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PUBLIC EXAMINATION

Honour School of Mathematical and Theoretical Physics (MMathPhys)

Master of Science in Mathematical and Theoretical Physics (MScMTP)

Groups and Representations

2015

Solutions

1.) Let G be a group.

a) Define the terms "representation of a group", "irreducible representation", "faithful representation" and "unitary representation". [5]

b) State Schur's Lemma for an irreducible representation $R : G \to \mathrm{Gl}(V)$ on a complex vector space V. Use Schur's Lemma to show that a complex irreducible representation of an Abelian group must be one-dimensional. [6]

c) Consider the Abelian group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ which consists of $\{(0,0),(1,0),(0,1),(1,1)\}$, with the group operation being addition modulo 2. Write down the irreducible complex representations and the character table for this group. [6]

d) Let $V = \{ax^2 + bxy + cy^2 \,|\, a,b,c \in \mathbb{C}\}$ be the vector space of quadratic polynomials p in two variables x, y with complex coefficients. A map $R : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Gl}(V)$ is defined by $R((1,0))(p)(x,y) := p(x,-y)$, $R((0,1))(p)(x,y) := p(y,x)$ and $R((1,1)) := R((1,0))R((0,1))$. Why is R a representation? Work out the representation matrices for $R((1,0))$ and $R((0,1))$ relative to the standard monomial basis $\{x^2, xy, y^2\}$ of V. Find the character and the irreducible representation content of R. [8]

Solution:

a) A representation of a group G is a map $R : G \to \mathrm{Gl}(V)$ from the group to the automorphisms of a vector space V which satisfies $R(g_1g_2) = R(g_1)R(g_2)$ for all $g_1, g_2 \in G$. [1]
A representation $R : G \to \mathrm{Gl}(V)$ is called irreducible if there is no sub vector space $U \subset V$, other than the two trivial sub vector spaces, such that $R(g)(U) \subset U$ for all $g \in G$. [2]
A representation $R : G \to \mathrm{Gl}(V)$ is called faithful if it is injective. [1]
A representation $R : G \to \mathrm{Gl}(V)$ is called unitary if all $R(g)$ are unitary relative to some scalar product on V. (Or, alternatively, if all $R(g)$ are unitary matrices.) [1]

b) If $R : G \to \mathrm{Gl}(V)$ is a complex irreducible representation and $P : V \to V$ a linear map with $[R(g), P] = 0$ for all $g \in G$, then Schur's Lemma states that P must be a multiple of the unit matrix. [2]
Let $R : G \to \mathrm{Gl}(V)$ be a complex irreducible representation of G. If G is Abelian, it follows from the representation property that $[R(g), R(\tilde{g})] = 0$ for all $g, \tilde{g} \in G$. If we fix g and set $P = R(g)$ it follows that $[P, R(\tilde{g}] = 0$ for all $\tilde{g} \in G$. Hence, from Schur's Lemma, $P = R(g) = \lambda(g)\mathbb{1}$, for some $\lambda(g) \in \mathbb{C}$ and this holds for all $g \in G$. However, this form of R is only consistent with R being irreducible if the dimension of V is one. [4]

c) Since G is an Abelian group each element is its own conjugacy class. Hence we have four classes and four irreducible, complex representations, denoted $R_{(q_1,q_2)}$, where $q_i \in \{0,1\}$. They are explicitly given by

$$R_{(q_1,q_2)}((k_1,k_2)) = (-1)^{q_1k_1+q_2k_2}\ .$$

Then, the character table is given by [3]

	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	1	1	1	1
(1,0)	1	-1	1	-1
(0,1)	1	1	-1	-1
(1,1)	1	-1	-1	1

[3]

d) The map R is a representation because applying $R((1,0)$ or $R((0,1)$ twice to the polynomials gives the identity map, in line with the group multiplication rules. [1]
We have $R((1,0))(x^2) = x^2$, $R((1,0))(xy) = -xy$ and $R((1,0))(y^2) = y^2$. Further, for the other generator, we have $R((0,1))(x^2) = y^2$, $R((0,1)(xy) = xy$ and $R((0,1))(y^2) = x^2$. From this we can easily read off the representation matrices

$$\tilde{R}((1,0)) = \mathrm{diag}(1,-1,1)\,, \quad \tilde{R}((0,1)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,.$$

