CALCULUS EXERCISES 5 — Further Differential Equations

1. Find all solutions of the following separable differential equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-xy}{xy-x},$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin^{-1}x}{y^2\sqrt{1-x^2}}, \qquad y(0)=0.$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(1+3x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \quad \text{where } y(1)=0 \text{ and } y'(1)=\frac{-1}{2}.$$

2. Use the method of integrating factors to solve the following equations with initial conditions

$$\frac{\mathrm{d}y}{\mathrm{d}x}+xy = x \quad \text{where} \quad y(0)=0,$$
$$2x^3\frac{\mathrm{d}y}{\mathrm{d}x}-3x^2y = 1 \quad \text{where} \quad y(1)=0,$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}-y\tan x = 1 \quad \text{where} \quad y(0)=1.$$

3. Find the most general solution of the following inhomogeneous constant coefficient differential equations:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y = x,$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y = \sin x,$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y = e^x,$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y = e^{-x}.$$

4. (a) By making the substitution $y(x) = xv(x)$ in the following *homogeneous polar equations*, convert them into separable differential equations involving v and x, which you should then solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2+y^2}{xy},$$
$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y+\sqrt{x^2+y^2}.$$

(b) Make substitutions of the form $x = X+a$, $y = Y+b$, to turn the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y-3}{x-y-1}$$

into a homogeneous polar differential equation in X and Y. Hence find the general solution of the above equation.

5. A particle P moves in the xy-plane. Its co-ordinates $x(t)$ and $y(t)$ satisfy the equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x+y \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = x-y,$$

and at time $t=0$ the particle is at $(1,0)$. Find, and solve, a homogeneous polar equation relating x and y.

By changing to polar co-ordinates $\left(r^2 = x^2+y^2,\ \tan\theta = y/x\right)$, sketch the particle's journey for $t \geq 0$.