Bridging Course in Mathematics

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0. NOTATION

0.0.1 The Greek Alphabet

A, α	alpha	H, η	eta	N, ν	nu	T, τ	tau
B, β	beta	Θ, θ	theta	Ξ, ξ	xi	Y, υ	upsilon
Γ, γ	gamma	I, ι	iota	O, o	omicron	Φ, ϕ, φ	phi
Δ, δ	delta	K, κ	kappa	Π, π	pi	X, χ	chi
E, ϵ	epsilon	Λ, λ	lambda	P, ρ, ϱ	rho	Ψ, ψ	psi
Z, ζ	zeta	M, μ	mu	$\Sigma, \sigma, \varsigma$	sigma	Ω, ω	omega

0.0.2 Set Theory and Functions

$\mathbb{R}$	the set of real numbers;
$\mathbb{C}$	the set of complex numbers;
$\mathbb{Q}$	the set of rational numbers — i.e. the fractions,
$\mathbb{Z}$	the set of integers — i.e. the whole numbers;
$\mathbb{N}$	the set of natural numbers — i.e. the non-negative whole numbers;
$\mathbb{R}^n$	n-dimensional real space — i.e. the set of all real n-tuples $(x_1, x_2, \ldots, x_n)$;
$\mathbb{R}[x]$	the set of polynomials in x with real coefficients;
$\in$	is an element of — e.g. $\sqrt{2} \in \mathbb{R}$ and $\pi \notin \mathbb{Q}$;
$\subset, \subseteq$	is a subset of — e.g. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$;
$\lvert X \rvert$	the *cardinality* (size) of the set X;
$X \cup Y$	the *union* of two sets — read 'cup' — $\{s : s \in X \text{ or } s \in Y\}$;
$X \cap Y$	the *intersection* of two sets — read 'cap' — $\{s : s \in X \text{ and } s \in Y\}$;
$X \times Y$	the *Cartesian product* of X and Y — $\{(x, y) : x \in X \text{ and } y \in Y\}$;
$X - Y$ or $X \backslash Y$	the *complement* of Y in X — $\{s : s \in X \text{ and } s \notin Y\}$;
$\varnothing$	the empty set.
$f : X \to Y$	f is a function, map, mapping from a set X to a set Y; X is called the *domain* and Y is called the *codomain;*
$f(X)$ or $f[X]$	the *image* or *range* of the function f — i.e. the set $\{f(x) : x \in X\}$;
$g \circ f$	the *composition* of the maps g and f – do f first then g;
f is *injective* or *1-1*	if $f(x) = f(y)$ then $x = y$;
f is *surjective* or *onto*	for each $y \in Y$ there exists $x \in X$ such that $f(x) = y$;
f is *bijective*	f is *1-1* and *onto*;
f is *invertible*	there exists a function $f^{-1} : Y \to X$ s.t.$f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$;

0.0.3 Logic

$:$ or $\mid$ or s.t.	such that;
$\forall$	for all;
$\exists$	there exists;
$\Longrightarrow$	implies, is sufficient for, only if;
$\Longleftarrow$	is implied by, is necessary for;
$\Longleftrightarrow$	if and only if, is logically equivalent to;
$\neg$	negation, not;
$\vee$	logical or, maximum;
$\wedge$	logical and, minimum;
$\square$ or QED	found at the end of a proof;

0.0.4 Miscellaneous

(a, b)	the real interval $a < x < b$;
$[a, b]$	the real interval $a \leq x \leq b$;
$\sum_{k=1}^{n} a_k$	the sum $a_1 + a_2 + \cdots + a_n$;
$\prod_{k=1}^{n} a_k$	the product $a_1 a_2 \cdots a_n$;
∇	grad, (also read as 'del' or 'nabla');
∂	partial differentiation;
$\oplus$	direct sum;
$\pm$	plus or minus;
$n!$	n factorial — i.e. $1 \times 2 \times 3 \times \cdots \times n$

1. COMPLEX NUMBERS

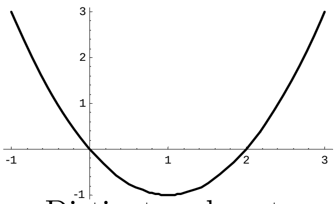
1.1 Their Algebra

1.1.1 The Need For Complex Numbers

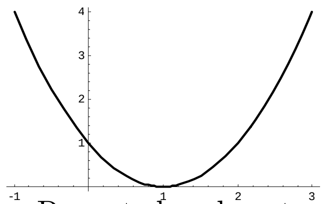
All of you will know that the two roots of the quadratic equation $ax^2+bx+c=0$ are

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \tag{1.1}$$

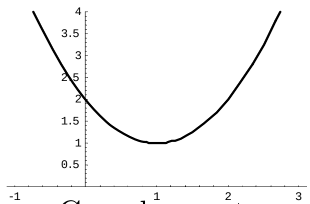
and solving quadratic equations is something that mathematicians have been able to do since the time of the Babylonians. When $b^2-4ac>0$ then these two roots are real and distinct; graphically they are where the curve $y=ax^2+bx+c$ cuts the x-axis. When $b^2-4ac=0$ then we have one real root and the curve just touches the x-axis here. But what happens when $b^2-4ac<0$? Then there are no real solutions to the equation as no real squares to give the negative b^2-4ac. From the graphical point of view the curve $y=ax^2+bx+c$ lies entirely above or below the x-axis.



Distinct real roots



Repeated real root



Complex roots

It is only comparatively recently that mathematicians have been comfortable with these roots when $b^2-4ac<0$. During the Renaissance the quadratic would have been considered unsolvable or its roots would have been called *imaginary.* (The term 'imaginary' was first used by the French Mathematician René Descartes (1596-1650). Whilst he is known more as a philosopher, Descartes made many important contributions to mathematics and helped found co-ordinate geometry – hence the naming of Cartesian co-ordinates.) If we imagine $\sqrt{-1}$ to exist, and that it behaves (adds and multiplies) much the same as other numbers then the two roots of the quadratic can be written in the form

$$x = A \pm B\sqrt{-1} \tag{1.2}$$

where

$$A = -\frac{b}{2a} \quad \text{and} \quad B = \frac{\sqrt{4ac-b^2}}{2a} \quad \text{are real numbers.}$$

But what meaning can such roots have? It was this philosophical point which pre-occupied mathematicians until the start of the 19th century when these 'imaginary' numbers started proving so useful (especially in the work of Cauchy and Gauss) that essentially the philosophical concerns just got forgotten about.

Notation 1 *We shall from now on write i for $\sqrt{-1}$, though many books, particularly those written for engineers and physicists use j instead. The notation i was first introduced by the Swiss mathematician Leonhard Euler (1707-1783). Much of our modern notation is due to him including e and π. Euler was a giant in 18th century mathematics and the most prolific mathematician ever. His most important contributions were in analysis (e.g. on infinite series, calculus of variations). The study of topology arguably dates back to his solution of the Königsberg Bridge Problem.*

Definition 2 *A complex number is a number of the form $a+bi$ where a and b are real numbers. If $z=a+bi$ then a is known as the real part of z and b as the imaginary part. We write $a=\operatorname{Re} z$ and $b=\operatorname{Im} z$. Note that real numbers are complex — a real number is simply a complex number with no imaginary part. The term 'complex number' is due to the German mathematician Carl Gauss (1777-1855). Gauss is considered by many the greatest mathematician ever. He made major contributions to almost every area of mathematics from number theory, to non-Euclidean geometry, to astronomy and magnetism. His name precedes a wealth of theorems and definitions throughout mathematics.*

Notation 3 *We write $\mathbb{C}$ for the set of all complex numbers.*

One of the first major results concerning complex numbers and which conclusively demonstrated their usefulness was proved by Gauss in 1799. From the quadratic formula (1.1) we know that all quadratic equations can be solved using complex numbers — what Gauss was the first to prove was the much more general result:

Theorem 4 *(FUNDAMENTAL THEOREM OF ALGEBRA). The roots of any polynomial equation $a_0+a_1x+a_2x^2+\cdots+a_nx^n=0$ with real (or complex) coefficients a_i are complex. That is there are n (not necessarily distinct) complex numbers $\gamma_1,\ldots,\gamma_n$ such that*

$$a_0+a_1x+a_2x^2+\cdots+a_nx^n=a_n\left(x-\gamma_1\right)\left(x-\gamma_2\right)\cdots\left(x-\gamma_n\right).$$

In particular the theorem shows that an n degree polynomial has, counting multiplicities, n roots in $\mathbb{C}$.

