; \;\; D_t(ru^\phi) = 0$$

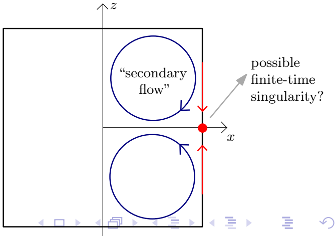
$$(u^r, u^z) = (-r^{-1}\partial_z\psi, r^{-1}\partial_r\psi),$$

$$L\psi = \frac{\omega^\phi}{r}, \;\; L\psi = -\frac{1}{r}\partial_r\left(\frac{1}{r}\partial_r\psi\right) - \frac{1}{r^2}\partial_z^2.$$



Fast growth of ω^ϕ is observed near a ring of boundary hyperbolic points of the flow.

That small scales first form at the boundary is not surprising - it is often at the boundary that turbulence is initiated.



Hyperbolic points: the most common “singularity”?

E. Saw et al ’16: an experimental study focusing on extreme dissipation regions. The flow is statistically axi-symmetric, and the camera is focused near the $r = 0$ axis. 75% of extreme dissipation regions are found to feature fronts/hyperbolic points (there are also jets, spirals, and cusps).

Hou-Luo scenario: The 2D inviscid Boussinesq system

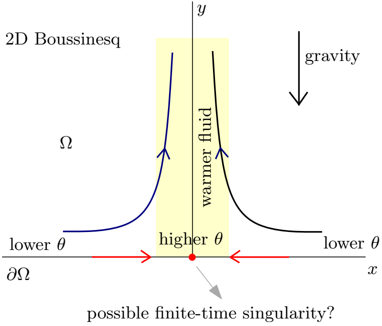
A good proxy for 3D axi-symmetric Euler equation away from the rotation axis is the 2D inviscid Boussinesq system.

$$\partial_t \omega + (u \cdot \nabla)\omega = \partial_{x_1}\theta$$
$$\partial_t \theta + (u \cdot \nabla)\theta = 0$$
$$u = \nabla^{\perp}(-\Delta)^{-1}\omega.$$

Global regularity for this system is on the list of “Eleven great problems of mathematical hydrodynamics” by Yudovich.

When θ is constant, get the 2D Euler equation. It makes sense to first understand the 2D Euler small scale creation in a geometry similar to Hou-Luo scenario.

The picture is essentially like for the 3D Euler equation rotated by $\pi/2$.



2D Euler equation: history

The 2D Euler equation:

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \quad u = \nabla^\perp (-\Delta_D)^{-1}\omega.$$

Trajectories

$$\frac{d\Phi_t(x)}{dt} = u(\Phi_t(x), t), \quad \Phi_0(x) = x.$$

Then $\omega(\Phi_t(x), t) = \omega_0(x)$, $\omega(x,t) = \omega_0(\Phi_t^{-1}(x))$. L^∞ norm is conserved!

Theorem (Wolibner; Hölder 1933)

Let D be smooth, compact, $\omega_0 \in C^1(D)$. Then there exists unique solution of the 2D Euler equation $\omega(x,t) \in C^1$. Moreover,

$$\frac{\|\nabla \omega(\cdot,t)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}} \leq \left(1 + \frac{\|\nabla \omega_0\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\right)^{\exp C\|\omega_0\|_{L^\infty} t}.$$

2D Euler equation: is double exp real?

Why double exponential? Kato inequality.
Recall $u = \nabla^\perp(-\Delta_D)^{-1}\omega$, so $\partial_j u_i$ are Riesz transforms of ω. Then

$$\|\nabla u\|_{L^\infty} \leq C\|\omega\|_{L^\infty}\left(1+\log\left(1+\frac{\|\nabla\omega\|_{L^\infty}}{\|\omega\|_{L^\infty}}\right)\right)$$

It is exactly the log term that leads to double exponential upper bound.

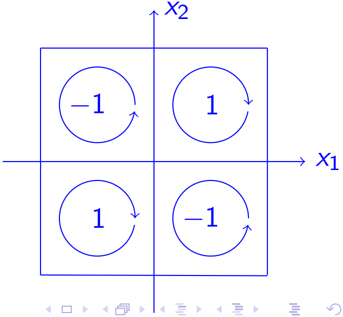
But can such fast growth actually happen?

Yudovich '74: some infinite growth of $\nabla\omega$.

Nadirashvili '91: linear growth of $\|\nabla\omega\|_{L^\infty}$.