Since $\mathrm{tr}(\tilde{R}((1,0))) = 1$, $\mathrm{tr}(\tilde{R}((0,1)) = 1$ and $\mathrm{tr}(\tilde{R}((1,0))\tilde{R}((0,1))) = -1$ the character of this representation is given by [3]

$$\chi = (3,1,1,-1)\,.$$

Dotting this character into the rows of the above character table shows that $(\chi_{(0,0)},\chi) = (\chi_{(0,1)},\chi) = (\chi_{(1,0)},\chi) = 1$ and $(\chi_{(1,1)},\chi) = 0$ and, hence, [2]

$$R = R_{(0,0)} \oplus R_{(1,0)} \oplus R_{(0,1)}\,.$$

[2]

2.) The quaternion group Q can be defined as a matrix group with the eight elements

$$Q = \{\pm\mathbb{1}_2, \pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3\}\ ,$$

where σ_i are the Pauli matrices, explicitly given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,.$$

a) Find the conjugacy classes of Q. How many irreducible, complex representations does Q have and what are the dimensions of these representations? [5]

b) Given the two one-dimensional representations

$$\begin{array}{llll} R_1(\pm\mathbb{1}_2) = 1 & R_1(\pm i\sigma_1) = 1 & R_1(\pm i\sigma_2) = -1 & R_1(\pm i\sigma_3) = -1 \\ R_2(\pm\mathbb{1}_2) = 1 & R_2(\pm i\sigma_1) = -1 & R_2(\pm i\sigma_2) = 1 & R_2(\pm i\sigma_3) = -1 \end{array}$$

of Q, write down the character table of Q and the remaining representation or representations. [8]

c) A four-dimensional representation R_4 of Q is given by $R_4(\pm \mathbb{1}_2) = \pm \mathbb{1}_4$ and

$$
\begin{aligned}
R_4(\pm i\sigma_1) &= \pm \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad R_4(\pm i\sigma_2) = \pm \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
R_4(\pm i\sigma_3) &= \pm \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Convince yourself that R_4 is indeed a representation of Q and determine its irreducible representation content. [6]

d) What is the irreducible representation content of $R_4 \oplus R_4$ and $R_4 \otimes R_4$? [6]

Solution:

a) Using the properties $\sigma_i^2 = \mathbb{1}_2$ and $\sigma_1\sigma_2 = i\sigma_3$ etc. of the Pauli matrices is it easy to show that there are five conjugacy classes $C_0 = \{\mathbb{1}_2\}$, $C_- = \{-\mathbb{1}_2\}$ and $C_i = \{\pm i\sigma_i\}$. [2]
Hence, there must be five complex, irreducible representations. The squares of their dimensions have to sum up to the group order 8 (and we have the trivial representation) so there must be four one-dimensional representations and one two-dimensional representation. [3]

b) Of course we have the trivial representation, R_0, and the two-dimensional representation ρ given by the matrices we have used to define the group as well as the two representations R_1, R_2 given in the question. Hence, we are missing one one-dimensional representations, which we denote R_3. Its associated character χ_3 must satisfy $\chi_3(\mathbb{1}_2) = 1$ and it must be orthogonal to all other four characters. This fixes the character table of Q:

	C_0	C_-	C_1	C_2	C_3
# elements	1	1	2	2	2
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_ρ	2	-2	0	0	0

[6]

For a one-dimensional representation, such as R_3, the representation "matrices" equal the character, so we have

$$R_3(\pm \mathbb{1}_2) = 1 \qquad R_3(\pm i\sigma_1) = -1 \qquad R_3(\pm i\sigma_2) = -1 \qquad R_3(\pm i\sigma_3) = 1$$

[2]

c) It is easy to show that these 4×4 matrices multiply in the same way as the group elements, for example $R_4(\pm i\sigma_1)^2 = -\mathbb{1}_4$, in line with $(\pm i\sigma_1)^2 = -\mathbb{1}_2$. [2]
By taking the trace it follows that the character of R_4 is given by

$$\chi_4 = (4, -4, 0, 0, 0) \ .$$

From the above character table we have $(\chi_0, \chi_4) = 0$, $(\chi_i, \chi_4) = 0$ for $i = 1, 2, 3$ and $(\chi_\rho, \chi_4) = 2$ which means that

$$R_4 = \rho \oplus \rho \ .$$

[5]

d) Of course $R_4 \oplus R_4$ contains four copies of ρ. [1]
Further $R_4 \otimes R_4 = (\rho \oplus \rho) \otimes (\rho \oplus \rho) = 4(\rho \otimes \rho)$. The character of $\rho \otimes \rho$ is given by $\chi_{\rho\otimes\rho}(g) = \chi_\rho(g)^2$ and hence, $\chi_{\rho\otimes\rho} = (4, 4, 0, 0, 0)$. Dotting into the character table this means $(\chi_0, \chi_{\rho\otimes\rho}) = 1$, $(\chi_i, \chi_{\rho\otimes\rho}) = 1$ for $i = 1, 2, 3$ and $(\chi_\rho, \chi_{\rho\otimes\rho}) = 0$, so that

$$\rho \otimes \rho = R_0 \oplus R_1 \oplus R_2 \oplus R_3 \ .$$

As a result, $R_4 \otimes R_4$ contains four copies of $R_0 \oplus R_1 \oplus R_2 \oplus R_3$. [5]