The proof of this theorem is far beyond the scope of this article. Note that the theorem only guarantees the *existence* of the roots of a polynomial somewhere in $\mathbb{C}$ unlike the quadratic formula which plainly gives us the roots. The theorem gives no hints as to where in $\mathbb{C}$ these roots are to be found.

1.1.2 Basic Operations

We add, subtract, multiply and divide complex numbers much as we would expect. We add and subtract complex numbers by adding their real and imaginary parts:-

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)\,i, \\ (a+bi)-(c+di) &= (a-c)+(b-d)\,i.\end{aligned}$$

We can multiply complex numbers by expanding the brackets in the usual fashion and using $i^2=-1$,

$$(a+bi)\,(c+di)=ac+bci+adi+bdi^2=(ac-bd)+(ad+bc)\,i.$$

To divide complex numbers we note firstly that $(c+di)\,(c-di)=c^2+d^2$ is real. So

$$\frac{a+bi}{c+di}=\frac{a+bi}{c+di}\times\frac{c-di}{c-di}=\left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i.$$

The number $c-di$ which we just used, as relating to $c+di$, has a special name and some useful properties — see Proposition 11.

Definition 5 *Let $z=a+bi$. The conjugate of z is the number $a-bi$ and this is denoted as $\bar{z}$ (or in some books as z^*).*

- Note from equation (1.2) that when the *real* quadratic equation

  $$ax^2+bx+c=0$$

  has complex roots then these roots are conjugates of each other. Generally if z_0 is a root of the polynomial $a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0=0$ where the a_i are real then so is its conjugate $\overline{z_0}$.

Problem 6 *Calculate, in the form $a+bi$, the following complex numbers:*

$$(1+3i)+(2-6i)\,,\quad (1+3i)-(2-6i)\,,\quad (1+3i)\,(2-6i)\,,\quad \frac{1+3i}{2-6i}.$$

Solution.

$$\begin{aligned}(1+3i)+(2-6i) &= (1+2)+(3+(-6))\,i=3-3i; \\ (1+3i)-(2-6i) &= (1-2)+(3-(-6))\,i=-1+9i. \\ (1+3i)\,(2-6i) &= 2+6i-6i-18i^2=2+18=20.\end{aligned}$$

Division takes a little more care, and we need to remember to multiply through by the conjugate of the denominator:

$$\frac{1+3i}{2-6i}=\frac{(1+3i)\,(2+6i)}{(2-6i)\,(2+6i)}=\frac{2+6i+6i+18i^2}{2^2+6^2}=\frac{-16+12i}{40}=\frac{-2}{5}+\frac{3}{10}i.$$

■

We present the following problem because it is a common early misconception involving complex numbers — if we need a new number i as the square root of -1, then shouldn't we need another one for the square root of i? But $z^2 = i$ is just another polynomial equation, with complex coefficients, and two (perhaps repeated) roots in $\mathbb{C}$ are guaranteed by the Fundamental Theorem of Algebra. They are also quite easy to calculate.

Problem 7 *Find all those z that satisfy $z^2 = i$.*

Solution. Suppose that $z^2 = i$ and $z = a + bi$, where a and b are real. Then

$$i = (a+bi)^2 = \left(a^2 - b^2\right) + 2abi.$$

Comparing the real and imaginary parts we see that

$$a^2 - b^2 = 0 \quad \text{and} \quad 2ab = 1.$$

So $b = \pm a$ from the first equation. Substituting $b = a$ into the second equation gives $a = b = 1/\sqrt{2}$ or $a = b = -1/\sqrt{2}$. Substituting $b = -a$ into the second equation of gives $-2a^2 = 1$ which has no real solution in a.

So the two z which satisfy $z^2 = i$, i.e. the two square roots of i, are

$$\frac{1+i}{\sqrt{2}} \quad \text{and} \quad \frac{-1-i}{\sqrt{2}}.$$

Notice, as with square roots of real numbers, that the two roots are negative one another. ■

Problem 8 *Use the quadratic formula to find the two solutions of*

$$z^2 - (3+i)\,z + (2+i) = 0.$$

Solution. We see that $a = 1$, $b = -3 - i$, and $c = 2 + i$. So

$$b^2 - 4ac = (-3-i)^2 - 4 \times 1 \times (2+i) = 9 - 1 + 6i - 8 - 4i = 2i.$$

Knowing

$$\sqrt{i} = \pm\frac{1+i}{\sqrt{2}}$$

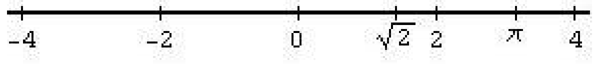
from the previous problem, we have

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{(3+i) \pm \sqrt{2i}}{2} = \frac{(3+i) \pm \sqrt{2}\sqrt{i}}{2} \\ &= \frac{(3+i) \pm (1+i)}{2} = \frac{4+2i}{2} \text{ or } \frac{2}{2} = 2+i \text{ or } 1. \end{aligned}$$

Note that the two roots are not conjugates of one another — this need not be the case when the coefficients a, b, c are not all real. ■

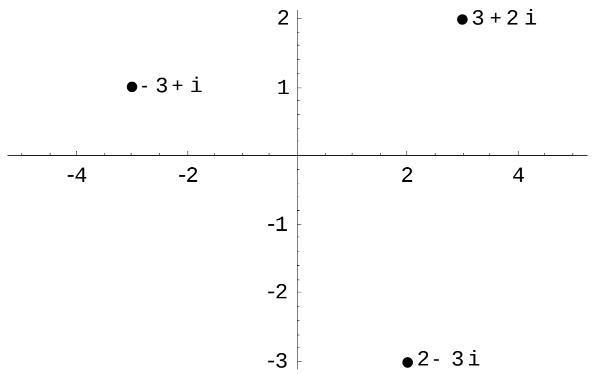
1.1.3 The Argand Diagram

The real numbers are often represented on the *real line* which increase as we move from left to right



The real number line

The complex numbers, having two components, their real and imaginary parts, can be represented as a plane; indeed $\mathbb{C}$ is sometimes referred to as the *complex plane,* but more commonly when we represent $\mathbb{C}$ in this manner we call it an *Argand diagram.* (After the Swiss mathematician Jean-Robert Argand (1768-1822)). The point (a, b) represents the complex number $a+bi$ so that the x-axis contains all the real numbers, and so is termed the *real axis,* and the y-axis contains all those complex numbers which are purely imaginary (i.e. have no real part) and so is referred to as the *imaginary axis.*



An Argand diagram

We can think of $z_0 = a + bi$ as a point in an Argand diagram but it is often useful to think of it as a vector as well. Adding z_0 to another complex number translates that number by the vector $\binom{a}{b}$. That is the map $z \mapsto z + z_0$ represents a translation a units to the right and b units up in the complex plane.

Note that the conjugate $\bar{z}$ of a point z is its mirror image in the real axis. So, $z \mapsto \bar{z}$ represents reflection in the real axis. We shall discuss in more detail the geometry of the Argand diagram in § 1.3.

