Induction

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Abstract

Induction is an extremely powerful method of proof used throughout mathematics. It deals with infinite families of statements which come in the form of lists. The idea behind induction is in showing how each statement follows from the previous one on the list — all that remains is to kick off this logical chain reaction from some starting point.

Notation 1 *Throughout this article we shall use the notation* $\mathbb{N}$ *to denote the set of natural numbers* $\{0, 1, 2, 3, \ldots\}$. *So when we write that a statement is true for all* $n \in \mathbb{N}$ *then we are simply saying that the statement is true in each of the cases* $n = 0, 1, 2, 3, \ldots$

The symbol $n!$ *read 'n factorial' denotes the product* $1 \times 2 \times 3 \times \cdots \times n$. *As a convention* $0! = 1$.

1 Introduction (M)

Mathematical statements can come in the form of a single proposition such as

$$3 < \pi \quad \text{or as} \quad 0 < x < y \implies x^2 < y^2,$$

but often they come as a family of statements such as

$$\begin{array}{ll} A & e^x > 0 \quad \text{for all real numbers } x; \\ B & 0 + 1 + 2 + \cdots + n = \dfrac{1}{2} n\,(n+1) \quad \text{for } n \in \mathbb{N}; \\ C & \displaystyle\int_0^\pi \sin^{2n} \theta \; \mathrm{d}\theta = \frac{(2n)!}{(n!)^2} \frac{\pi}{2^{2n}} \quad \text{for } n \in \mathbb{N}; \\ D & 2n + 4 \text{ can be written as the sum of two primes for all } n \in \mathbb{N}. \end{array}$$

Induction, or more exactly mathematical induction, is a particularly useful method of proof for dealing with families of statements which are indexed by the natural numbers, such as the last three statements above. We shall prove both statements B and C using induction (see below and Example 6). Statement B (and likewise statement C) can be approached with induction because in each case knowing that the nth statement is true helps enormously in showing that the $(n+1)$th statement is true — this is the crucial idea behind induction. Statement D, on the other hand, is a famous problem known as *Goldbach's Conjecture*[1] and is still an open problem. If we let $D(n)$ be the statement that $2n + 4$ can be written as the sum of two primes, then it is currently known that $D(n)$ is true for $n < 4 \times 10^{14}$. What makes statement D different, and more intractable to induction, is that in trying to verify $D(n+1)$ we can't

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[1]Christian Goldbach (1690–1764), who was a professor of mathematics at St. Petersburg, made this conjecture in a letter to Euler in 1742.

generally make much use of knowledge of $D(n)$ and so we can't build towards a proof. For example, we can verify $D(17)$ and $D(18)$ by noting that

$$\begin{aligned} 38 &= 7+31 = 19+19, \\ 40 &= 3+37 = 11+29 = 17+23. \end{aligned}$$

Here, knowing that 38 can be written as a sum of two primes, is no help in verifying that 40 can be, as none of the primes we might use for the latter was previously used in splitting 38.

By way of an example we shall prove statement B by induction, before giving a formal definition of just what induction is. For any $n \in \mathbb{N}$, let $B(n)$ be the statement

$$0+1+2+\cdots+n = \frac{1}{2}n(n+1).$$

We shall prove two facts:

(i) $B(0)$ is true and (ii) for any $n \in \mathbb{N}$, if $B(n)$ is true then $B(n+1)$ is also true.

The first fact is the easy part as we just need to note that

$$\text{LHS of } B(0) = 0 = \frac{1}{2} \times 0 \times 1 = \text{RHS of } B(0).$$

To verify (ii) we need to prove *for each* n that $B(n+1)$ is true *assuming* $B(n)$ to be true. Now

$$\text{LHS of } B(n+1) = 0+1+\cdots+n+(n+1).$$

But, assuming $B(n)$ to be true, we know that the terms from 0 through to n add up to $n(n+1)/2$ and so

$$\begin{aligned} \text{LHS of } B(n+1) &= \frac{1}{2}n(n+1)+(n+1) \\ &= (n+1)\left(\frac{n}{2}+1\right) \\ &= \frac{1}{2}(n+1)(n+2) = \text{RHS of } B(n+1). \end{aligned}$$

This verifies (ii). Be sure that you understand the above calculation, it contains the important steps common to any proof by induction. Note in the final step that we have retrieved our original formula of $n(n+1)/2$, but with $n+1$ now replacing n everywhere; this was the expression that we always had to be working towards.

With induction we now know that B is true, i.e. that $B(n)$ is true for any $n \in \mathbb{N}$. How does this work? Well suppose we want to be sure $B(2)$ is correct — above we have just verified the following three statements:

$B(0)$ is true, if $B(0)$ is true then $B(1)$ is true, if $B(1)$ is true then $B(2)$ is true

and so putting the three together, we see that $B(2)$ is true: the first statement tells us that $B(0)$ is true and the second two are stepping stones, first to the truth about $B(1)$, and then on to proving $B(2)$.

Formally then the Principle of Induction is as follows:

Theorem 2 *(THE PRINCIPLE OF INDUCTION) Let $P(n)$ be a family of statements indexed by the natural numbers. Suppose that*

$P(0)$ is true, and
for any $n \in \mathbb{N}$, if $P(n)$ is true then $P(n+1)$ is also true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let S denote the subset of $\mathbb{N}$ consisting of all those n for which $P(n)$ is *false*. We aim to show that S is empty, i.e. that no $P(n)$ is false.

Suppose for a contradiction that S is non-empty. Any non-empty subset of $\mathbb{N}$ has a minimum element; let's write m for the minimum element of S. As $P(0)$ is true then $0 \notin S$, and so m is at least 1.

Consider now $m-1$. As $m \geq 1$ then $m-1 \in \mathbb{N}$ and further, as $m-1$ is smaller than the minimum element of S, then $m-1 \notin S$, i.e. $P(m-1)$ is true. But, as $P(m-1)$ is true, then induction tells us that $P(m)$ is also true. This means $m \notin S$, which contradicts m being the minimum element of S. This is our required contradiction, an absurd conclusion. We see that S being non-empty just doesn't hold water. If S being non-empty leads to a contradiction, then S must be empty. ■

It is not hard to see how we might amend the hypotheses of the theorem above to show

Corollary 3 *Let $N \in \mathbb{N}$ and let $P(n)$ be a family of statements for $n = N, N+1, N+2, \ldots$ Suppose that*

$$\begin{array}{l} P(N) \text{ is true, and} \\ \text{for any } n \geq N, \text{ if } P(n) \text{ is true then } P(n+1) \text{ is also true,} \end{array}$$

then $P(n)$ is true for all $n \geq N$.

