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PUBLIC EXAMINATION

Honour School of Mathematical and Theoretical Physics (MMathPhys)

Master of Science in Mathematical and Theoretical Physics (MScMTP)

Groups and Representations

2016

Solutions

1.) Let G be a group.

a) Provide a set of necessary and sufficient conditions for a subset H of G to be a sub-group. Define the term "normal sub-group". Show that, for a normal sub-group $H \subset G$, the quotient G/H is a group. [5]

b) From now on focus on the group $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ with the group multiplication being the addition modulo 4. Write down all the complex irreducible representations and the character table of this group and check explicitly that the characters are ortho-normal. [6]

c) Show that $H = \{0, 2\}$ is a normal subgroup of G and write down the complex irreducible representations of H. For each complex irreducible representation of G from part b), find the H representation it branches to under the restriction of G to H. [6]

d) Given the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

show that $R : G \to \mathrm{Gl}(\mathbb{C}^4)$ defined by $R(g) := M^g$ is a representation of G and determine which irreducible representations of G it contains. Under the restriction of G to H, which irreducible H representations does R contain? [8]

Solution:

a) $H \subset G$ is a sub-group of G iff $e \in H$, for every $h \in H$ we have $h^{-1} \in H$ and H is closed under multiplication. [1]
A sub-group $H \subset G$ is called normal iff $gH = Hg$ for all $g \in G$. [1]
The quotient G/H consists of the cosets gH with multiplication defined by $(g_1H)(g_2H) := (g_1g_2)H$. For two other representatives g_1h_1 and g_2h_2 of the same cosets we have

$$(g_1h_1)H(g_2h_2)H = (g_1h_1g_2h_2)H = (g_1h_1g_2)H = g_1h_1Hg_2 = g_1Hg_2 = (g_1g_2)H$$

where normality of H has been used in the final steps. Hence the multiplication is independent of the choice of representative and well-defined. [2]
The so-defined multiplication is clearly associative, eH is the neutral element and the inverse for gH is $g^{-1}H$. Hence G/H is a group. [1]

b) The four complex, irreducible representations of $\mathbb{Z}_4$ are one-dimensional and given by

$$R_q(g) = i^{qg} , \qquad q = 0, 1, 2, 3 ,$$

[3]

The associated characters are $\chi_q(g) = \mathrm{tr}(R_q(g)) = i^{qg}$ which leads to the character table

	0	1	2	3
R_0	1	1	1	1
R_1	1	i	-1	-i
R_2	1	-1	1	-1
R_3	1	-i	-1	i

It follows that $\langle \chi_q, \chi_p \rangle = \frac{1}{4} \sum_g \chi_q(g)^* \chi_p(g) = \delta_{qp}$. [3]

c) Clearly, $H = \{0, 2\}$ is closed under addition mod 4, contains the unit element and an inverse for each element and is, hence, a sub-group which is isomorphic to $\mathbb{Z}_2$. The normality condition $gH = Hg$ is trivially satisfied in the Abelian case so H is a normal sub-group. [1]
The complex irreducible representations of H are the standard $\mathbb{Z}_2$ representations

$$\tilde{R}_p(g) = (-1)^{pg/2} \ , \qquad p = 0, 1$$

For $g \in H$ we have $R_q(g) = (-1)^{qg/2} = \tilde{R}_p(g)$, where $p = q \bmod 2$. Hence, the branching is [2]

$$R_q \to \tilde{R}_{q \bmod 2} \ . \tag{1}$$

[3]

d) Since $M^4 = 1$ we have

$$R(g_1)R(g_2) = M^{g_1} M^{g_2} = M^{g_1+g_2} = M^{(g_1+g_2) \bmod 4} = R(g_1 g_2)$$

and, hence, R is a representation. [2]
It is easy to verify that $\mathrm{tr}(M) = \mathrm{tr}(M^2) = \mathrm{tr}(M^3) = 0$ so the character of R is given by $\chi_R = (4, 0, 0, 0)$. Dotting this into the rows of the character table shows that

$$R = R_0 \oplus R_1 \oplus R_2 \oplus R_3 \ .$$

Combining this with the result (1) from c) it follows that R branches as [4]

$$R \to 2\tilde{R}_0 \oplus 2\tilde{R}_1 \ .$$

[2]