A complex number z in the complex plane can be represented by Cartesian co-ordinates, its real and imaginary parts, but equally useful is the representation of z by polar co-ordinates. If we let r be the distance of z from the origin and, if $z \neq 0$, we let θ be the angle that the line connecting z to the origin makes with the positive real axis then we can write

$$z = x + iy = r\cos\theta + ir\sin\theta. \tag{1.3}$$

The relations between z's Cartesian and polar co-ordinates are simple — we see that

$$\begin{aligned} x &= r\cos\theta \text{ and } y = r\sin\theta, \\ r &= \sqrt{x^2+y^2} \text{ and } \tan\theta = \frac{y}{x}. \end{aligned}$$

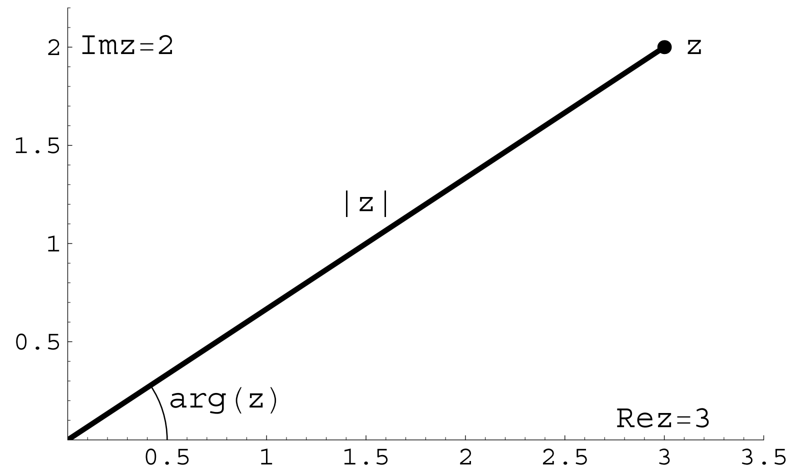
Definition 9 *The number r is called the modulus of z and is written $|z|$. If $z = x + iy$ then*

$$|z| = \sqrt{x^2+y^2}.$$

Definition 10 *The number θ is called the argument of z and is written* $\arg z$*. If $z = x + iy$ then*

$$\sin\arg z = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\arg z = \frac{x}{\sqrt{x^2+y^2}} \quad \textit{and} \quad \tan\arg z = \frac{y}{x}.$$

Note that the argument of 0 is undefined. Note also that $\arg z$ *is defined only up to multiples of 2π. For example, the argument of $1+i$ could be $\pi/4$ or $9\pi/4$ or $-7\pi/4$ etc.. For simplicity we shall give all arguments in the range $0 \leq \theta < 2\pi$, so that $\pi/4$ would be the preferred choice here.*



A complex number's Cartesian and polar co-ordinates

We now prove some important formulae about properties of the modulus, argument and conjugation:–

Proposition 11 *The modulus, argument and conjugate functions satisfy the following properties. Let $z,w\in\mathbb{C}$. Then*

$$
\begin{aligned}
|zw| &= |z|\,|w|, && (1.4)\\
\left|\frac{z}{w}\right| &= \frac{|z|}{|w|} \text{ if } w\neq 0, && (1.5)\\
\overline{z\pm w} &= \overline{z}\pm\overline{w}, && (1.6)\\
\overline{zw} &= \overline{z}\,\overline{w}, && (1.7)\\
\arg(zw) &= \arg z+\arg w \text{ if } z,w\neq 0, && (1.8)\\
z\overline{z} &= |z|^2, && (1.9)\\
\arg\left(\frac{z}{w}\right) &= \arg z-\arg w \quad \text{if } \ z,w\neq 0, && (1.10)\\
\overline{\left(\frac{z}{w}\right)} &= \frac{\overline{z}}{\overline{w}} \quad \text{if } \ w\neq 0, && (1.11)\\
|\overline{z}| &= |z|, && (1.12)\\
\arg\overline{z} &= -\arg z, && (1.13)\\
|z+w| &\leq |z|+|w|, && (1.14)\\
||z|-|w|| &\leq |z-w|. && (1.15)
\end{aligned}
$$

Proof. Identity (1.4) $|zw|=|z|\,|w|$.
Let $z=a+bi$ and $w=c+di$. Then $zw=(ac-bd)+(bc+ad)\,i$ so that

$$
\begin{aligned}
|zw| &= \sqrt{(ac-bd)^2+(bc+ad)^2}\\
&= \sqrt{a^2c^2+b^2d^2+b^2c^2+a^2d^2}\\
&= \sqrt{(a^2+b^2)(c^2+d^2)}\\
&= \sqrt{a^2+b^2}\sqrt{c^2+d^2}=|z|\,|w|.
\end{aligned}
$$

■

Proof. Identity (1.8) $\arg(zw)=\arg z+\arg w$.
Let $z=r(\cos\theta+i\sin\theta)$ and $w=R(\cos\Theta+i\sin\Theta)$. Then

$$
\begin{aligned}
zw &= rR(\cos\theta+i\sin\theta)(\cos\Theta+i\sin\Theta)\\
&= rR((\cos\theta\cos\Theta-\sin\theta\sin\Theta)+i(\sin\theta\cos\Theta+\cos\theta\sin\Theta))\\
&= rR(\cos(\theta+\Theta)+i\sin(\theta+\Theta)).
\end{aligned}
$$

We can read off that $|zw|=rR=|z|\,|w|$, which is a second proof of the previous part, and also that

$$\arg(zw)=\theta+\Theta=\arg z+\arg w, \quad \text{up to multiples of } 2\pi.$$

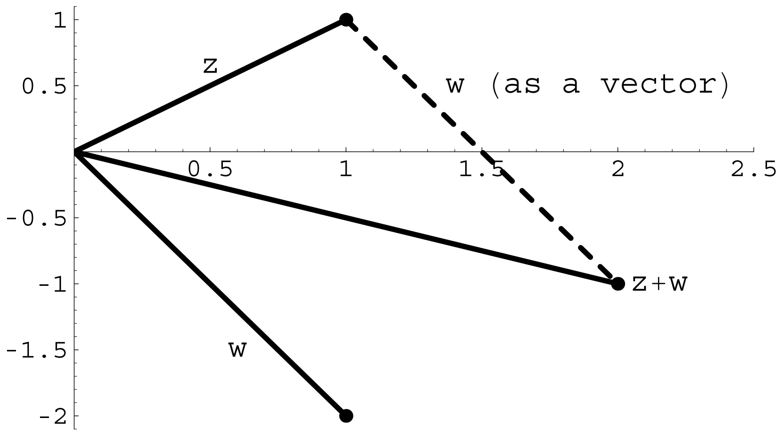
■

Proof. Identity (1.7) $\overline{zw} = \overline{z}\,\overline{w}$.
Let $z = a + bi$ and $w = c + di$. Then

$$\begin{aligned} \overline{zw} &= \overline{(ac - bd) + (bc + ad)\, i} \\ &= (ac - bd) - (bc + ad)\, i \\ &= (a - bi)\,(c - di) = \overline{z}\,\overline{w}. \end{aligned}$$

■

Proof. Identity (1.14): the Triangle Inequality $|z + w| \le |z| + |w|$. A diagrammatic proof of this is simple and explains the inequality's name:



Note that the shortest distance between 0 and $z + w$ is the modulus of $z + w$. This is shorter in length than the path which goes from 0 to z to $z + w$. The total length of this second path is $|z| + |w|$. For an algebraic proof, note that for any complex number

$$z + \bar{z} = 2\,\mathrm{Re}\, z \quad \text{and} \quad \mathrm{Re}\, z \le |z|\,.$$

So for $z, w \in \mathbb{C}$,

$$\frac{z\bar{w} + \bar{z}w}{2} = \mathrm{Re}\,(z\bar{w}) \le |z\bar{w}| = |z|\,|\bar{w}| = |z|\,|w|\,.$$

Then

$$\begin{aligned} |z + w|^2 &= (z + w)\,\overline{(z + w)} \\ &= (z + w)\,(\bar{z} + \bar{w}) \\ &= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\ &\le |z|^2 + 2\,|z|\,|w| + |w|^2 = (|z| + |w|)^2\,, \end{aligned}$$

to give the required result.