This is really just induction again, but we have started the ball rolling at a later stage. Here is another version of induction, which is usually referred to a the *Strong Form Of Induction*:

Theorem 4 *(STRONG FORM OF INDUCTION) Let $P(n)$ be a family of statements for $n \in \mathbb{N}$. Suppose that*

$$\begin{array}{l} P(0) \text{ is true, and} \\ \text{for any } n \in \mathbb{N}, \text{ if } P(0), P(1), P(2), \ldots, P(n) \text{ are all true then so is } P(n+1), \end{array}$$

then $P(n)$ is true for all $n \in \mathbb{N}$.

To reinforce the need for proof, and to show how patterns can at first glance delude us, consider the following example. Take two points on the circumference of a circle and take a line joining them; this line then divides the circle's interior into two regions. If we take three points on the perimeter then the lines joining them will divide the disc into four regions. Four points can result in a maximum of eight regions — surely then, we can confidently predict that n points will maximally result in 2^{n-1} regions. Further investigation shows our conjecture to be true for $n = 5$, but to our surprise, however we take six points on the circle, the maximum number of regions attained is 31. Indeed the maximum number of regions attained from n points on the perimeter is given by the formula [2, p.18]

$$\frac{1}{24}\left(n^4 - 6n^3 + 23n^2 - 18n + 24\right).$$

Our original guess was way out!

There are other well-known 'patterns' that go awry in mathematics: for example, the number

$$n^2 - n + 41$$

is a prime number for $n = 1, 2, 3, \ldots, 40$ (though this takes some tedious verifying), but it is easy to see when $n = 41$ that $n^2 - n + 41 = 41^2$ is not prime. A more amazing example comes from the study of Pell's equation $x^2 = py^2 + 1$ in number theory, where p is a prime number and x and y are natural numbers. If $P(n)$ is the statement that

$$991n^2 + 1 \text{ is not a perfect square (i.e. the square of a natural number),}$$

then the first counter-example to $P(n)$ is staggeringly found at [1, pp. 2–3]

$$n = 12,055,735,790,331,359,447,442,538,767.$$

Hint: In many of the exercises that follow it is useful to keep in mind what expression you are working towards, namely some formula where each n has now been replaced by $n+1$.

Exercise 1 ***A.*** *Let a and r be real numbers with $r \neq 1$. Prove by induction, that*

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{a\left(r^n - 1\right)}{r-1} \quad \textit{for } n = 1, 2, 3 \ldots$$

Exercise 2 ***A.*** *Use the formula from statement B to show that the sum of an arithmetic progression with initial value a, common difference d and n terms, is*

$$\frac{n}{2}\left\{2a + (n-1)\,d\right\}.$$

Exercise 3 ***A.*** *Prove Bernoulli's Inequality which states that*

$$(1+x)^n \geq 1 + nx \quad \textit{for } x \geq -1 \textit{ and } n \in \mathbb{N}.$$

Exercise 4 ***A.*** *Show by induction that $n^2 + n \geq 42$ when $n \geq 6$ and $n \leq -7$.*

Exercise 5 ***B.*** *Prove for $n \geq 2$that,*

$$\sum_{r=2}^{n} \frac{1}{r^2 - 1} = \frac{(n-1)(3n+2)}{4n\,(n+1)}.$$

Exercise 6 ***C.*** *Let*

$$S(n) = \sum_{r=0}^{n} r^2 \quad \textit{for} \quad n \in \mathbb{N}.$$

Show that there is a unique cubic $f(n) = an^3 + bn^2 + cn + d$, whose coefficients a, b, c, d you should determine, such that $f(n) = S(n)$ for $n = 0, 1, 2, 3$. Prove by induction that $f(n) = S(n)$ for $n \in \mathbb{N}$.

Exercise 7 ***B.*** *Show that*

$$n + 3\sum_{r=1}^{n} r + 3\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n}\left\{(r+1)^3 - r^3\right\} = (n+1)^3 - 1.$$

Hence, using the formula from statement B, find an expression for $\sum_{r=1}^{n} r^2$.

Exercise 8 ***B.*** *Use induction to show that*

$$\sum_{k=1}^{n} \cos(2k-1)\,x = \frac{\sin 2nx}{2\sin x}.$$

Exercise 9 ***A.*** *What is wrong with the following 'proof' that all people are of the same height?*

"Let $P(n)$ be the statement that n persons must be of the same height. Clearly $P(1)$ is true as a person is the same height as him/herself. Suppose now that $P(k)$ is true for some natural number k and we shall prove that $P(k+1)$ is also true. If we have a crowd of $k+1$ people then we can invite one person to briefly leave so that k remain — from $P(k)$ we know that these people must all be equally tall. If we invite back the missing person and someone else leaves, then these k persons are also of the same height. Hence the $k+1$ persons were all of equal height and so $P(k+1)$ follows. By induction everyone is of the same height."

Exercise 10 ***B.*** *Below are certain families of statements $P(n)$ (indexed by $n \in \mathbb{N}$), which satisfy rules that are similar (but not identical) to the hypotheses required for induction. In each case say for which $n \in \mathbb{N}$ the truth of $P(n)$ must follow from the given rules.*

(a) $P(0)$ is true; if $P(n)$ is true then $P(n+2)$ is true for $n \in \mathbb{N}$;
(b) $P(1)$ is true; if $P(n)$ is true then $P(2n)$ is true for $n \in \mathbb{N}$;
(c) $P(0)$ and $P(1)$ are true; if $P(n)$ is true then $P(n+2)$ is true for $n \in \mathbb{N}$;
(d) $P(0)$ and $P(1)$ are true; if $P(n)$ and $P(n+1)$ are true then $P(n+2)$ is true for $n \in \mathbb{N}$;
(e) $P(0)$ is true; if $P(n)$ is true then $P(n+2)$ and $P(n+3)$ are both true for $n \in \mathbb{N}$;
(f) $P(0)$ is true; if $P(n)$ is true then $P(n+1)$ is true for $n \geq 1$.

2 Examples (M)

On a more positive note though, many of the patterns found in mathematics won't trip us at some later stage and here are some further examples of proof by induction.