Bahouri-Chemin '94. "Singular cross" flow.
ω odd in x_1, x_2, $\equiv 1$ on $(0,\pi)^2$, periodic.

$$u_1(x_1,0) = cx_1 \log x_1 + O(x_1)!$$



2D Euler growth examples

Denisov 2010s: example set on $\mathbb{T}^2$ where $\|\nabla\omega\|_{L^\infty}$ grows superlinearly. Also, given $T > 0$, $\lambda > 1$, one can find ω_0^T such that

$$\|\nabla\omega^T(\cdot, T)\|_{L^\infty} \geq \lambda^{e^T - 1}\|\nabla\omega_0^T\|_{L^\infty}.$$

Theorem (K-Sverak '14)

Let D be unit disk. There exist $\omega_0 \in C^\infty(\bar{D})$ with $\|\nabla\omega_0\|_{L^\infty} > \|\omega_0\|_{L^\infty}$ such that

$$\frac{\|\nabla\omega(\cdot, t)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}} \geq \left(\frac{\|\nabla\omega_0\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\right)^{c \exp c\|\omega_0\|_{L^\infty} t} \quad \text{for all } t \geq 0.$$

Inspired by Luo-Hou numerical experiments for 3D Euler (more later). Growth happens at the boundary!

Main Lemma: Biot-Savart

Assume ω_0 is odd with respect to x_1.
Analyze $u = \nabla^\perp(-\Delta_D)^{-1}\omega$.

Lemma (Main Lemma)

Fix small $\gamma > 0$. For $x \in D_1^\gamma$, $|x| \le \delta$, we have

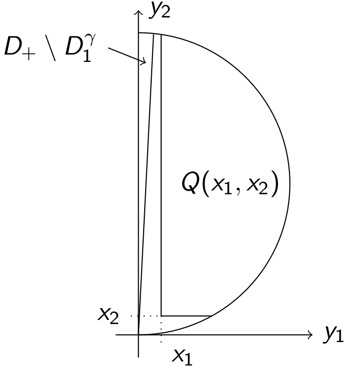
$$u_1(x) = -\frac{4}{\pi}x_1 \int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4}\omega(y,t)\,dy + B_1 x_1.$$

Moreover, if $x \in D_2^\gamma$, $|x| \le \delta$, we have

$$u_2(x) = \frac{4}{\pi}x_2 \int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4}\omega(y,t)\,dy + B_2 x_2.$$

Here $\|B_{1,2}\|_{L^\infty} \le C(\gamma)\|\omega_0\|_{L^\infty}$.

$D_+ = \{(x_1,x_2) \in D \,|\, x_1 \ge 0\}$.
Set the origin at the bottom of the disk!



$Q(x_1,x_2) = \{y \in D_+ :\ y_1 \ge x_1,\ y_2 \ge x_2\}$

The 2D Euler example

Denote

$$\Omega(x,t) = \frac{4}{\pi}\int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4}\omega(y,t)\,dy.$$

Corollary (of Main Lemma)

Exponential growth is easy!

Take $0 \leq \omega_0 \leq 1$ in D_+, and $\omega_0(x) = 1$ if $x_1 \geq \delta$.

Then by incompressibility $\Omega(x,t) \geq c \log \delta^{-1}$ if $|x| \leq \delta$, for all times.

If δ is chosen small enough, Main Lemma gives $u_1(x) \leq -Cx_1$ for all times, all $|x| \leq \delta$. In particular the characteristic along the boundary converges to the origin at an exponential rate.

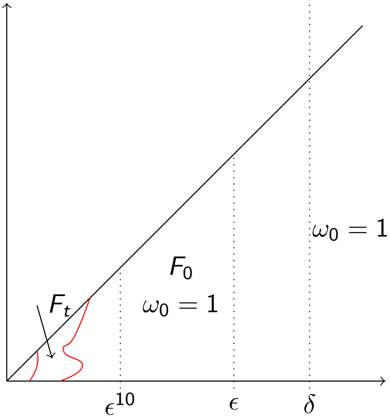
The double exponential

The argument to show double exponential growth is truly nonlinear: $\Omega(x,t)$ grows due to larger values of ω approaching the origin.

The approach of F_t to the origin leads to growth of Ω; one needs to control it!

A key role in the proof plays hidden "comparison principle" in the structure of

$$\Omega(x,t) = \frac{4}{\pi}\int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4}\omega(y,t)\,dy.$$



The initial data

1D models of the Hou-Luo scenario

Main difficulties in analysis of the Boussinesq system (relative to Euler):

- Vorticity may grow, affecting the "error" terms estimates.
- Vorticity is no longer sign definite.

