CALCULUS EXERCISES 4 — Differential Equations

1. Find the general solutions of the following separable differential equations.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos^2 x}{\cos^2 2y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = e^{x+2y}.$$

2. Find the solution of the following initial value problems. On separate axes sketch the solution to each problem.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1-2x}{y}, \qquad y(1) = -2, \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x\left(x^2+1\right)}{4y^3}, \qquad y(0) = \frac{-1}{\sqrt{2}}, \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1+y^2}{1+x^2} \text{ where } y(0) = 1. \end{aligned}$$

3. The equation for *Simple Harmonic Motion,* with constant frequency ω, is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x.$$

Show that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

where $v = \mathrm{d}x/\mathrm{d}t$ denotes velocity. Find and solve a separable differential equation in v and x given that $x = a$ when $v = 0$.

Hence show that

$$x(t) = a\sin(\omega t + \varepsilon)$$

for some constant ε.

4. Find the most general solution of the following homogeneous constant coefficient differential equations:

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y &= 0, \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y &= 0, \quad \text{where } y(0) = y'(0) = 1, \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y &= 0, \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y &= 0, \quad \text{where } y(0) = y'(0) = 1. \end{aligned}$$

5. Write the left hand side of the differential equation

$$(2x+y) + (x+2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x,y)) = 0,$$

where $F(x,y)$ is a polynomial in x and y. Hence find the general solution of the equation.

Use this method to find the general solution of

$$(y\cos x + 2xe^y) + \left(\sin x + x^2e^y - 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$