CALCULUS EXERCISES 2 — Numerical Methods and Estimation

1. Use calculus, or trigonometric identities, to prove the following inequalities for θ in the range $0 < \theta < \frac{\pi}{2}$:

- $\sin\theta < \theta$;
- $\theta < \tan\theta$;
- $\cos 2\theta < \cos^2\theta$.

Hence, without directly calculating the following integrals, rank them in order of size.

$$\text{(a)} \int_0^1 x^3 \cos x \, \mathrm{d}x, \qquad \text{(b)} \int_0^1 x^3 \cos^2 x \, \mathrm{d}x, \qquad \text{(c)} \int_0^1 x^2 \sin x \cos x \, \mathrm{d}x, \qquad \text{(d)} \int_0^1 x^3 \cos 2x \, \mathrm{d}x.$$

2. Show that the equation

$$\sin x = \frac{1}{2}x$$

has three roots. Using Newton-Raphson, or a similar numerical method, find the positive root to 6 d.p.

The equation $\sin x = \lambda x$ has three real roots when $\lambda = \alpha$ or when $\beta < \lambda < 1$ for two real numbers $\alpha < 0 < \beta$. Plot, on the same axes, the curves

$$y = \sin x, \qquad y = \alpha x, \qquad y = \beta x.$$

3. Let S denote the circle in the xy-plane with centre (0,0) and radius 1. A regular m-sided polygon I_m is inscribed in S and a regular n-sided polygon C_n is circumscribed about S.

(a) By considering the perimeter of I_m and the area bounded by C_n, prove that:

$$m \sin\left(\frac{\pi}{m}\right) < \pi < n \tan\left(\frac{\pi}{n}\right),$$

for all natural numbers $m, n \geq 3$.

(b) Archimedes showed (using this method) that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. What are the smallest values of m and n needed to verify Archimedes' inequality?

4. Find the coefficients of $1, x, x^2, x^3, x^4$ in the power series expansion (Taylor's series expansion) for $f(x) = \sec x$.

Use this approximation to make an estimate for $\sec\frac{1}{10}$. With the aid of a calculator, find to how many decimal places the approximation is accurate.

5. Show that $\int \ln x \, \mathrm{d}x = x\ln x - x + \text{constant}$

Sketch the graph of the equation $y = \ln x$. By consideration of areas on your graph, show that

$$n \ln n - n + 1 < \sum_1^n \ln r < (n+1)\ln(n+1) - n.$$

Let $G_n = \sqrt[n]{n!}$ denote the geometric mean of 1, 2, ..., n. Show that G_n/n approaches $1/e$ as n becomes large