The 1D models of the Hou-Luo scenario.

$$\partial_t\omega + u\partial_x\omega = \partial_x\theta; \quad \partial_t\theta + u\partial_x\theta = 0.$$

$$\text{Hou} - \text{Luo model}: u_x = H\omega; \text{ CKY model}: u(x) = -x\int_x^1 \frac{\omega(y)}{y}\,dy.$$

Derivation of the Hou-Luo model is based on boundary layer assumption on structure of vorticity. It is used to close the Biot-Savart law: $\omega(x_1, x_2, t) = \omega(x_1, t)\chi_{[0,a]}(x_2)$.

The CKY model is "almost local" and its Biot-Savart law is inspired by 2D Euler Main Lemma.

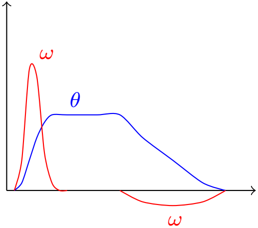
1D models of the Hou-Luo scenario

Theorem (Choi-K-Yao '15; Choi-Hou-K-Luo-Sverak-Yao '16)

Both models are locally well-posed, but there exist initial data leading to finite time blow up. For the CKY model, $\int_0^T \|\omega(\cdot,t)\|_{L^\infty}\,dt = \infty$ *at blow up time; for Hou-Luo model,* $\int_0^T \|u_x(\cdot,t)\|_{L^\infty}\,dt = \infty$.

Recall trajectories

$$\frac{d\Phi_t(x)}{dt} = u(\Phi_t(x),t), \quad \Phi_0(x) = x.$$



If $\psi(x,t) = -\log \Phi_t(x)$, then $\partial_t^2 \psi \sim e^{c\psi}$!

Other work on Hou-Luo scenario: Hou-Liu '15, Do-K-Xu '17, K-Yang '18 - 1d models;

Hoang, Orcan, Radosz, Yang '17, K-Tan '17 - 2d models.

Elgindi-Jeong '17 - singular solutions in domains with corners.

Alternative recent singular constructions

Tao '16 - Fourier side-inspired models. Motivated in part by dyadic models (Katz-Pavlovic '05). Model equations that are designed to replicate most of the properties of 3D Euler. Blow up has self-similar flavor.

Brenner, Harmoz, Pumir '16. A specific self-similar mechanism: vortex sheet break up into vortex tubes $\mapsto$ vortex tube flattening $\mapsto$ repeated break up on smaller scale.

The modified SQG equation: history

One of the key difficulties in the Hou-Luo scenario for the actual 2D Boussinesq system: growth of ω making error terms in the Main Lemma uncontrollable. Heuristic computations using numerical data suggest that "helpful" for blow up and "opposing" terms are of the same order.

A similar story in a different setting: the modified SQG patches.

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \;\; u = \nabla^\perp (-\Delta)^{-1+\alpha}\omega, \;\; 0 \leq \alpha \leq 1/2.$$

The value $\alpha = 0$ - 2D Euler, $\alpha = 1/2$ - SQG. For $\alpha > 0$ finite time blow vs global regularity question is open!

Constantin-Majda-Tabak '94; Cordoba '98, Cordoba-Fefferman '02.

A special class of initial data: patches, $\omega_0 = \sum_{j=1}^N \theta_j \chi_{\Omega_j}(x)$.

The Euler and SQG patches - history

The regularity question in patch context refers to regularity of the domain boundaries $\partial\Omega_j(t)$ and lack of touching by different patches.

2D Euler patches: existence and uniqueness by Yudovich '63. Global well-posedness in $\mathbb{R}^2$ or $\mathbb{T}^2$: Chemin '93, Bertozzi-Constantin '93.

With boundary - limited results, Depauw '99, Dutrifoy '03.

Patches for modified SQG: local regularity Rodrigo '05, Gancedo '08.

Let us consider patches in half-plane D, with $u \cdot n|_{\partial D} = 0$.

Theorem (K-Ryzhik-Yao-Zlatos '16)

Let $\alpha = 0$ (2D Euler), half-plane setting. If $\omega_0(x)$ is $C^{1,\gamma}$ for some $\gamma > 0$, then there exists a unique global $C^{1,\gamma}$ patch solution $\omega(x,t)$.

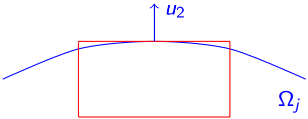
Patches are allowed to touch the boundary!

Theorem (K-Yao-Zlatos '16)

If $0 < \alpha < 1/24$ the following holds. If ω_0 is an H^3 patch, then there exists a unique local H^3 patch solution.

