DYNAMICS 1 – Basic Definitions. Newton's Second Law.

1. A particle, of mass m, has position vector

$$\mathbf{r}(t) = (x(t), y(t)) = (3\sin 2t + 4\cos 2t, 3t + 2)\,.$$

at time t.

(i) Determine the particle's *momentum* $m(\mathrm{d}\mathbf{r}/\mathrm{d}t)$ at time t.

(ii) Determine the particle's *kinetic energy* $\frac{1}{2}m\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|^2$ at time t.

(iii) At what times is the particle's kinetic energy maximal?

(iv) Determine the particle's *acceleration* $\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$ at time t. Show that

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -4x(t)\mathbf{i}.$$

2. Consider a particle of mass m moving in one vertical dimension with height $y(t)$. It moves under gravity, so that its acceleration always satisfies $\mathrm{d}^2y/\mathrm{d}t^2 = -g$. Initially the particle is projected from ground-level with speed v. That is, at $t = 0$, we have $y = 0$ and $\mathrm{d}y/\mathrm{d}t = v$.

(i) Determine $y(t)$.

(ii) What is the greatest height achieved by the particle?

(iii) Find the time taken to return to ground-level.

(iv) Show that the quantity

$$E = \frac{1}{2}m\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + mgy$$

is constant throughout the motion.

3. A particle of mass m moves along the x-axis under the force $F(t)$, at time t, given below.

$$F(t) = \begin{cases} 3 & 0 \leqslant t < 2, \\ 1 & 2 \leqslant t < 3, \\ 2 & 3 \leqslant t \leqslant 5, \end{cases}$$

and otherwise moves under no force. Initially, at $t = 0$, the particle is at rest at $x = 0$.

Newton's Second Law states that

$$F(t) = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}.$$

Determine x and $\mathrm{d}x/\mathrm{d}t$. On separate axes, sketch graphs of x and $\mathrm{d}x/\mathrm{d}t$ against t.

4. A particle of mass m moves along the x-axis under a force $F(x)$, when at position x, given below

$$F(x) = \begin{cases} -kx^3 & -a < x < a, \\ 0 & |x| \geqslant a. \end{cases}$$

Initially, at $t = 0$, we have $x = 0$ and $\mathrm{d}x/\mathrm{d}t = u \geqslant 0$.

(i) Let $v = \mathrm{d}x/\mathrm{d}t$. Show that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}x}.$$

(ii) From Newton's Second Law, show that

$$\frac{1}{2}mv^2 + \frac{1}{4}kx^4 = E$$

is constant throughout the motion.

(iii) Find the minimum value U of u such that the particle moves outside of the interval $-a < x < a$. If $u < U$ what is the maximum value of x?

(iv) Show that if $u = U$ then the time T taken for the particle to reach $x = a$ equals

$$T = \sqrt{\frac{2m}{k}}\int_0^a \frac{\mathrm{d}x}{\sqrt{a^4 - x^4}}.$$