INDUCTION EXERCISES 1

1. Factorials are defined inductively by the rule

$$0! = 1 \quad \text{and} \quad (n+1)! = n! \times (n+1).$$

Then binomial coefficients are defined for $0 \le k \le n$ by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Prove *from these definitions* that

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1},$$

and deduce the Binomial Theorem: that for any x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

2. Prove that

$$\sum_{r=1}^{n} \frac{1}{r^2} \le 2 - \frac{1}{n}.$$

3. Prove that for $n = 1, 2, 3, \ldots$

$$\sqrt{n} \le \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \le 2\sqrt{n} - 1.$$

4. Let $A = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$. Show that

$$A^n = 3^{n-1} \begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix}$$

for $n = 1, 2, 3, \ldots$ Can you find a matrix B such that $B^2 = A$?

5. Let k be a positive integer. Prove by induction on n that

$$\sum_{r=1}^{n} r(r+1)(r+2)\cdots(r+k-1) = \frac{n(n+1)(n+2)\cdots(n+k)}{k+1}.$$

Show now by induction on k that

$$\sum_{r=1}^{n} r^k = \frac{n^{k+1}}{k+1} + E_k(n)$$

where $E_k(n)$ is a polynomial in n of degree at most k.