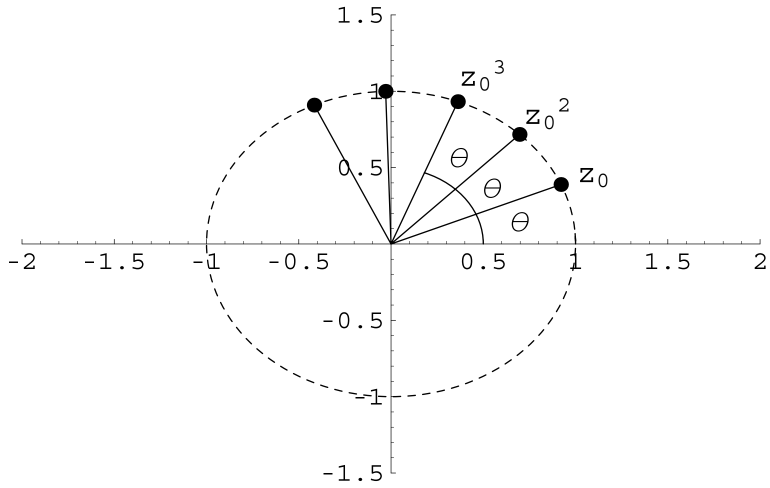
The remaining identities are left to Exercise 9 ■

1.1.4 Roots Of Unity

Consider the complex number

$$z_0 = \cos\theta + i\sin\theta$$

where θ is some real number in the range $0 \leqslant \theta < 2\pi$. The modulus of z_0 is 1 and the argument of z_0 is θ.



Powers of z_0

In Proposition 11 we proved for $z, w \neq 0$ that

$$|zw| = |z|\,|w| \quad \text{and} \quad \arg(zw) = \arg z + \arg w.$$

So for any integer n, and any $z \neq 0$, we have that

$$|z^n| = |z|^n \quad \text{and} \quad \arg(z^n) = n\arg z.$$

Then the modulus of $(z_0)^n$ is 1, and the argument of $(z_0)^n$ is $n\theta$ up to multiples of 2π. Putting this another way, we have the famous theorem due to De Moivre:

Theorem 12 *(DE MOIVRE'S THEOREM) For a real number θ and integer n we have that*

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

(De Moivre (1667-1754), a French protestant who moved to England, is best remembered for this formula but his major contributions were in probability and appeared in his *The Doctrine Of Chances* (1718)).

We apply these ideas now to the following:

Example 13 *Let $n \geqslant 1$ be a natural number. Find all those complex z such that $z^n = 1$.*

Solution. We know from the Fundamental Theorem of Algebra that there are (counting multiplicities) n solutions — these are known as *the nth roots of unity*. Let's first solve $z^n = 1$ directly for $n = 2, 3, 4$.

- When $n = 2$ we have

$$0 = z^2 - 1 = (z - 1)(z + 1)$$

  and so the square roots of 1 are ± 1.

- When $n = 3$ we can factorise as follows

$$0 = z^3 - 1 = (z - 1)\left(z^2 + z + 1\right).$$

  So 1 is a root and completing the square we see

$$0 = z^2 + z + 1 = \left(z + \frac{1}{2}\right)^2 + \frac{3}{4}$$

  which has roots $-1/2 \pm \sqrt{3}i/2$. So the cube roots of 1 are

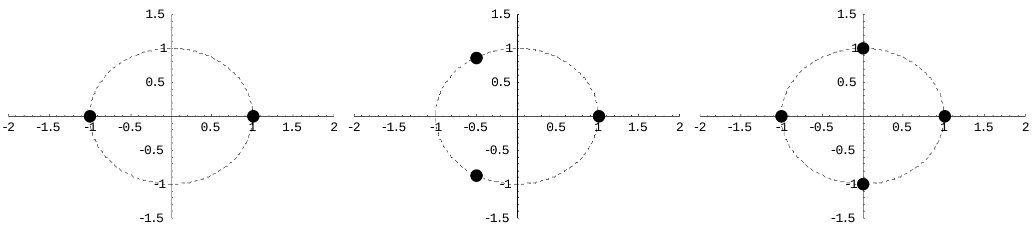
$$1 \quad \text{and} \quad \frac{-1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad \frac{-1}{2} - \frac{\sqrt{3}}{2}i.$$

- When $n = 4$ we can factorise as follows

$$0 = z^4 - 1 = \left(z^2 - 1\right)\left(z^2 + 1\right) = (z - 1)(z + 1)(z - i)(z + i),$$

  so that the fourth roots of 1 are $1, -1, i$ and $-i$.

Plotting these roots on Argand diagrams we can see a pattern developing



Square Roots Cube Roots Fourth Roots

Returning to the general case suppose that

$$z = r(\cos\theta + i\sin\theta) \quad \text{and satisfies} \quad z^n = 1.$$

Then by the observations preceding De Moivre's Theorem z^n has modulus r^n and has argument $n\theta$ whilst 1 has modulus 1 and argument 0. Then comparing their moduli

$$r^n = 1 \implies r = 1.$$

Comparing arguments we see $n\theta = 0$ up to multiples of 2π. That is, $n\theta = 2k\pi$ for some integer k giving $\theta = 2k\pi/n$. So we see that if $z^n = 1$ then z has the form

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \quad \text{where } k \text{ is an integer.}$$

At first glance there seems to be an infinite number of roots but, as cos and sin have period 2π, then these z repeat with period n. ■

Hence we have shown

Proposition 14 *The nth roots of unity, that is the solutions of the equation* $z^n = 1$, *are*

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \quad \textit{where} \quad k = 0, 1, 2, \ldots, n-1.$$

Plotted on an Argand diagram these nth roots of unity form a regular n-gon inscribed within the unit circle with a vertex at 1.

Problem 15 *Find all the solutions of the cubic* $z^3 = -2 + 2i$.

Solution. If we write $-2 + 2i$ in its polar form we have

$$-2 + 2i = \sqrt{8}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right).$$

So if $z^3 = -2 + 2i$ and z has modulus r and argument θ then

$$r^3 = \sqrt{8} \quad \text{and} \quad 3\theta = \frac{3\pi}{4} \quad \text{up to multiples of } 2\pi,$$

which gives

$$r = \sqrt{2} \quad \text{and} \quad \theta = \frac{\pi}{4} + \frac{2k\pi}{3} \quad \text{for some integer } k.$$

As before we need only consider $k = 0, 1, 2$ (as other values of k lead to repeats) and so the three roots are

$$\begin{aligned}
\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) &= 1 + i, \\
\sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right) &= \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}\right) + i\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right), \\
\sqrt{2}\left(\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right) &= \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}\right) + i\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\right).
\end{aligned}$$

■

1.2 Their Analysis

1.2.1 The Complex Exponential Function

The real exponential function e^x (or $\exp x$) can be defined in several different ways. One such definition is by power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

The above infinite sum converges for all real values of x. What this means, is that for any real value of our *input* x, as we add more and more of the terms from the infinite sum above we generate a list of numbers which get closer and closer to some value — this value we denote e^x. Different inputs will mean the sum converges to different answers. As an example, let's consider the case when $x = 2$:

1 term:	1	$= 1.0000$	6 terms:	$1 + \cdots + \frac{32}{120}$	$\cong 7.2667$
2 terms:	$1 + 2$	$= 3.0000$	7 terms	$1 + \cdots + \frac{64}{720}$	$\cong 7.3556$
3 terms:	$1 + 2 + \frac{4}{2}$	$= 5.0000$	8 terms	$1 + \cdots + \frac{128}{5040}$	$\cong 7.3810$
4 terms:	$1 + \cdots + \frac{8}{6}$	$\cong 6.3333$	9 terms	$1 + \cdots + \frac{256}{40320}$	$\cong 7.3873$
5 terms:	$1 + \cdots + \frac{16}{24}$	$= 7.0000$	∞ terms	e^2	$\cong 7.3891$

This idea of a power series defining a function should not be too alien — it is likely that you have already seen that the infinite geometric progression

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

converges to $(1 - x)^{-1}$, at least when $|x| < 1$. This is another example of a power series defining a function.

Proposition 16 *Let x be a real number. Then*

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

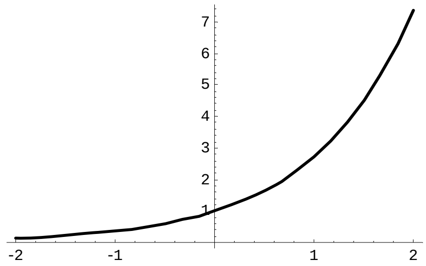
converges to a real value which we shall denote as e^x. The function e^x has the following properties

$$\text{(i)} \qquad \frac{d}{dx}e^x = e^x, \qquad e^0 = 1;$$

$$\text{(ii)} \qquad e^{x+y} = e^x e^y \quad \text{for any real } x, y;$$

$$\text{(iii)} \qquad e^x > 0 \text{ for any real } x.$$

and a sketch of the exponential's graph is given below.