2.) Define $\alpha = \exp(2\pi i/5)$ and the two matrices

$$\tau = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} , \qquad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

a) Show that τ and σ generate a group G of order 10 with elements

$$G = \{\tau^k \,|\, k = 0, \ldots, 4\} \cup \{\sigma\tau^k \,|\, k = 0, \ldots, 4\} \ .$$

Further, show that G has four conjugacy classes given by $C_1 = \{\mathbb{1}_2\}$, $C_\tau = \{\tau, \tau^4\}$, $C_{\tau^2} = \{\tau^2, \tau^3\}$ and $C_\sigma = \{\sigma\tau^k \,|\, k = 0, \ldots, 4\}$. Since G is given in terms of 2×2 matrices it trivially defines a two-dimensional complex representation R. Show that R is irreducible. [6]

b) How many complex, irreducible representations does G have and what are their dimensions? Find the character table of G. [8]

c) Fing the characters of the tensor representation $R \otimes R$ and find its irreducible representation content. [8]

d) Show that the complex conjugate representation R^* is equivalent to R. [3]

Solution:

a) It is clear that G must contain the 10 matrices given. It has to be shown that multiplication of these 10 matrices closes within the same set. To do this, first note that $\tau^5 = \mathbb{1}_2$, $\sigma^2 = \mathbb{1}_2$ and $\sigma\tau^k\sigma = \tau^{5-k}$. It follows

$$\begin{aligned} \tau^k\tau^l &= \tau^{k+l} \in G \\ \tau^k\sigma\tau^l &= \sigma\sigma\tau^k\sigma\tau^l = \sigma\tau^{5-k}\tau^l = \sigma\tau^{5-k+l} \in G \\ \sigma\tau^k\tau^l &= \sigma\tau^{k+l} \in G \\ \sigma\tau^k\sigma\tau^l &= \tau^{5-k}\tau^l = \tau^{5-k+l} \in G \, , \end{aligned}$$

so the multiplication does indeed close and G is a group of order 10. [2]
The unit $C_1 = \{\mathbb{1}_2\}$ always forms an equivalence class by itself. Since $\sigma\tau\sigma = \tau^4$ and $\sigma\tau^2\sigma = \tau^3$ we have the two classes $C_\tau = \{\tau, \tau^4\}$ and $C_{\tau^2} = \{\tau^2, \tau^3\}$. To see that the remaining five elements are conjugate work out $\tau^k(\sigma\tau^l)\tau^{5-k} = \sigma\tau^{l-k}$. [2]
If R was reducible there would be a one-dimensional invariant sub-space and, in particular, a non-zero vector v within this sub-space which is an eigenvector under all 10 matrices. It is easy to show that $\sigma v = \lambda v$ requires $v = (a, \pm a)^T$ and that such a v cannot be an eigenvector of τ. [2]

b) Since G has four conjugacy classes it has four complex, irreducible representations R_i, where $i = 1, 2, 3, 4$. Given that $\sum_i \dim(R_i)^2 = |G| = 10$ the only possibility is to have two one-dimensional and two two-dimensional representations. [2]
We order characters according to their values on $\{C_1, C_\tau, C_{\tau^2}, C_\sigma\}$. Then, we have the trivial representation R_1 with character $\chi_1 = (1, 1, 1, 1)$, the one-dimensional representation $R_2(g) = \det(g)$ with character $\chi_2 = (1, 1, 1, -1)$ and the two-dimensional representation $R_3 = R$ with character $\chi_3 = (2, \alpha + \alpha^4, \alpha^2 + \alpha^3, 0)$. There is one remaining two-dimensional representation R_4 with character $\chi_4 = (2, a, b, c)$ and we can determine a, b, c by demanding that $\langle \chi_i, \chi_4 \rangle = 0$ for $i = 1, 2, 3$. This leads to $\chi_4 = (2, \alpha^2 + \alpha^3, \alpha + \alpha^4, 0)$. Hence, the character table is

	C_1	C_τ	C_{τ^2}	C_σ
R_1	1	1	1	1
R_2	1	1	1	-1
$R_3 = R$	2	$\alpha + \alpha^4$	$\alpha^2 + \alpha^3$	0
R_4	2	$\alpha^2 + \alpha^3$	$\alpha + \alpha^4$	0