Example 5 *Show that n lines in the plane, no two of which are parallel and no three meeting in a point, divide the plane into $n(n+1)/2+1$ regions.*

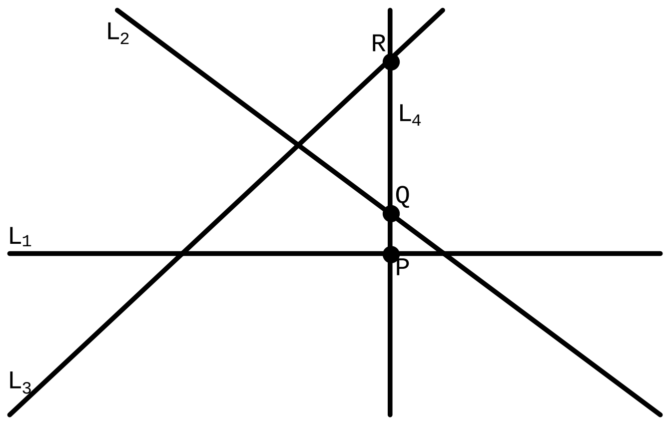
Proof. When we have no lines in the plane then clearly we have just one region, as expected from putting $n=0$ into the formula $n\,(n+1)\,/2+1$.

Suppose now that we have n lines dividing the plane into $n\,(n+1)\,/2+1$ regions and we will add a $(n+1)$th line. This extra line will meet each of the previous n lines because we have assumed it to be parallel with none of them. Also, it meets each of these n lines in a distinct point, as we have assumed that no three lines are concurrent.

These n points of intersection divide the new line into $n+1$ segments. For each of these $n+1$ segments there are now two regions, one on either side of the segment, where previously there had been only one region. So by adding this $(n+1)$th line we have created $n+1$ new regions. In total the number of regions we now have is

$$\frac{n\,(n+1)}{2}+1+(n+1)=\frac{(n+1)\,(n+2)}{2}+1.$$

This is the correct formula when we replace n with $n+1$, and so the result follows by induction. ■



An example when $n=3$.

Here the four segments, 'below P', PQ, QR and 'above R' on the fourth line L_4, divide what were four regions previously, into eight new ones.

Example 6 *Prove for $n\in\mathbb{N}$ that*

$$\int_0^\pi \sin^{2n}\theta\ d\theta=\frac{(2n)!}{(n!)^2}\frac{\pi}{2^{2n}}. \tag{1}$$

Proof. Let's denote the integral on the LHS of equation (1) as I_n. The value of I_0 is easy to calculate, because the integrand is just 1, and so $I_0=\pi$. We also see

$$\text{RHS}\,(n=0)=\frac{0!}{(0!)^2}\frac{\pi}{2^0}=\pi,$$

verifying the initial case.

We now prove a *reduction formula* connecting I_n and I_{n+1}, so that we can use this in our induction.

$$\begin{aligned} I_{n+1} &= \int_0^{\pi} \sin^{2(n+1)}\theta \ \mathrm{d}\theta \\ &= \int_0^{\pi} \sin^{2n+1}\theta \times \sin\theta \ \mathrm{d}\theta \\ &= \left[\sin^{2n+1}\theta \times (-\cos\theta)\right]_0^{\pi} - \int_0^{\pi} (2n+1)\sin^{2n}\theta\cos\theta \times (-\cos\theta) \ \mathrm{d}\theta \ \ \text{[using integration by parts]} \\ &= 0 + (2n+1)\int_0^{\pi} \sin^{2n}\theta\left(1-\sin^2\theta\right) \ \mathrm{d}\theta \ \ \text{[using } \cos^2\theta = 1-\sin^2\theta\text{]} \\ &= (2n+1)\int_0^{\pi}\left(\sin^{2n}\theta - \sin^{2(n+1)}\theta\right) \ \mathrm{d}\theta \\ &= (2n+1)\left(I_n - I_{n+1}\right). \end{aligned}$$

Rearranging gives

$$I_{n+1} = \frac{2n+1}{2n+2} I_n.$$

Suppose now that equation (1) gives the right value of I_k for some natural number k. Then, turning to equation (1) with $n = k+1$, and using our assumption and the reduction formula, we see:

$$\begin{aligned} \text{LHS} &= I_{k+1} = \frac{2k+1}{2(k+1)} \times \frac{(2k)!}{(k!)^2} \times \frac{\pi}{2^{2k}} \\ &= \frac{2k+2}{2(k+1)} \times \frac{2k+1}{2(k+1)} \times \frac{(2k)!}{(k!)^2} \times \frac{\pi}{2^{2k}} \\ &= \frac{(2k+2)!}{((k+1)!)^2} \times \frac{\pi}{2^{2(k+1)}}, \end{aligned}$$

which equals the RHS of equation (1) with $n = k+1$. The result follows by induction. ■

Example 7 *Show for $n = 1, 2, 3\ldots$ and $k = 1, 2, 3, \ldots$ that*

$$\sum_{r=1}^{n} r(r+1)(r+2)\cdots(r+k-1) = \frac{n(n+1)(n+2)\cdots(n+k)}{k+1}. \tag{2}$$

Remark 8 *This problem differs from our earlier examples in that our family of statements now involves two variables n and k, rather than just the one variable. If we write $P(n,k)$ for the statement in equation (2) then we can use induction to prove all of the statements $P(n,k)$ in various ways:*

- *we could prove $P(1,1)$ and show how $P(n+1,k)$ and $P(n,k+1)$ both follow from $P(n,k)$ for $n,k = 1,2,3,\ldots$;*
- *we could prove $P(1,k)$ for all $k = 1,2,3,\ldots$ and show how knowledge of $P(n,k)$ for all k, leads to proving $P(n+1,k)$ for all k — effectively this reduces the problem to one application of induction, but to a family of statements at a time*
- *we could prove $P(n,1)$ for all $n = 1,2,3,\ldots$ and show how knowing $P(n,k)$ for all n, leads to proving $P(n,k+1)$ for all n — in a similar fashion to the previous method, now inducting through k and treating n as arbitrary.*

What these different approaches rely on, is that all the possible pairs (n,k) are somehow linked to our initial pair (or pairs). Let

$$S = \{(n,k) : n,k \geq 1\}$$

be the set of all possible pairs (n,k).

The first method of proof uses the fact that the only subset T of S satisfying the properties

$$\begin{array}{rcl} (1,1) & \in & T, \\ \text{if } (n,k) & \in & T \text{ then } (n,k+1) \in T, \\ \text{if } (n,k) & \in & T \text{ then } (n+1,k) \in T, \end{array}$$

is S itself. Starting from the truth of $P(1,1)$, and deducing further truths as the second and third properties allow, then every $P(n,k)$ must be true. The second and third methods of proof rely on the fact that the whole of S is the only subset having similar properties.