The graph of $y = e^x$.

That these properties hold true of e^x is discussed in more detail in the appendices at the end of this chapter.

- Property (i) uniquely characterises the exponential function. That is, there is a unique real-valued function e^x which differentiates to itself, and which takes the value 1 at 0.

- Note that when $x = 1$ this gives us the identity

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots \cong 2.718.$$

We can use either the power series definition, or one equivalent to property (i), to define the *complex* exponential function.

Proposition 17 *Let z be a complex number. Then*

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

converges to a complex value which we shall denote as e^z. The function e^z has the following properties

$$\begin{array}{ll} \textit{(i)} & \frac{d}{dz}e^z = e^z, \qquad e^0 = 1; \\ \textit{(ii)} & e^{z+w} = e^z e^w \ \ \textit{for any complex} \ \ z, w; \\ \textit{(iii)} & e^z \neq 0 \ \ \textit{for any complex} \ z. \end{array}$$

Analytically we can differentiate complex functions in much the same way as we differentiate real functions. The product, quotient and chain rules apply in the usual way, and z^n has derivative nz^{n-1} for any integer n.

By taking more and more terms in the series, we can calculate e^z to greater and greater degrees of accuracy as before. For example, to calculate e^{1+i} we see

1 term:	1	$= 1.0000$
2 terms:	$1+(1+i)$	$= 2.0000 + 1.0000i$
3 terms:	$1+(1+i)+\frac{2i}{2}$	$= 2.0000 + 2.0000i$
4 terms:	$1+\cdots+\frac{-2+2i}{6}$	$\cong 1.6667 + 2.3333i$
5 terms:	$1+\cdots+\frac{-4}{24}$	$\cong 1.5000 + 2.3333i$
6 terms:	$1+\cdots+\frac{-4-4i}{120}$	$\cong 1.4667 + 2.3000i$
7 terms:	$1+\cdots+\frac{-8i}{720}$	$\cong 1.4667 + 2.2889i$
8 terms:	$1+\cdots+\frac{8-8i}{5040}$	$\cong 1.4683 + 2.2873i$
9 terms:	$1+\cdots+\frac{16}{40320}$	$\cong 1.4687 + 2.2873i$
∞ terms:	e^{1+i}	$\cong 1.4687 + 2.2874i$

Their are two other important functions, known as hyperbolic functions, which are closely related to the exponential function — namely *hyperbolic cosine* $\cosh z$ and *hyperbolic sine* $\sinh z$.

Definition 18 *Let z be a complex number. Then we define*

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

Corollary 19 *Hyperbolic sine and hyperbolic cosine have the following properties (which can easily be derived from the properties of the exponential function given in Proposition 17). For complex numbers z and w:*

$$\text{(i)} \qquad \cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots$$

$$\text{(ii)} \qquad \sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n+1}}{(2n+1)!} + \cdots$$

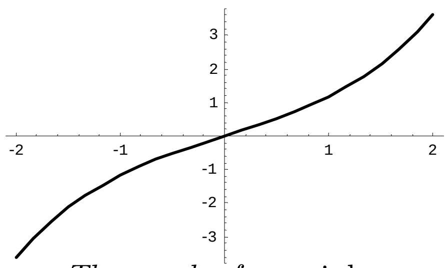
$$\text{(iii)} \qquad \frac{d}{dz}\cosh z = \sinh z \quad \text{and} \quad \frac{d}{dz}\sinh z = \cosh z,$$

$$\text{(iv)} \qquad \cosh(z+w) = \cosh z \cosh w + \sinh z \sinh w,$$

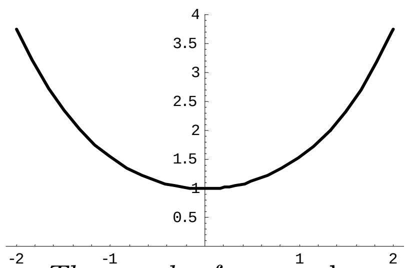
$$\text{(v)} \qquad \sinh(z+w) = \sinh z \cosh w + \cosh z \sinh w,$$

$$\text{(vi)} \qquad \cosh(-z) = \cosh z \quad \text{and} \quad \sinh(-z) = -\sinh z.$$

and graphs of the sinh *and* cosh *are sketched below for real values of x*



The graph of $y = \sinh x$



The graph of $y = \cosh x$

The real functions *sine* and *cosine* can similarly be defined by power series and other characterising properties. Note that these definitions give us the sine and cosine of x *radians.*

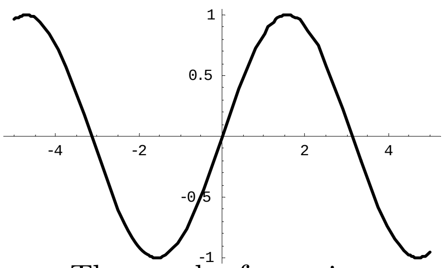
Proposition 20 *Let x be a real number. Then*

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots, \quad \textit{and}$$
$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots$$

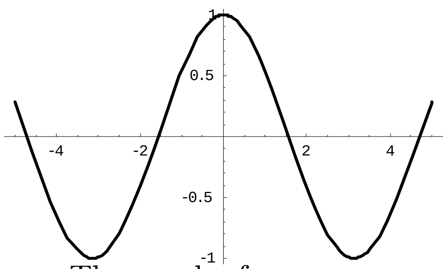
converge to real values which we shall denote as $\cos x$ *and* $\sin x$*. The functions* $\cos x$ *and* $\sin x$ *have the following properties*

$$\begin{array}{ll}
\textit{(i)} & \frac{d^2}{dx^2}\cos x=-\cos x, \qquad \cos 0=1, \quad \cos' 0=0,\\
\textit{(ii)} & \frac{d^2}{dx^2}\sin x=-\sin x, \qquad \sin 0=0, \quad \sin' 0=1,\\
\textit{(iii)} & \frac{d}{dx}\cos x=-\sin x, \quad \textit{and} \quad \frac{d}{dx}\sin x=\cos x,\\
\textit{(iv)} & -1\leq\cos x\leq 1 \quad \textit{and} \quad -1\leq\sin x\leq 1,\\
\textit{(v)} & \cos(-x)=\cos x \quad \textit{and} \quad \sin(-x)=-\sin x.
\end{array}$$

- Property (i) above characterises $\cos x$ and property (ii) characterises $\sin x$ — that is $\cos x$ and $\sin x$ are the unique real functions with these respective properties.



The graph of $y=\sin x$



The graph of $y=\cos x$

As before we can extend these power series to the complex numbers to define the complex trigonometric functions.