3.) Consider the group $SU(4)$ of 4×4 special unitary matrices.

a) Determine the Lie-algebra and the Cartan sub-algebra of $SU(4)$. What is the dimension and the rank of this Lie algebra? Write down a simple basis for the Cartan sub-algebra. [6]

b) For the fundamental representation, $\mathbf{4}$, of $SU(4)$, find the weights of the standard unit vectors $\mathbf{e}_i$, where $i = 1, \ldots, 4$, in $\mathbb{C}^4$. What are the weights of the complex conjugate of the fundamental representation, $\bar{\mathbf{4}}$? [4]

c) Using Young tableaux, find the irreducible representations in $\mathbf{4} \otimes \mathbf{4}$ and $\mathbf{4} \otimes \bar{\mathbf{4}}$. For $\mathbf{4} \otimes \mathbf{4}$, identify the tensors associated to these irreducible representations. [8]

d) Consider the $SU(3)$ sub-group of $SU(4)$ defined by the embedding

$$U = \begin{pmatrix} U_3 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

where $U_3 \in SU(3)$. How do the representations $\mathbf{4}$, $\bar{\mathbf{4}}$ and $\mathbf{4} \otimes \mathbf{4}$ branch under this $SU(3)$ sub-group? [7]

Solution:

a) Writing $U = \mathbb{1}_4 + iT + \cdots$ and inserting into $U^\dagger U = \mathbb{1}_4$ and $\det(U) = 1$ leads to $T = T^\dagger$ and $\mathrm{tr}(T) = 0$. So

$$\mathcal{L}(SU(4)) = \{T \,|\, T = T^\dagger \text{ and } \mathrm{tr}(T) = 0\} \ .$$

[2]

The Cartan sub-algebra is given by the diagonal matrices $T = \mathrm{diag}(a_1, a_2, a_3, a_4)$ with $a_i \in \mathbb{R}$ and $\sum_{i=1}^4 a_i = 0$. [1]
There are 6 complex entries above the diagonal of T, making for 12 real degrees of freedom, and four real ones along the diagonal, subject to the trace condition, so another three for a total of 15. Hence, $\dim(\mathcal{L}(SU(4))) = 15$. [1]
The dimension of the Cartan sub-algebra is clearly three, so $\mathrm{rk}(\mathcal{L}(SU(4))) = 3$. [1]
A simple basis for the Cartan sub-algebra is

$$Y_1 = \mathrm{diag}(1, -1, 0, 0)\,, \quad Y_2 = \mathrm{diag}(0, 1, -1, 0)\,, \quad Y_3 = \mathrm{diag}(0, 0, 1, -1)\;.$$

[1]

b) The weights $\mathbf{w}_i$ for the unit vector $\mathbf{e}_i$ can be read off from the equations $Y_a \mathbf{e}_i = w_{ia} \mathbf{e}_i$ and this leads to

$$\mathbf{w}_1 = (1, 0, 0)\,, \quad \mathbf{w}_2 = (-1, 1, 0)\,, \quad \mathbf{w}_3 = (0, -1, 1)\,, \quad \mathbf{w}_4 = (0, 0, -1)\;.$$

These are the weights in the fundamental representation $\mathbf{4}$. [3]
The complex conjugate $\bar{\mathbf{4}}$ simply has weights $-\mathbf{w}_i$. [1]

c) Following the rules for Young-tableaux we have

$$\mathbf{4} \otimes \mathbf{4} \quad \sim \quad \ydiagram{1} \quad \otimes \quad \begin{ytableau} a \end{ytableau} \quad = \quad \begin{ytableau} \ \\ a \end{ytableau} \quad \oplus \quad \begin{ytableau} \ & a \end{ytableau} \quad = \quad \mathbf{6} \oplus \mathbf{10}$$

$$\bar{\mathbf{4}} \otimes \mathbf{4} \quad \sim \quad \begin{ytableau} \ \\ \ \\ \ \\ \ \end{ytableau} \quad \otimes \quad \begin{ytableau} a \end{ytableau} \quad = \quad \begin{ytableau} \ \\ \ \\ \ \\ \ \\ a \end{ytableau} \quad \oplus \quad \begin{ytableau} \ & a \\ \ \\ \ \\ \ \end{ytableau} \quad = \quad \mathbf{1} \oplus \mathbf{15}$$ [6]