Proof. In this case is the second method of proof seems easiest, that is: we will prove that $P(1,k)$ holds for each $k = 1, 2, 3, \ldots$ and show that assuming statements $P(N,k)$ for a particular N and all k, is sufficient to prove the statements $P(N+1,k)$ for all k. Firstly we note

$$\begin{array}{rcl} \text{LHS of } P(1,k) & = & 1 \times 2 \times 3 \times \cdots \times k, \text{ and} \\ \text{RHS of } P(1,k) & = & \dfrac{1 \times 2 \times 3 \times \cdots \times (k+1)}{k+1} = 1 \times 2 \times 3 \times \cdots \times k, \end{array}$$

are equal, proving $P(1,k)$ for all $k \geq 1$. Then, assuming $P(N,k)$ for all $k = 1, 2, 3, \ldots$, we have

$$\begin{array}{rcl} \text{LHS of } P(N+1,k) & = & \displaystyle\sum_{r=1}^{N+1} r(r+1)(r+2)\cdots(r+k-1) \\ & = & \dfrac{N(N+1)\ldots(N+k)}{k+1} + (N+1)(N+2)\cdots(N+k) \\ & = & (N+1)(N+2)\cdots(N+k)\left(\dfrac{N}{k+1} + 1\right) \\ & = & \dfrac{(N+1)(N+2)\cdots(N+k)(N+k+1)}{k+1} = \text{RHS of } P(N+1,k), \end{array}$$

proving $P(N+1,k)$ simultaneously for each k. This verifies all that is required for the second method. ∎

We end with one example which makes use of the Strong Form of Induction. Recall that a natural number $n \geq 2$ is called *prime* if the only natural numbers which divide it are 1 and n. (Note that 1 is not considered prime.) The list of prime numbers begins $2, 3, 5, 7, 11, 13, \ldots$ and has been known to be infinite since the time of Euclid[2]. The prime numbers are, in a sense, the atoms of the natural numbers under multiplication as every natural number $n \geq 2$ can be written as a product of primes in what is essentially a unique way — this fact is known as the Fundamental Theorem of Arithmetic. Here we prove the existence of such a product; for a complete proof of the theorem see the article entitled *The Integers.*

Example 9 *Every natural number $n \geq 2$ can be written as a product of prime numbers.*

Proof. We begin at $n = 2$ which is prime. As our inductive hypothesis we assume that every number $2 \leq k \leq N$ is a prime number or can be written as a product of prime numbers. Consider then $N+1$; we need to show this is a prime number, or else a product of prime numbers. Either $N+1$ is prime or it is not. If $N+1$ is prime then we are done. If $N+1$ is not prime, then it has a factor $2 \leq m < N+1$ which divides $N+1$. Note that $m \leq N$ and $(N+1)/m \leq N$, as m is at least 2. So, by hypothesis, we know m and $(N+1)/m$ are both either prime or the product of prime numbers. Hence we can write

$$m = p_1 \times p_2 \times \cdots \times p_k, \quad \text{and} \quad \frac{N+1}{m} = P_1 \times P_2 \times \cdots \times P_K,$$

where $p_1, \ldots, p_k$ and $P_1, \ldots, P_K$ are prime numbers. Finally we have that

$$N+1 = m \times \frac{N+1}{m} = p_1 \times p_2 \times \cdots \times p_k \times P_1 \times P_2 \times \cdots \times P_K,$$

showing $N+1$ to be a product of primes. The result follows using the strong form of induction. ∎

[2] Euclid was an Alexandrian Greek living c. 300 B.C. His most famous work is *The Elements*, thirteen books which present much of the mathematics discovered by the ancient Greeks, and which was a hugely influential text on the teaching of mathematics even into the twentieth century. The work presents its results in a rigorous fashion, laying down basic assumptions, called *axioms,* and carefully proving his theorems from these axioms.

Exercise 11 ***B.*** *Assuming only the product rule of differentiation, show that*

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \quad \text{for } n = 1, 2, 3, \ldots$$

Exercise 12 ***B.*** *Let*

$$A = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}.$$

Show that

$$A^n = 3^{n-1}\begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix}$$

for $n = 1, 2, 3, \ldots$. *Can you find a matrix* B *such that* $B^2 = A$*?*

Exercise 13 ***B.*** *Show that there are* 2^n *subsets of the set* $\{1, 2, ..., n\}$. *[Be sure to include the empty set.]*

Exercise 14 ***B.*** *Show for* $n \geq 1$ *and* $0 \leq k \leq n$ *that*

$$\frac{n!}{k!\,(n-k)!} < 2^n.$$

[Hint: you may find it useful to note the symmetry in the LHS which takes the same value at $k = k_0$ *as it does at* $k = n - k_0$*.]*

Exercise 15 ***B.*** *Show that every natural number* $n \geq 1$ *can be written in the form* $n = 2^k l$ *where* k, l *are natural numbers and* l *is odd.*

Exercise 16 ***C.*** *Show that every integer* n *can be written as a sum* $3a + 5b$ *where* a *and* b *are integers.*

Exercise 17 ***B.*** *Show that* $3^{3n} + 5^{4n+2}$ *is divisible by* 13 *for all natural numbers* n*.*

Exercise 18 ***B.*** *By setting up an identity between* I_n *and* I_{n-2} *show that*

$$I_n = \int_0^\pi \frac{\sin nx}{\sin x}\; dx$$

equals π *when* n *is odd. What value does* I_n *take when* n *is even?*

Exercise 19 ***B.*** *Euler's*[3] *Gamma function* $\Gamma(a)$ *is defined for all* $a > 0$ *by the integral*

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\; dx.$$

Show that $\Gamma(a+1) = a\Gamma(a)$ *for* $a > 0$*, and deduce that*

$$\Gamma(n+1) = n! \quad \text{for } n \in \mathbb{N}.$$

Exercise 20 ***B.*** *Euler's Beta function* $B(a, b)$ *is defined for all positive* a, b *by the integral*

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\; dx.$$

Set up a reduction formula involving B*, and deduce that if* m *and* n *are natural numbers then*

$$B(m+1, n+1) = \frac{m!\,n!}{(m+n+1)!}.$$

Exercise 21 ***C.*** *Let* k *be a natural number. Deduce from Example 7 that*

$$\sum_{r=1}^{n} r^k = \frac{n^{k+1}}{k+1} + E_k(n)$$

where $E_k(n)$ *is a polynomial in* n *of degree at most* k*.*

[3] The Swiss mathematician Leonhard Euler (1707-1783) was the colossus of eighteenth century mathematics and the most prolific mathematician ever. His name accompanies constants, polynomial, angles and formulae scattered throughout mathematics. Much of our modern notation is due to him including e, π and i. His most important contributions were in analysis (eg. on infinite series, calculus of variations), where he introduced the Gamma function as a kind of continuous factorial function. The study of topology arguably dates back to his solution of the *Königsberg Bridge Problem.*

3 Application: The Binomial Theorem (M)

All of you will have met the identity

$$(x+y)^2 = x^2 + 2xy + y^2$$

and may even have met identities like

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

It may even have been pointed out to you that these coefficients $1, 2, 1$ and $1, 3, 3, 1$ are simply the numbers that appear in *Pascal's Triangle.* This is the infinite triangle of numbers that has 1s down both sides and a number internal to some row of the triangle is calculated by adding the two numbers above it in the previous row. So the triangle grows as follows:

$$\begin{array}{lccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

From the triangle we could say read off the identity

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

Of course we haven't *proved* this identity yet — these identities, for general n other than just $n = 6$, are the subject of the Binomial Theorem. We introduce now the binomial coefficients; their connection with Pascal's triangle will become clear soon.