Proposition 21 *Let z be a complex number. Then the series*

$$1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots+(-1)^n\frac{z^{2n}}{(2n)!}+\cdots, \quad \textit{and}$$
$$z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots+(-1)^n\frac{z^{2n+1}}{(2n+1)!}+\cdots$$

converge to complex values which we shall denote as $\cos z$ *and* $\sin z$*. The functions* $\cos$ *and* $\sin$ *have the following properties*

$$\begin{array}{ll} (i) & \dfrac{d^2}{dz^2}\cos z = -\cos z, \qquad \cos 0 = 1, \quad \cos' 0 = 0, \\ (ii) & \dfrac{d^2}{dz^2}\sin z = -\sin z, \qquad \sin 0 = 0, \quad \sin' 0 = 1, \\ (iii) & \dfrac{d}{dz}\cos z = -\sin z, \quad \text{and} \quad \dfrac{d}{dz}\sin z = \cos z, \\ (iv) & \text{Neither } \sin \text{ nor } \cos \text{ is bounded on the complex plane,} \\ (v) & \cos(-z) = \cos z \quad \text{and} \quad \sin(-z) = -\sin z. \end{array}$$

Example 22 *Prove that* $\cos^2 z + \sin^2 z = 1$ *for all complex numbers* z*. (Note that, as we are dealing with complex numbers, this does not imply that* $\cos z$ *and* $\sin z$ *have modulus less than or equal to 1.)*

Solution. Define

$$F(z) = \sin^2 z + \cos^2 z.$$

Differentiating F, using the previous proposition and the product rule we see

$$F'(z) = 2\sin z \cos z + 2\cos z \times (-\sin z) = 0.$$

As the derivative $F' = 0$ then F must be constant. We note that

$$F(0) = \sin^2 0 + \cos^2 0 = 0^2 + 1^2 = 1$$

and hence $F(z) = 1$ for all z. ■

Contrast this with:

Example 23 *Prove that* $\cosh^2 - \sinh^2 z = 1$ *for all complex numbers* z*.*

Solution. We could argue similarly to the above. Alternatively as

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

and using $e^z e^{-z} = e^{z-z} = e^0 = 1$ from Proposition 17 we see

$$\begin{aligned} \cosh^2 z - \sinh^2 z &= \left[\frac{(e^z)^2 + 2e^z e^{-z} + (e^z)^2}{4}\right] - \left[\frac{(e^z)^2 - 2e^z e^{-z} + (e^{-z})^2}{4}\right] \\ &= \frac{4e^z e^{-z}}{4} = 1. \end{aligned}$$

■

It is for these reasons that the functions cosh and sinh are called *hyperbolic functions* and the functions sin and cos are often referred to as the *circular functions.* From the first example above we see that the point $(\cos t, \sin t)$ lies on the circle $x^2 + y^2 = 1$. As we vary t between 0 and 2π this point moves once anti-clockwise around the unit circle. In contrast, the point $(\cosh t, \sinh t)$ lies on the curve $x^2 - y^2 = 1$. This is the equation of a *hyperbola.* As t varies through the reals then $(\cosh t, \sinh t)$ maps out all of the right branch of the hyperbola. We can obtain the left branch by varying the point $(-\cosh t, \sinh t)$.

1.2.3 Identities

From looking at the graphs of $\exp x, \sin x, \cos x$ for real values of x it seems unlikely that all three functions can be related. The $\sin x$ and $\cos x$ are just out-of-phase but the exponential is unbounded unlike the trigonometric functions and has no periodicity. However, once viewed as functions of a complex variable, it is relatively easy to demonstrate a fundamental identity connecting the three. The following is due to Euler, dating from 1740.

Theorem 24 *Let z be a complex number. Then*

$$e^{iz} = \cos z + i \sin z.$$

Proof. Note that the sequence i^n of powers of i goes $1, i, -1, -i, 1, i, -1, -i, \ldots$ repeating forever with period 4. So, recalling the power series definitions of the exponential and trigonometric functions from Propositions 17 and 21, we see

$$\begin{aligned} e^{iz} &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \cdots \\ &= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} + \cdots \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\ &= \cos z + i \sin z. \end{aligned}$$

■

- Note that $\cos z \neq \operatorname{Re} e^{iz}$ and $\sin z \neq \operatorname{Im} e^{iz}$ in general for complex z.
- When we put $z = \pi$ into this proposition we find

  $$e^{i\pi} = -1.$$

  This is referred to as Euler's Equation, and is often credited as being the most beautiful equation in all of mathematics because it relates the fundamental constants $1, i, \pi, e$.
- Note that the complex exponential function has period $2\pi i$. That is

  $$e^{z+2\pi i} = e^z \quad \text{for all complex numbers } z.$$
- More generally when θ is a real number we see that

  $$e^{i\theta} = \cos\theta + i \sin\theta$$

  and so the polar form of a complex number from equation (1.3) is often written as

  $$z = re^{i\theta}.$$

  Moreover in these terms, De Moivre's Theorem (see Theorem 12) is the less surprising identity

  $$\left(e^{i\theta}\right)^n = e^{i(n\theta)}.$$

- If $z = x + iy$ then

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y$$

  and so

$$|e^z| = e^x \quad \text{and} \quad \arg e^z = y.$$

As a corollary to the previous theorem we can now express $\cos z$ and $\sin z$ in terms of the exponential. We note

Corollary 25 *Let z be a complex number. Then*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \textit{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and

$$\begin{aligned} \cosh z &= \cos iz \quad \textit{and} \quad i \sinh z = \sin iz \\ \cos z &= \cosh iz \quad \textit{and} \quad i \sin z = \sinh iz. \end{aligned}$$

Proof. As cos is even and sin is odd then

$$e^{iz} = \cos z + i \sin z \quad \text{and} \quad e^{-iz} = \cos z - i \sin z.$$

Solving for $\cos z$ and $\sin z$ from these simultaneous equations we arrive at the required expressions. The others are easily verified from our these new expressions for cos and sin and our previous ones for cosh and sinh. ■

1.2.4 Applications

We can now turn these formula towards some applications and calculations. The following demonstrates, for one specific case, how formulae for $\cos nz$ and $\sin nz$ can be found in terms of powers of $\sin z$ and $\cos z$. The second problem demonstrates a specific case of the reverse process — writing powers of $\cos z$ or $\sin z$ as combinations of $\cos nz$ and $\sin nz$ for various n.

Example 26 *Show that*

$$\cos 5z = 16 \cos^5 z - 20 \cos^3 z + 5 \cos z.$$

Solution. Recall from De Moivre's Theorem that

$$(\cos z + i \sin z)^5 = \cos 5z + i \sin 5z.$$

Now if x and y are real then by the Binomial Theorem

$$(x + iy)^5 = x^5 + 5ix^4y - 10x^3y^2 - 10ix^2y^3 + 5xy^4 + iy^5.$$

Hence

$$\begin{aligned}
\cos 5\theta &= \operatorname{Re}(\cos\theta + i\sin\theta)^5 \\
&= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\
&= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta\left(1-\cos^2\theta\right)^2 \\
&= (1+10+5)\cos^5\theta + (-10-10)\cos^3\theta + 5\cos\theta \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.
\end{aligned}$$

This formula in fact holds true when θ is a general complex argument and not necessarily real. ■

Example 27 *Let z be a complex number. Prove that*

$$\sin^4 z = \frac{1}{8}\cos 4z - \frac{1}{2}\cos 2z + \frac{3}{8}.$$

Hence find the power series for $\sin^4 z$.

Solution. We have that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

So

$$\begin{aligned}
\sin^4 z &= \frac{1}{(2i)^4}\left(e^{iz} - e^{-iz}\right)^4 \\
&= \frac{1}{16}\left(e^{4iz} - 4e^{2iz} + 6 - 4e^{-2iz} + e^{-4iz}\right) \\
&= \frac{1}{16}\left(\left(e^{4iz} + e^{-4iz}\right) - 4\left(e^{2iz} + e^{-2iz}\right) + 6\right) \\
&= \frac{1}{16}(2\cos 4z - 8\cos 2z + 6) \\
&= \frac{1}{8}\cos 4z - \frac{1}{2}\cos 2z + \frac{3}{8},
\end{aligned}$$

as required. Now $\sin^4 z$ has only even powers of z^{2n} in its power series. From our earlier power series for $\cos z$ we see, when $n > 0$, the coefficient of z^{2n} equals

$$\frac{1}{8} \times (-1)^n \frac{4^{2n}}{(2n)!} - \frac{1}{2} \times (-1)^n \frac{2^{2n}}{(2n)!} = (-1)^n \frac{2^{4n-3} - 2^{2n-1}}{(2n)!} z^{2n}$$

which we note is zero when $n = 1$. Also when $n = 0$ we see that the constant term is $1/8 - 1/2 + 3/8 = 0$. So the required power series is

$$\sin^4 z = \sum_{n=2}^{\infty} (-1)^n \frac{2^{4n-3} - 2^{2n-1}}{(2n)!} z^{2n}.$$