The $\mathbf{6}$ representation corresponds to an anti-symmetric two-index tensor and the $\mathbf{10}$ representation to a symmetric two-index tensor. [2]

d) Clearly, from the given embedding we have the branchings $\mathbf{4} \to \mathbf{3} \oplus \mathbf{1}$ and $\bar{\mathbf{4}} \to \bar{\mathbf{3}} \oplus \mathbf{1}$. [3]
Further

$$\mathbf{4} \otimes \mathbf{4} = (\mathbf{3} \oplus \mathbf{1}) \otimes (\mathbf{3} \oplus \mathbf{1}) = (\mathbf{3} \otimes \mathbf{3}) \oplus \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{1} = \bar{\mathbf{3}} \oplus \mathbf{6} \oplus \mathbf{3} \oplus \mathbf{3} \oplus \mathbf{1}\;.$$

[4]

4.) Consider the group $SO(7)$ of 7×7 special orthogonal matrices.

a) Determine the Lie algebra, the Cartan sub-algebra, the dimension and the rank for this group. [5]

b) The Cartan matrix for the associated algebra B_3 is given by

$$A(B_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} .$$

Find the weight systems for the representations with highest Dynkin weight $(1,0,0)$. [7]

c) Find the weight system for the representations with highest Dynkin weight $(0,0,1)$. [7]

d) The algebra A_3, associated to $SU(4)$, can be embedded into B_3 via the projection matrix

$$P(A_3 \subset B_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} .$$

Using this matrix, determine the branching under $SU(4)$ of the $SO(7)$ representations with highest weight $(0,0,1)$ from part **4.**) c). [6]

Solution:

a) Writing $R = 1 + T + \cdots$ the orthogonality condition $R^T R = \mathbb{1}_7$ implies that $T = -T^T$, so

$$\mathcal{L}(SO(7)) = \{T \,|\, T = -T^T\} .$$

[1]

The Cartan sub-algebra consists of matrices of the form

$$\mathrm{diag}\left(\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, 0\right) ,$$

where $a, b, c \in \mathbb{R}$. [2]
The 7×7 anti-symmetric matrices form a 21-dim. vector space so $\dim(\mathcal{L}(SO(7))) = 21$. [1]
Clearly, $\mathrm{rk}((\mathcal{L}(SO(7))) = 3$. [1]

b) The three rows of the Cartan matrix are the three positive simple roots α_1, α_2 and α_3. For the representation with highest weight $(1,0,0)$ we have

$$(1,0,0) \stackrel{\alpha_1}{\to} (-1,1,0) \stackrel{\alpha_2}{\to} (0,-1,2) \stackrel{\alpha_3}{\to} (0,0,0) \stackrel{\alpha_3}{\to} (0,1,-2) \stackrel{\alpha_2}{\to} (1,-1,0) \stackrel{\alpha_1}{\to} (-1,0,0) .$$

[7]

c) For the representation with highest weight $(0,0,1)$ we have

$$(0,0,1) \stackrel{\alpha_3}{\to} (0,1,-1) \stackrel{\alpha_2}{\to} (1,-1,1) \stackrel{\alpha_1,\alpha_3}{\to} \begin{matrix} (1,0,-1) \\ (-1,0,1) \end{matrix} \stackrel{\alpha_1,\alpha_3}{\to} (-1,1,-1) \stackrel{\alpha_2}{\to} (0,-1,1) \stackrel{\alpha_3}{\to} (0,0,-1) .$$

[7]

d) Acting with the given matrix $P(A_3 \subset B_3)$ on the eight weights above gives the A_3 weights

$$(0,0,1),\ (1,0,0),\ (-1,1,0),\ (0,1,-1),\ (0,-1,1),\ (1,-1,0),\ (-1,0,0),\ (0,0,-1)\ .$$

Using the Cartan matrix [3]

$$A(A_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

of A_3 one can generate the weights of the representations $\mathbf{4}$, starting from the highest weight $(1,0,0)$, and the weights of the representation $\bar{\mathbf{4}}$, starting from the highest weight $(0,0,1)$. Together, these two weight systems coincide with the above list of weights so that

$$\mathbf{8}_{B_3} \to (\mathbf{4} \oplus \bar{\mathbf{4}})_{A_3}\ .$$

[3]