Definition 10 *The (n, k)th binomial coefficient is the number*

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

where $n = 0, 1, 2, 3, \ldots$ and $0 \leq k \leq n$. It is read as 'n choose k' and in some books is denoted as ${}^nC_k$. As a convention we set $\binom{n}{k}$ to be zero when $n < 0$ or when $k < 0$ or $k > n$.

Note some basic identities concerning the binomial coefficients

$$\binom{n}{k} = \binom{n}{n-k}, \quad \binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = \binom{n}{n-1} = n.$$

The following lemma demonstrates that the binomial coefficients are precisely the numbers that appear in Pascal's triangle.

Lemma 11 *Let $n \in \mathbb{N}$ and $1 \leq k \leq n$. Then*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Proof. Putting the LHS over a common denominator

$$\begin{aligned}
\text{LHS} &= \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!} \\
&= \frac{n!}{k!\,(n-k+1)!}\{k + (n-k+1)\} \\
&= \frac{(n+1)!}{k!\,(n+1-k)!} \\
&= \binom{n+1}{k} = \text{RHS}.
\end{aligned}$$

■

Corollary 12 *The kth number in the nth row of Pascal's triangle is* $\binom{n}{k}$ *(remembering to count from* $n = 0$ *and* $k = 0$*). In particular the binomial coefficients are whole numbers.*

Proof. We shall prove this by induction. Note that $\binom{0}{0} = 1$ gives the 1 at the apex of Pascal's triangle, proving the initial step.

Suppose now that the numbers $\binom{N}{k}$ are the numbers that appear in the Nth row of Pascal's triangle. The first and last entries of the next, $(N+1)$th, row (associated with $k = 0$ and $k = N+1$) are

$$1 = \binom{N+1}{0}, \quad \text{and} \quad 1 = \binom{N+1}{N+1}$$

as required. For $1 \leq k \leq N$, then the kth entry on the $(N+1)$th row is formed by adding the $(k-1)$th and kth entries from the Nth row. By our hypothesis about the Nth row their sum is

$$\binom{N}{k-1} + \binom{N}{k} = \binom{N+1}{k},$$

using the previous lemma, and this verifies that the $(N+1)$th row also consists of binomial coefficients. So the $(N+1)$th row checks out, and the result follows by induction. ■

Finally, we come to the binomial theorem:

Theorem 13 *(THE BINOMIAL THEOREM): Let* $n \in \mathbb{N}$ *and* x, y *be real numbers. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. Let's check the binomial theorem first for $n = 0$. We can verify this by noting

$$\text{LHS} = (x+y)^0 = 1, \qquad \text{RHS} = \binom{0}{0} x^0 y^0 = 1.$$

We aim now to show the theorem holds for $n = N+1$ assuming it to be true for $n = N$. In this case

$$\text{LHS} = (x+y)^{N+1} = (x+y)(x+y)^N = (x+y)\left(\sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k}\right)$$

writing in our assumed expression for $(x+y)^N$. Expanding the brackets gives

$$\sum_{k=0}^{N} \binom{N}{k} x^{k+1} y^{N-k} + \sum_{k=0}^{N} \binom{N}{k} x^k y^{N+1-k},$$

which we can rearrange to

$$x^{N+1} + \sum_{k=0}^{N-1} \binom{N}{k} x^{k+1} y^{N-k} + \sum_{k=1}^{N} \binom{N}{k} x^k y^{N+1-k} + y^{N+1}$$

by taking out the last term from the first sum and the first term from the second sum. In the first sum we now make a change of variable. We set $k = l - 1$, noting that as k ranges over $0, 1, ..., N-1$, then l ranges over $1, 2, ..., N$. So the above equals

$$x^{N+1} + \sum_{l=1}^{N} \binom{N}{l-1} x^l y^{N+1-l} + \sum_{k=1}^{N} \binom{N}{k} x^k y^{N+1-k} + y^{N+1}.$$

We may combine the sums as they are over the same range, obtaining

$$x^{N+1} + \sum_{k=1}^{N} \left\{ \binom{N}{k-1} + \binom{N}{k} \right\} x^k y^{N+1-k} + y^{N+1}$$

which, using Lemma 11, equals

$$x^{N+1}+\sum_{k=1}^{N}\binom{N+1}{k}x^k y^{N+1-k}+y^{N+1}=\sum_{k=0}^{N+1}\binom{N+1}{k}x^k y^{N+1-k}=\text{RHS}.$$

The result follows by induction. ■

There is good reason why $\binom{n}{k}$ is read as 'n choose k' — there are $\binom{n}{k}$ ways of choosing k elements from the set$\{1,2,\ldots,n\}$ (when showing no interest in the order that the k elements are to be chosen). Put another way, there are $\binom{n}{k}$ subsets of $\{1,2,\ldots,n\}$ with k elements in them. To show this, let's think about how we might go about choosing k elements.

For our 'first' element we can choose any of the n elements, but once this has been chosen it can't be put into the subset again. So for our second element any of the remaining $n-1$ elements may be chosen, for our third any of the $n-2$ that are left, and so on. So choosing a set of k elements from $\{1,2,\ldots,n\}$ in a particular order can be done in

$$n\times(n-1)\times(n-2)\times\cdots\times(n-k+1)=\frac{n!}{(n-k)!}\quad\text{ways.}$$

But there are lots of different orders of choice that could have produced this same subset. Given a set of k elements there are $k!$ ways of ordering this (see Exercise 23 below) — that is to say, for each subset with k elements there are $k!$ different orders of choice that will lead to that subset. So the number $n!/(n-k)!$ is an 'overcount' by a factor of $k!$. Hence the number of subsets of size k equals

$$\frac{n!}{k!\,(n-k)!}$$

as required.

