COMPLEX NUMBERS EXERCISES

1. By writing $\omega = a + ib$ (where a and b are real), solve the equation

$$\omega^2 = -5 - 12i.$$

Hence find the two roots of the quadratic equation

$$z^2 - (4+i)\,z + (5+5i) = 0.$$

2. By substituting $z = x + iy$ or $z = re^{i\theta}$ into the following equations and inequalities, sketch the following regions of the complex plane on separate Argand diagrams:

- $|z - 3 - 4i| < 5$,
- $\arg(z) = \pi/3$
- $0 \leq \mathrm{Re}((iz+3)/2) < 2$,
- $e^z = 1$,
- $\mathrm{Im}(z^2) < 0$.

3. Find the image of the point $z = 2 + it$ under each of the following transformations.

- $z \mapsto iz$,
- $z \mapsto z^2$,
- $z \mapsto e^z$,
- $z \mapsto 1/z$.

By letting t vary over all real values find the image of the line $\mathrm{Re}\, z = 2$ under the same transformations.

4. (a) Given that $e^{i\theta} = \cos\theta + i\sin\theta$, prove that

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

(b) Use *De Moivre's Theorem* to show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

5. (a) Let $z = \cos\theta + i\sin\theta$ and let n be an integer. Show that

$$2\cos\theta = z + \frac{1}{z} \quad \text{and that} \quad 2i\sin\theta = z - \frac{1}{z}.$$

Find expressions for $\cos n\theta$ and $\sin n\theta$ in terms of z.

(b) Show that

$$\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$$

and hence find $\int_0^{\pi/2} \cos^5\theta \; \mathrm{d}\theta$.