■

Example 28 *Prove for any complex numbers z and w that*

$$\sin(z+w) = \sin z \cos w + \cos z \sin w.$$

Solution. Recalling the expressions for sin and cos from Corollary 25 we have

$$\begin{aligned}
\text{RHS} &= \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iw}+e^{-iw}}{2}\right)+\left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{iw}-e^{-iw}}{2i}\right) \\
&= \frac{2e^{iz}e^{iw}-2e^{-iz}e^{-iw}}{4i} \\
&= \frac{e^{i(z+w)}-e^{-i(z+w)}}{2i} = \sin(z+w) = \text{LHS}.
\end{aligned}$$

■

Example 29 *Prove that for complex z and w*

$$\sin(z+iw) = \sin z \cosh w + i \cos z \sinh w.$$

Solution. Use the previous problem recalling that $\cos(iw) = \cosh w$ and $\sin(iw) = i \sinh w$. ■

Example 30 *Let x be a real number and n a natural number. Show that*

$$\sum_{k=0}^{n} \cos kx = \frac{\cos \frac{n}{2}x \sin \frac{n+1}{2}x}{\sin \frac{1}{2}x} \quad \textit{and} \quad \sum_{k=0}^{n} \sin kx = \frac{\sin \frac{n}{2}x \sin \frac{n+1}{2}x}{\sin \frac{1}{2}x}$$

Solution. As $\cos kx + i \sin kx = \left(e^{ix}\right)^k$ then these sums are the real and imaginary parts of a geometric series, with first term 1, common ration e^{ix} and $n+1$ terms in total. So recalling

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1},$$

we have

$$\begin{aligned}
\sum_{k=0}^{n} \left(e^{ix}\right)^k &= \frac{e^{(n+1)ix}-1}{e^{ix}-1} \\
&= \frac{e^{inx/2}\left(e^{(n+1)ix/2}-e^{-(n+1)ix/2}\right)}{e^{ix/2}-e^{-ix/2}} \\
&= e^{inx/2}\frac{2i \sin \frac{n+1}{2}x}{2i \sin \frac{1}{2}x} \\
&= \left(\cos \frac{nx}{2} + i \sin \frac{nx}{2}\right)\frac{\sin \frac{n+1}{2}x}{\sin \frac{1}{2}x}.
\end{aligned}$$

The results follow by taking real and imaginary parts. Again this identity holds for complex values of x as well. ■

1.3 Their Geometry

1.3.1 Distance and Angles in the Complex Plane

Let $z = z_1 + iz_2$ and $w = w_1 + iw_2$ be two complex numbers. By Pythagoras' Theorem the distance between z and w as points in the complex plane equals

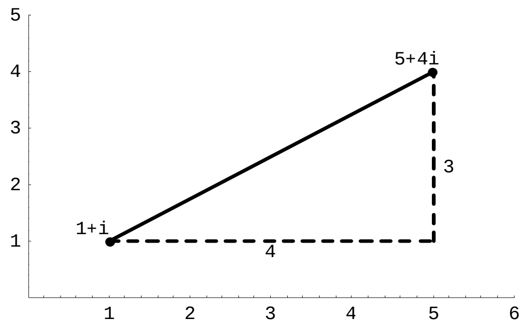
$$\begin{aligned} \text{distance} &= \sqrt{(z_1 - w_1)^2 + (z_2 - w_2)^2} \\ &= |(z_1 - w_1) + i(z_2 - w_2)| \\ &= |(z_1 + iz_2) - (w_1 + iw_2)| \\ &= |z - w|. \end{aligned}$$

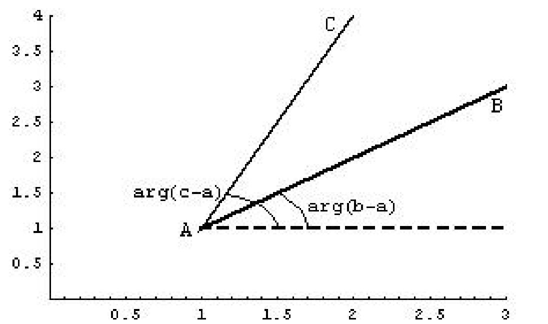
Let $a = a_1 + ia_2$, $b = b_1 + ib_2$, and $c = c_1 + ic_2$ be three points in the complex plane representing three points A, B and C. To calculate the angle $\measuredangle BAC$ as in the diagram we see

$$\measuredangle BAC = \arg(c - a) - \arg(b - a) = \arg\left(\frac{c - a}{b - a}\right).$$

Note that if in the diagram B and C we switched then we get the larger angle

$$\arg\left(\frac{c - a}{b - a}\right) = 2\pi - \measuredangle BAC.$$





The distance here is $\sqrt{3^2 + 4^2} = 5$ The angle is $\arg\left(\frac{1+3i}{2+i}\right) = \arg(1 + i) = \frac{1}{4}\pi$

Problem 31 *Find the smaller angle* $\measuredangle BAC$ *where* $a = 1 + i$, $b = 3 + 2i$, *and* $c = 4 - 3i$.

Solution. The angle $\measuredangle BAC$ is given by

$$\arg\left(\frac{b - a}{c - a}\right) = \arg\left(\frac{2 + i}{3 - 4i}\right) = \arg\left(\frac{2 + 11i}{25}\right) = \tan^{-1}\left(\frac{11}{2}\right).$$

■

1.3.2 A Selection of Geometric Theory

When using complex numbers to prove geometric theorems it is prudent to choose complex co-ordinates so as to make any calculations as simple as possible. If we put co-ordinates on the plane (which after all begins as featureless and blank, like any blackboard or sheet of paper) we can choose

- where to put the origin;
- where the real and imaginary axes go;
- what unit length to use.

Practically this is not that surprising: any astronomical calculation involving the sun and the earth might well begin by taking the sun as the origin and is more likely to use miles or astronomical units than it is centimetres; modelling a projectile shot from a gun would probably take time t from the moment the gun is shot and z as the height above the ground and metres and seconds are more likely to be employed rather than miles and years. Similarly in a geometrical situation, if asked to prove a theorem about a circle then we could take the centre of the circle as the origin. We could also choose our unit length to be that of the radius of the circle. If we had labelled points on the circle to consider then we can take one of the points to be the point 1; in this case these choices (largely) use up all our *degrees of freedom* and any other points would need to be treated generally. If we were considering a triangle, then we could choose two of the vertices to be 0 and 1 but the other point (unless we know something special about the triangle, say that it is equilateral or isosceles) we need to treat as an arbitrary point z.

We now prove a selection of basic geometric facts. Here is a quick reminder of some identities which will prove useful in their proofs.

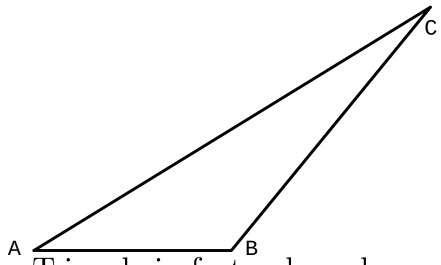
$$\operatorname{Re} z = \frac{z+\overline{z}}{2}, \qquad z\overline{z} = |z|^2, \qquad \cos\arg z = \frac{\operatorname{Re} z}{|z|}.$$

Theorem 32 *(THE COSINE RULE). Let ABC be a triangle. Then*

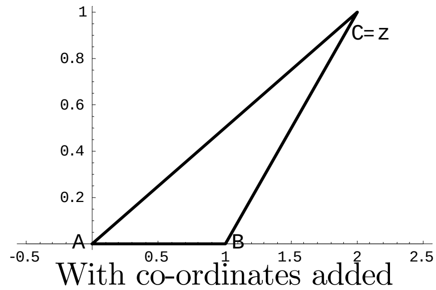
$$|BC|^2 = |AB|^2 + |AC|^2 - 2\,|AB|\,|AC|\cos\hat{A}. \tag{1.16}$$