Remark 14 *There is a Trinomial Theorem and further generalisations of the binomial theorem to greater numbers of variables. Given three real numbers x,y,z and a natural number n we can apply the binomial theorem twice to obtain*

$$\begin{aligned}(x+y+z)^n &= \sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}x^k\,(y+z)^{n-k}\\ &= \sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\sum_{l=0}^{n-k}\frac{(n-k)!}{l!\,(n-k-l)!}x^k y^l z^{n-k-l}\\ &= \sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{n!}{k!l!\,(n-k-l)!}x^k y^l z^{n-k-l}.\end{aligned}$$

This is a somewhat cumbersome expression; it's easier on the eye, and has a nicer symmetry, if we write $m=n-k-l$ and then we can rewrite the above as

$$(x+y+z)^n=\sum_{\substack{k+l+m=n\\k,l,m\geq 0}}\frac{n!}{k!\,l!\,m!}x^k\,y^l\,z^m.$$

Again the number $n!/(k!\,l!\,m!)$, where $k+l+m=n$ and $k,l,m\geq 0$, is the number of ways that n elements can be apportioned into three subsets associated with the numbers x,y and z.

Exercise 22 ***A***. *Show that Lemma 11 holds true for general integers n and k, remembering the convention that $\binom{n}{k}$ is zero when $n<0$ or $k<0$ or $k>n$.*

Exercise 23 ***A***. *Show by induction that there are $n!$ ways of ordering a set with n elements.*

Exercise 24 ***B***. *Interpret Lemma 11 in terms of subsets of $\{1,2,\ldots,n\}$ and $\{1,2,\ldots,n+1\}$ to give a new proof of the lemma. [Hint: consider subsets of $\{1,2,\ldots,n+1\}$ containing k elements, and whether they do, or do not, contain the final element $n+1$.]*

Exercise 25 ***B***. *Let n be a natural number. Show that*

$$(a)\qquad \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n;$$

$$(b)\qquad \binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots=2^{n-1}.$$

Interpret part (a) in terms of the subsets of $\{1,2,\ldots,n\}$. [Note that the sums above are not infinite as the binomial coefficients $\binom{n}{k}$ eventually become zero once k becomes sufficiently large.]

Exercise 26 ***C***. *Let n be a natural number. Show that*

$$\binom{2n}{0}+\binom{2n}{4}+\binom{2n}{8}+\cdots=\begin{cases} 2^{2n-2}+(-1)^{n/2}\,2^{n-1} & \text{if } n \text{ is even;}\\ 2^{2n-2} & \text{if } n \text{ is odd.}\end{cases}$$

Distinguishing cases, find the value of

$$\binom{2n}{1}+\binom{2n}{5}+\binom{2n}{9}+\cdots$$

[Hint: one way to approach this question makes use of complex numbers.]

Exercise 27 ***B***. *Use the identity $(1+x)^{2n}=(1+x)^n\,(1+x)^n$ to show that*

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}.$$

Exercise 28 ***B***. *By differentiating the binomial theorem, show that*

$$\sum_{k=1}^{n}k\binom{n}{k}=n2^{n-1}.$$

Exercise 29 ***C***. *Show that*

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}=\frac{2^{n+1}-1}{n+1}.$$

Exercise 30 ***B***. *Use induction to prove Leibniz's rule for the differentiation of a product — this says that for functions $u(x)$ and $v(x)$ of a variable x, and $n\in\mathbb{N}$, then*

$$\frac{d^n}{dx^n}\left(u(x)\,v(x)\right)=\sum_{k=0}^{n}\binom{n}{k}\frac{d^k u}{dx^k}\frac{d^{n-k}v}{dx^{n-k}}.$$

Exercise 31 ***C***. *Use the method of Exercise 7 and the binomial theorem to show that*

$$\sum_{r=1}^{n}r^k=\frac{n^{k+1}}{k+1}+E_k(n),$$

where $E_k(n)$ is a polynomial in n of degree at most k.

4 Application: The Fibonacci Numbers (M++)

The Fibonacci numbers F_n are defined recursively

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for} \quad n \geq 2,$$

with initial values $F_0 = 0$ and $F_1 = 1$. So the sequence of Fibonacci numbers begins as

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

and continues to grow, always producing whole numbers and increasing by a factor of roughly 1.618 each time.

This sequence was first studied by Leonardo of Pisa (c.1170–c.1250), who called himself Fibonacci.[4] The numbers were based on a model of rabbit reproduction: the model assumes that we begin with a pair of rabbits in the first month, which every month produces a new pair of rabbits, which in turn begin producing when they are one month old. If we look at the F_n pairs we have at the start of the nth month, then these consist of $F_{n-1} - F_{n-2}$ pairs, which have just become mature but were immature the previous month, and F_{n-2} pairs which were already mature and their new F_{n-2} pairs of offspring. In other words $F_n = (F_{n-1} - F_{n-2}) + 2F_{n-2}$ which rearranges to the recursion above.

We now use induction to prove some of the theory of Fibonacci numbers.

Proposition 15 *For every integer $n \geq 0$,*

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \tag{3}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \textit{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Proof. As our expression for F_n is in terms of F_{n-2} and F_{n-1}, then we will need to know equation (3) holds for *two* consecutive Fibonacci numbers in order to be able to deduce anything about the next Fibonacci number. So to get the ball rolling we need to check the $n = 0$ and $n = 1$ cases. Now

$$\begin{aligned} \text{when } n &= 0\text{: LHS of equation (3)} = F_0 = 0 = \frac{1-1}{\sqrt{5}} = \frac{\alpha^0 - \beta^0}{\sqrt{5}} = \text{RHS, and} \\ \text{when } n &= 1\text{: LHS of equation (3)} = F_1 = 1 = \frac{\sqrt{5}}{\sqrt{5}} = \frac{\alpha - \beta}{\sqrt{5}} = \text{RHS.} \end{aligned}$$

To keep the ball rolling, we assume that equation (3) holds for $n = k-1$ and $n = k$ and aim to prove (3) holds for $n = k+1$. Based on our assumptions we may write

$$F_{k+1} = F_k + F_{k-1} = \frac{\alpha^k - \beta^k}{\sqrt{5}} + \frac{\alpha^{k-1} - \beta^{k-1}}{\sqrt{5}} = \frac{\alpha^{k-1}(\alpha+1)}{\sqrt{5}} - \frac{\beta^{k-1}(\beta+1)}{\sqrt{5}}.$$

At this point we note that α and β are the two roots of the quadratic $1 + x = x^2$ (check this!) giving

$$F_{k+1} = \frac{\alpha^{k-1}\alpha^2}{\sqrt{5}} - \frac{\beta^{k-1}\beta^2}{\sqrt{5}} = \frac{\alpha^{k+1} - \beta^{k+1}}{\sqrt{5}}$$

which is the correct form. So based on our assumption (3) holds when $n = k+1$.