[6]

c) We have $\chi_{R\otimes R}(g) = \chi_3(g)^2$ which means that $\chi_{R\otimes R} = (4, 2+\alpha^2+\alpha^3, 2+\alpha+\alpha^4, 0)$. [3]
Working out $n_i = \langle \chi_i, \chi_{R\otimes R}\rangle$ gives $(n_1, n_2, n_3, n_4) = (1, 1, 0, 1)$ and, hence,

$$R \otimes R = R_1 \oplus R_2 \oplus R_4 \ .$$

[5]

d) By taking the complex conjugate of the matrices defining G we see immediately that $\chi_{R^*} = \chi_R$. Since R and R^* have the same character they must be equivalent. [3]

3.) Consider the group $SU(6)$ of 6×6 unitary matrices with determinant one.

a) Find the Lie algebra $su(6)$ and the Cartan sub-algebra of $SU(6)$. What is the dimension of this Lie algebra and what is the rank of $SU(6)$? [4]

b) For the $SU(6)$ representations with highest Dynkin weights $(1\,0\,0\,0\,0)$, $(0\,0\,0\,0\,1)$ and $(0\,1\,0\,0\,0)$ write down the associated Young tableaux, the associated tensors in index notation and the dimensions. [5]

c) Construct all possible $SU(6)$ singlets which are cubic in the tensors found in part b). [4]

d) Consider the sub-group $SU(5) \times U(1)$ of $SU(6)$, where the $SU(5)$ factor is embedded in the standard way as

$$SU(5) \ni U_5 \to \begin{pmatrix} U_5 & 0 \\ 0 & 1 \end{pmatrix} \in SU(6) \ .$$

Find the branching of the $SU(6)$ representations in part b) under this $SU(5) \times U(1)$ sub-group. [8]

e) Write the singlets found in part c) in terms of the $SU(5) \times U(1)$ representations in d). [4]

Solution:

a) Writing $U = 1 + T + \cdots$, inserting this into $U \dagger U = \mathbb{1}$ and $\det(U) = 1$ and working these relations out to linear order in T one finds that $T = -T^\dagger$ and $\mathrm{tr}(T) = 0$. Hence

$$su(6) = \{T \,|\, T = -T^\dagger \text{ and } \mathrm{tr}(T) = 0\} \ .$$

Counting real degree of freedoms in 6×6 anti-hermitian, traceless matrices gives $\dim(su(6)) = 35$. [2]
The Cartan sub-algebra $\mathcal{H}$ of $su(6)$ is given by

$$\mathcal{H} = \left\{ \mathrm{diag}(ia_1, \ldots, ia_6) \,|\, \sum_{i=1}^{6} a_i = 0 \right\} .,$$

so that $\mathrm{rk}(su(6)) = 5$. [2]

b) Using the standard relations between these objects we get:

Dynkin label	Young tableau	tensor	dimension
(1 0 0 0 0)	$\ydiagram{1}$	ψ_a	6
(0 1 0 0 0)	$\ydiagram{1,1,1,1,1}$	$\chi_{abcde} \leftrightarrow \chi^a$	6
(0 1 0 0 0)	$\ydiagram{2}$	$\phi_{[ab]}$	15

[5]

c) The two possibilities are $\epsilon^{abcdef}\phi_{ab}\phi_{cd}\phi_{ef}$ and $\chi_1^a\chi_2^b\phi_{ab}$ so we have singlets in $\mathbf{15}^3$ and in $\bar{\mathbf{6}}^2 \otimes \mathbf{15}$. [4]

d) Splitting up indices up as $a = (i, 6)$, where $i = 1, \ldots 5$ we have

$$\begin{aligned} \psi_a &\to (\psi_i, \psi_6) & \mathbf{6} &\to \mathbf{5} \oplus \mathbf{1} \\ \chi^a &\to (\chi^i, \chi^6) & \bar{\mathbf{6}} &\to \bar{\mathbf{5}} \oplus \mathbf{1} \\ \phi_{[ab]} &\to (\phi_{[ij]}, \phi_{i6}) & \mathbf{15} &\to \mathbf{10} \oplus \mathbf{5} \end{aligned}$$

The $U(1)$ in $SU(5) \times U(1)$ is embedded into $SU(6)$ via [3]

$$g = \mathrm{diag}(e^{i\alpha}, \ldots, e^{i\alpha}, e^{-5i\alpha}) \in SU(6)\ .$$

From this the $U(1)$ charges are easily read off and we get [2]

$$\mathbf{6} \to \mathbf{5}_1 \oplus \mathbf{1}_{-5}\ , \qquad \bar{\mathbf{6}} \to \bar{\mathbf{5}}_{-1} \oplus \mathbf{1}_5\ , \qquad \mathbf{15} \to \mathbf{10}_2 \oplus \mathbf{5}_{-4}\ .$$