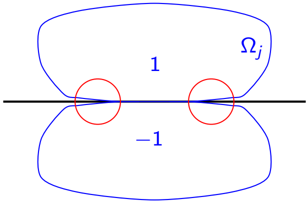
Why local well-posedness for patches with boundary is hard?
If $\alpha > 0$, $u \in C^{1-2\alpha}$ only, not Lipschitz.

However: can show normal to patch component of u has better regularity.



$$u_2(x_1, x_2) = \int_{\Omega_j} \frac{x_1 - y_1}{|x-y|^{2+2\alpha}} \omega(y)\, dy$$

Due to the no-penetration boundary condition, a patch touching the boundary is equivalent to reflected patch touching the original. Estimates near the point of touching are hard!

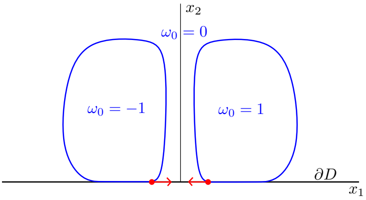


The modified SQG patches: finite time blow up

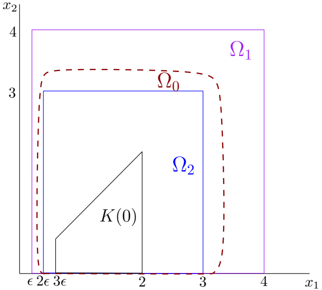
Theorem (K-Ryzhik-Yao-Zlatos '16)

If $0 < \alpha < 1/24$, there exist initial data $\omega_0 \in H^3$ such that the corresponding patch solution blows up in finite time.

The initial data: odd, two patches.



Patch structure:



On the right, $K(0)$ is the initial barrier that evolves according to

$$K(t) = \{x : x_2 \le x_1, \ X(t) \le x_1 \le 1\}, \ X' = -\frac{1}{50}X^{1-2\alpha}, \ X(0) = 3\epsilon.$$

Note that $K(t)$ arrives at $x_1 = 0$ in time $\tau \sim \epsilon^{2\alpha}$.

Plan: show that $K(t) \subset \Omega(t)$ while the patch $\Omega(t)$ remains regular.

Let D^+ be the first quadrant in $\mathbb{R}^2$. Focus on estimating u_1 (u_2 is similar).

$$u_1(x) = -\int_{D^+} K_1(x,y)\omega(y)dy,$$

$$K_1(x,y) = \frac{y_2 - x_2}{|x-y|^{2+2\alpha}} - \frac{y_2 - x_2}{|x-\tilde{y}|^{2+2\alpha}} - \frac{y_2 + x_2}{|x+y|^{2+2\alpha}} + \frac{y_2 + x_2}{|x-\bar{y}|^{2+2\alpha}}$$

$$\geq \frac{y_2 - x_2}{|x-y|^{2+2\alpha}} - \frac{y_2 - x_2}{|x-\tilde{y}|^{2+2\alpha}}, \quad \tilde{y} = (-y_1, y_2), \quad \bar{y} = (y_1, -y_2).$$

Lemma

Suppose $0 \leq \omega \leq 1$ *in* D_+. *If* $x \in D^+$ *and* $x_2 \leq x_1$, *then*

$$u_1^{bad}(x) \equiv \int_{\mathbb{R}^+ \times (0,x_2)} K_1(x,y)\omega(y)\, dy \leq \frac{1}{\alpha}\left(\frac{1}{1-2\alpha} - 2^{-\alpha}\right) x_1^{1-2\alpha}.$$

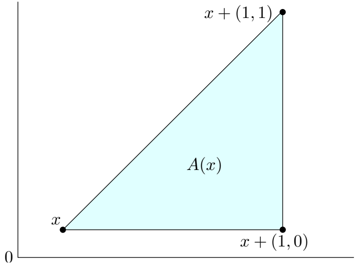
To estimate the "good" part of u_1, assume that $\omega(x) = 1$ on the set

$$A(x) := \{y \,:\, y_1 \in (x_1, x_1+1),\;\; y_2 \in (x_2, x_2 + y_1 - x_1)\}$$

Lemma

Assume that for some x we have $\omega \geq \chi_{A(x)}$. There exists $\delta_\alpha > 0$ s. t. if $x_1 \leq \delta_\alpha$, then

$$u_1^{good}(x) \leq -\frac{1}{6 \cdot 20^\alpha \alpha} x_1^{1-2\alpha}.$$



After cancellations, u_1^{good} and u_1^{bad} can be bounded by integrals of $\frac{x_2 - y_2}{|x-y|^{2+2\alpha}}$ over G and B respectively.

For small α, kernel is long range and G wins. For α closer to $1/2$, B wins!