We now have two new consecutive numbers, k and $k+1$, for which (3) is true, which allow us to prove the $k+2$ case etc. etc. and the result follows by induction. ■

[4] The meaning of the name 'Fibonacci' is somewhat uncertain; it may have meant 'son of Bonaccio' or may have been a nickname meaning 'lucky son'.

In the previous proof, we didn't use induction as described in Theorems 2 and 4. Rather, I aimed to stress how we need to know the truth of (3) in two consecutive cases to keep induction working. If we wished to be more precise we could apply induction, as it appears in Theorem 2, to the statements $P(n)$ for $n = 0, 1, 2, 3, \ldots$, where $P(N)$ is the statement

$$\text{equation (3) holds true for } n = N \text{ and } n = N + 1.$$

Now that we have introduced the idea here, we will be more formal in the next proof. Note that the identity below involves two variables, something we have tackled with induction (see Remark 8). In the following example we proceed by treating at each stage m as arbitrary and inducting through n, but noting, as in the previous proposition, that we need two consecutive cases to be true to be able to move onto the next one.

Proposition 16 *For* $m, n \in \mathbb{N}$

$$F_{n+m+1} = F_n F_m + F_{n+1} F_{m+1}. \tag{4}$$

Proof. For $n \in \mathbb{N}$, we shall take $P(n)$ to be the statement that:

$$\text{equation (4) holds true for all } \; m \in \mathbb{N} \; \text{ in the cases } \; k = n \; \text{ and } \; k = n + 1.$$

So we are using induction to progress through n and dealing with m simultaneously at each stage. To verify $P(0)$, we note that

$$\begin{aligned} F_{m+1} &= F_0 F_m + F_1 F_{m+1}, \\ F_{m+2} &= F_1 F_m + F_2 F_{m+1} \end{aligned}$$

for all m, as $F_0 = 0$, $F_1 = F_2 = 1$. For the inductive step we assume $P(n)$, i.e. that for all $m \in \mathbb{N}$,

$$\begin{aligned} F_{n+m+1} &= F_n F_m + F_{n+1} F_{m+1}, \\ F_{n+m+2} &= F_{n+1} F_m + F_{n+2} F_{m+1.} \end{aligned}$$

To prove $P(n+1)$ it remains to show that for all $m \in \mathbb{N}$,

$$F_{n+m+3} = F_{n+2} F_m + F_{n+3} F_{m+1}. \tag{5}$$

From our $P(n)$ assumptions and the definition of the Fibonacci numbers,

$$\begin{aligned} \text{LHS of (5)} &= F_{n+m+3} \\ &= F_{n+m+2} + F_{n+m+1} \\ &= F_n F_m + F_{n+1} F_{m+1} + F_{n+1} F_m + F_{n+2} F_{m+1} \\ &= (F_n + F_{n+1}) F_m + (F_{n+1} + F_{n+2}) F_{m+1} \\ &= F_{n+2} F_m + F_{n+3} F_{m+1} \\ &= \text{RHS of (5).} \end{aligned}$$

■

The following exercises contain further properties of the Fibonacci numbers.

Exercise 32 ***A***. *Use Proposition 16 to show that*

$$F_{2n+1} = (F_{n+1})^2 + (F_n)^2 \qquad (n \in \mathbb{N}).$$

Deduce that

$$F_{2n} = (F_{n+1})^2 - (F_{n-1})^2 \qquad (n = 1, 2, 3...).$$

Exercise 33 ***B***. *Prove by induction the following identities involving the Fibonacci numbers*

$$\begin{array}{ll} (a) & F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2}, \\ (b) & F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1, \\ (c) & (F_1)^2 + (F_2)^2 + \cdots + (F_n)^2 = F_n F_{n+1}. \end{array}$$

Exercise 34 ***B***. *(a) The Lucas numbers L_n $(n \in \mathbb{N})$ are defined by*

$$L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2,$$

and by $L_0 = 2$ and $L_1 = 1$. Prove that

$$L_n = 2F_{n-1} + F_n \text{ for } n \geq 1.$$

More generally, show that if a sequence of numbers G_n, for $n \in \mathbb{N}$, is defined by $G_n = G_{n-1} + G_{n-2}$ for $n \geq 2$,and by $G_0 = a$ and $G_1 = b$, show that $G_n = aF_{n-1} + bF_n$ for $n \geq 1$.

Exercise 35 ***A***. *Show that $F_{2n} = F_n L_n$ for $n \in \mathbb{N}$, where L_n denotes the nth Lucas number, described in the previous exercise. Deduce then the identity*

$$F_{2^n} = L_2 L_4 L_8 \cdots L_{2^{n-1}}.$$

Exercise 36 ***C***. *Show, for $0 \leq k \leq n$, with k even, that*

$$F_{n-k} + F_{n+k} = L_k F_n.$$

Can you find, and prove, a similar expression for $F_{n-k} + F_{n+k}$ when k is odd?

Exercise 37 ***B***. *Use Lemma 11 to prove by induction that*

$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots \quad \text{for } n \in \mathbb{N}.$$

[Note that the series is not infinite as the terms in the sum will eventually become zero.]

Exercise 38 ***C***. *Use Proposition 16 to show that*

$$F_{(m+1)k} = F_{mk+1} F_k + F_{k-1} F_{mk}$$

and deduce that if k divides n then F_k divides F_n.

Exercise 39 ***C***. *Let $\alpha = \left(1 + \sqrt{5}\right)/2$ and $\beta = \left(1 - \sqrt{5}\right)/2$. Show that α and β are roots of $1 + x = x^2$. Use the identity from Proposition 15 and the binomial theorem to show that*

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}.$$

5 Further Exercises

Exercise 40 ***A.*** *Prove that*

$$\sum_{r=n}^{2n-1} 2r+1 = 3n^2 \quad \textit{for} \quad n = 1, 2, 3 \ldots$$

Exercise 41 ***B.*** *Prove for* $n \in \mathbb{N}$, *that*

$$\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1.$$

Exercise 42 ***B.*** *Show that*

$$\sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n} \quad \textit{for } n = 1, 2, 3, \ldots$$

Exercise 43 ***B.*** *Extend the method of Exercise 7 to find expressions for* $\sum_{r=1}^{n} r^3$ *and* $\sum_{r=1}^{n} r^4$.