[3]

e) $\epsilon^{abcdef}\phi_{ab}\phi_{cd}\phi_{ef} = 6\epsilon^{ijklm}\phi_{ij}\phi_{kl}\phi_{m6}$ means that the singlet in $\mathbf{15}^3$ goes to the singlet in $\mathbf{10}_2 \otimes \mathbf{10}_2 \otimes \mathbf{5}_{-4}$. [2]
Further, $\chi_1^a\chi_2^b\phi_{ab} = \chi_1^i\chi_2^j\phi_{ij} + (\chi_1^i\chi_2^6 - \chi_1^6\chi_2^i)\phi_{i6}$ implies that the singlet in $\bar{\mathbf{6}}^2 \otimes \mathbf{15}$ goes to the singlets in $\bar{\mathbf{5}}_{-1} \otimes \bar{\mathbf{5}}_{-1} \otimes \mathbf{10}_2$ and $\mathbf{1}_5 \otimes \mathbf{5}_{-4} \otimes \bar{\mathbf{5}}_{-1}$. [2]

4.) The Cartan matrix $A(G_2)$ and the metric $G(G_2)$ of the exceptional Lie algebra G_2 are given by

$$A(G_2) = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\ , \qquad G(G_2) = \frac{1}{3}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}\ .$$

a) Construct the weight systems of the G_2 representations with highest Dynkin weights (1 0) and (0 1). What are the dimensions of these representations (assuming the (0 0) weight in the representation (1 0) has degeneracy 2 and all other weights are non-degenerate)? [8]

b) What are the values of quadratic Casimir for the representations in a)? [5]

c) The projection matrix for the sub-algebra $A_2 \subset G_2$ is given by

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} .$$

Find the A_2 weights contained in the branching under A_2 of the G_2 representations from part a). [6]

d) From the results in part c), identify the $SU(3)$ representations which are contained in the G_2 representations from part a). [6]

Solution:

a) For the representation $(1\,0)$, applying the algorithm leads to the 13 weights

$\{$(1 0),(-1 3),(0 1),(1 -1),(-1 2),(2 -3),(0 0),(1 -2),(-2 3),(-1 1),(0 -1),(1 -3),(-1 0)$\}$

Since $(0\,0)$ has degeneracy two the dimension of this representation is **14**. [4]
For $(0\,1)$ the same methods leads to the weights

$\{$(0 1),(1 -1),(-1 2),(0 0),(1 -2),(-1 1),(0 -1)$\}$

of a seven-dimensional representation, **7**. [4]

b) The Casimir of a representation is given by $C = \langle \Lambda, \Lambda + 2\delta \rangle$, where Λ is the highest weight and $\delta = (1, \ldots, 1)$ in the Dynkin basis. [1]
This means

$$C(1\,0) = (1,0)\,G \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 8 \,, \qquad C(0\,1) = (0,1)\,G \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 4 \,.$$

[4]

c) Projecting the 13 weights for $(1\,0)$ using the matrix P one finds the A_2 weights

$\{$(-2 1),(-1 -1),(-1 0),(-1 1),(-1 2),(0 -1),(0 0),(0 1),(1 -2),(1 -1),(1 0),(1 1),(2 -1) [3]

For the representation $(0\,1)$ one finds analogously

$\{$(-1 0),(-1 1),(0 -1),(0 0),(0 1),(1 -1),(1 0)$\}$ [3]

d) With the $SU(3)$ Cartan matrix

$$A(SU(3)) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

it is easy to reconstruct the weight systems of the basic $SU(3)$ representations. The are

$(1\,0) \sim \mathbf{3}$: $\{$(1 0), (-1 1), (0 -1)$\}$
$(0\,1) \sim \bar{\mathbf{3}}$: $\{$(0 1), (1 -1), (-1 0)$\}$
$(1\,1) \sim \mathbf{8}$: $\{$(1 1),(2 -1),(-1 2),(0 0),(-2 1),(1 -2),(-1 -1)$\}$ [2]

Comparing with the projected G_2 weights for $(1\,0) \sim \mathbf{14}$ in part c) we find that $\mathbf{14} \to \mathbf{8} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$. [2]
For the G_2 representation $(0\,1) \sim \mathbf{7}$ we find in the same way that $\mathbf{7} \to \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1}$. [2]