Exercise 44 ***B.*** *Bertrand's Postulate*[5] *states that for* $n \geq 3$ *there is prime number* p *satisfying*

$$\frac{n}{2} < p < n.$$

Use this postulate and the strong form of induction to show that every positive integer can be written as a sum of prime numbers, all of which are distinct. (For the purposes of this exercise you will need to regard 1 *as prime number.)*

Exercise 45 ***C.*** *Use induction to show that*

$$\sum_{k=1}^{n} \sin kx = \frac{\sin\left\{\frac{1}{2}(n+1)x\right\} \sin\left\{\frac{1}{2}nx\right\}}{\sin\left\{\frac{1}{2}x\right\}}.$$

Exercise 46 ***B.*** *Show that*

$$\int_0^{\pi/2} \cos^{2n+1} x \; dx = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

Exercise 47 ***C.***

(a) Show that $7^{m+3} - 7^m$ *and* $11^{m+3} - 11^m$ *are both divisible by* 19 *for all* $m \geq 0$.
(b) Calculate the remainder when $7^m - 11^n$ *is divided by* 19, *for the cases* $0 \leq m \leq 2$ *and* $0 \leq n \leq 2$.
(c) Deduce that $7^m - 11^n$ *is divisible by* 19, *precisely when* $m + n$ *is a multiple of* 3.

Exercise 48 ***B.*** *Use induction to show that there are* $\binom{n}{k}$ *subsets of* $\{1, 2, \ldots, n\}$ *with* k *elements.*

Exercise 49 ***B.*** *Prove for* $m, n \in \mathbb{N}$ *that*

$$\sum_{k=0}^{n} \binom{m+k}{m} = \binom{n+m+1}{m+1}.$$

Exercise 50 ***B.*** *By considering the identity* $(1+x)^{m+n} = (1+x)^m (1+x)^n$, *or otherwise, prove that*

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} \quad \textit{for } m, n, r \in \mathbb{N}.$$

Exercise 51 ***C.*** *Show that for* $n = 1, 2, 3, \ldots$

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = \frac{(2n-1)!}{\{(n-1)!\}^2}.$$

[5] Joseph Bertrand (1822-1900) made this conjecture in 1845 and it was first proved by the Russian mathematician Pafnutii Chebyshev (1821-1894) in 1851. Research goes on to this day on how much smaller this region can be made whilst still being guaranteed to contain a prime number.

Exercise 52 ***B.*** *Show that*

$$\sum_{i=m}^{n} F_i = F_{n+2} - F_{m+1} \quad \textit{for } m, n \in \mathbb{N}.$$

Exercise 53 ***C.*** *Show that*

$$F_{i+j+k} = F_{i+1}F_{j+1}F_{k+1} + F_iF_jF_k - F_{i-1}F_{j-1}F_{k-1} \textit{ for } i, j, k = 1, 2, 3, \ldots$$

Exercise 54 ***C.*** *Show by induction that* $\alpha = \left(1+\sqrt{5}\right)/2$ *and* $\beta = \left(1-\sqrt{5}\right)/2$ *are roots of the equation*

$$F_{m-1} + F_m x = x^m$$

for $m = 1, 2, 3, \ldots$. *Hence generalise the result of Exercise 39 to show that*

$$\sum_{k=0}^{n} \binom{n}{k} (F_m)^k (F_{m-1})^{n-k} F_k = F_{mn}.$$

Exercise 55 ***C.*** *(For those with some knowledge of infinite geometric progressions). Use Proposition 15 to show that*

$$\sum_{k=0}^{\infty} F_k x^k = \frac{x}{1-x-x^2}.$$

This is the generating function of the Fibonacci numbers. For what values of x *does the infinite series above converge?*

Exercise 56 ***B.*** *The sequence of numbers* x_n *is defined recursively by*

$$x_n = x_{n-1} + 2x_{n-2} \textit{ for } n \geq 2,$$

and by $x_0 = 1$, *and* $x_1 = -1$. *Calculate* x_n *for* $n \leq 6$, *and make an estimate for the value of* x_n *for general* n. *Use induction to prove your estimate correct.*

Exercise 57 ***B.*** *The sequence of numbers* x_n *is defined recursively by*

$$x_n = 2x_{n-1} - x_{n-2} \textit{ for } n \geq 2,$$

and by $x_0 = a$ *and* $x_1 = b$. *Calculate* x_n *for* $n \leq 6$, *and make an estimate for the value of* x_n *for general* n. *Use induction to prove your estimate correct.*

Exercise 58 ***B.*** *The sequence of numbers* x_n *is defined recursively by*

$$x_n = 3x_{n-2} + 2x_{n-3} \textit{ for } n \geq 3,$$

and by $x_0 = 1$, $x_1 = 3$, $x_2 = 5$. *Show that*

$$2^n < x_n < 2^{n+1} \textit{ for } n \geq 1$$

and that

$$x_{n+1} = 2x_n + (-1)^n.$$

Exercise 59 ***B.*** *The sequences* u_n *and* v_n *are defined recursively by*

$$u_{n+1} = u_n + 2v_n \quad \textit{and} \quad v_{n+1} = u_n + v_n,$$

with initial values $u_1 = v_1 = 1$. *Show that*

$$(u_n)^2 - 2(v_n)^2 = \begin{cases} 1 & \textit{if } n \textit{ is even,} \\ -1 & \textit{if } n \textit{ is odd.} \end{cases}$$

Show further, for $n \geq 1$, *that*

$$u_n = \frac{\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n}{2} \quad \textit{and} \quad v_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}.$$

Exercise 60 ***C.*** *The Hermite polynomials $H_n(x)$ for $n = 0, 1, 2, \ldots$ are defined recursively as*

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \ \text{ for } n \geq 1,$$

with $H_0(x) = 1$ and $H_1(x) = 2x$.

(a) Calculate $H_n(x)$ for $n = 2, 3, 4, 5$.
(b) Show by induction that

$$H_{2k}(0) = (-1)^k \frac{(2k)!}{k!} \quad \text{and} \quad H_{2k+1}(0) = 0.$$

(c) Show by induction that

$$\frac{dH_n}{dx} = 2nH_{n-1}.$$

(d) Deduce the $H_n(x)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0.$$

(e) Use Leibniz's rule for differentiating a product (see Example 30) to show that the polynomials

$$(-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

satisfy the same recursion as $H_n(x)$ with the same initial conditions and deduce that

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \quad \text{for } n = 0, 1, 2, \ldots$$

References

[1] Joseph Rotman, *Journey Into Mathematics — An Introduction to Proofs*, Prentice Hall, 1998.

[2] David Acheson, *1089 And All That,* Oxford University Press, 2002.