Proof. We can choose our co-ordinates in the plane so that A is at the origin and B is at 1. Let C be at the point z. So in terms of our co-ordinates:

$$|AB| = 1, \qquad |BC| = |z-1|, \qquad |AC| = |z|, \qquad \hat{A} = \arg z.$$



Triangle in featureless plane



With co-ordinates added

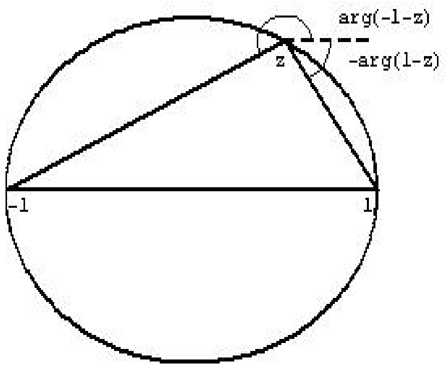
So

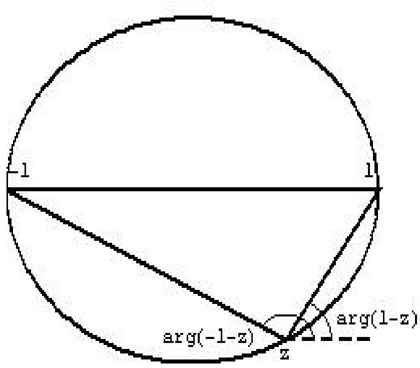
$$\begin{aligned}
\text{RHS of (1.16)} &= |z|^2 + 1 - 2|z| \cos \arg z \\
&= z\overline{z} + 1 - 2|z| \times \frac{\operatorname{Re} z}{|z|} \\
&= z\overline{z} + 1 - 2 \times \frac{(z + \overline{z})}{2} \\
&= z\overline{z} + 1 - z - \overline{z} \\
&= (z-1)(\overline{z}-1) \\
&= |z-1|^2 = \text{LHS of (1.16)}.
\end{aligned}$$

■

Theorem 33 *The diameter of a circle subtends a right angle at the circumference.*

Proof. We can choose our co-ordinates in the plane so that the circle has unit radius with its centre at the origin and with the diameter in question having endpoints 1 and -1. Take an arbitrary point z in the complex plane — for the moment we won't assume z to be on the circumference.





From the diagrams we see that below the diameter we want to show

$$\arg(-1-z) - \arg(1-z) = \frac{\pi}{2},$$

and above the diameter we wish to show that

$$\arg(-1-z) - \arg(1-z) = \frac{3\pi}{2}.$$

Recalling that $\arg(z/w) = \arg z - \arg w$ we see that we need to prove that

$$\arg\left(\frac{-1-z}{1-z}\right) = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

or equivalently we wish to show that $(-1-z)/(1-z)$ is *purely* imaginary — i.e. it has no real part. To say that a complex number w is purely imaginary is equivalent to saying that $w = -\overline{w}$, i.e. that

$$\left(\frac{-1-z}{1-z}\right) = -\overline{\left(\frac{-1-z}{1-z}\right)}.$$

which is the same as saying

$$\frac{-1-z}{1-z} = \frac{1+\bar{z}}{1-\bar{z}}.$$

Multiplying up we see this is the same as

$$(-1-z)(1-\bar{z}) = (1+\bar{z})(1-z).$$

Expanding this becomes

$$-1 - z + \bar{z} + z\bar{z} = 1 + \bar{z} - z - z\bar{z}.$$

A final rearranging gives $z\bar{z} = 1$, but as $|z|^2 = z\bar{z}$ we see we must have

$$|z| = 1.$$

We have proved the required theorem. In fact we've proved more than this also demonstrating its converse: that the diameter subtends a right angle at a point on the circumference and subtends right angles nowhere else. ■

1.3.3 Transformations of the Complex Plane

We now describe some transformations of the complex plane and show how they can be written in terms of complex numbers.

- **Translations:** A translation of the plane is one which takes the point (x, y) to the point $(x + a, y + b)$ where a and b are two real constants. In terms of complex co-ordinates this is the map $z \mapsto z + z_0$ where $z_0 = a + ib$.

- **Rotations:** Consider rotating the plane about the origin anti-clockwise through an angle α. If we take an arbitrary point in polar form $re^{i\theta}$ then this will rotate to the point $re^{i(\theta+\alpha)} = re^{i\theta}e^{i\alpha}$. So this particular rotation, about the origin, is represented in complex co-ordinates as the map

$$z \mapsto ze^{i\alpha}.$$

  More generally, any rotation of $\mathbb{C}$, not necessarily about the origin has the form $z \mapsto az + b$ where $a, b \in \mathbb{C}$, with $|a| = 1$ and $a \neq 1$.

- **Reflections:** We have already commented that $z \mapsto \overline{z}$ denotes reflection in the real axis.

  More generally, any reflection about the origin has the form $z \mapsto a\overline{z} + b$ where $a, b \in \mathbb{C}$ and $|a| = 1$.

What we have listed here are the three types of *isometry* of $\mathbb{C}$. An isometry of $\mathbb{C}$ is a map $f : \mathbb{C} \to \mathbb{C}$ which preserves distance — that is for any two points z and w in $\mathbb{C}$ the distance between $f(z)$ and $f(w)$ equals the distance between a and b. Mathematically this means

$$|f(z) - f(w)| = |z - w|$$

for any complex numbers z and w. The following theorem, the proof of which is omitted here, characterises the isometries of $\mathbb{C}$.

Theorem 34 *Let $f : \mathbb{C} \to \mathbb{C}$ be an isometry. Then there exist complex numbers a and b with $|a| = 1$ such that*

$$f(z) = az + b \quad \text{or} \quad f(z) = a\overline{z} + b$$

for each $z \in \mathbb{C}$.

Example 35 *Express in the form $f(z) = a\overline{z} + b$, reflection in the line $x + y = 1$.*

Solution. Method One: Knowing from the theorem that the reflection has the form $f(z) = a\overline{z} + b$ we can find a and b by considering where two points go to. As 1 and i both lie on the line of reflection then they are both fixed. So

$$\begin{aligned} a1 + b &= a\overline{1} + b = 1, \\ -ai + b &= a\overline{i} + b = i. \end{aligned}$$

Substituting $b = 1 - a$ into the second equation we find

$$a = \frac{1 - i}{1 + i} = -i,$$

and $b = 1 + i$. Hence

$$f(z) = -i\overline{z} + 1 + i.$$

Method Two: We introduce as alternative method here — the idea of changing co-ordinates. We take a second set of complex co-ordinates in which the point $z = 1$ is the origin and for which the line of reflection is the real axis. The second complex co-ordinate w is related to the first co-ordinate z by

$$w = (1 + i)(z - 1).$$

For example when $z = 1$ then $w = 0$, when $z = i$ then $w = -2$, when $z = 2 - i$ then $w = 2$, when $z = 2 + i$ then $w = 2i$. The real axis for the w co-ordinate has equation $x+y=1$ and the imaginary axis has equation $y=x-1$ in terms of our original co-ordinates.

The point to all this is that as w's real axis is the line of reflection then the transformation we're interested in is given by $w \mapsto \overline{w}$ in the new co-ordinates. Take then a point with complex co-ordinate z in our original co-ordinates system. Its w-co-ordinate is $(1+i)(z-1)$ — note we haven't moved the point yet, we've just changed co-ordinates. Now if we reflect the point we know the w-co-ordinate of the new point is $\overline{(1+i)(z-1)} = (1-i)(\overline{z}-1)$. Finally to get from the w-co-ordinate of the image point to the z-co-ordinate we reverse the co-ordinate change to get

$$\frac{(1-i)(\overline{z}-1)}{1+i} + 1 = -i(\overline{z}-1) + 1 = -i\overline{z} + i + 1$$

as required. ■

1.4 Appendices

1.4.1 Appendix 1 — Properties of the Exponential

In Proposition 17 we stated the following for any complex number z.

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

converges to a complex value which we shall denote as e^z. The function e^z has the following properties

$$\begin{array}{ll} (i) & \frac{\mathrm{d}}{\mathrm{d}z} e^z = e^z, \qquad e^0 = 1, \\ (ii) & e^{z+w} = e^z e^w \text{ for any complex } z, w, \\ (iii) & e^z \neq 0 \text{ for any complex } z. \end{array}$$

For the moment we shall leave aside any convergence issues.

To prove property (i) we *assume* that we can differentiate a power series *term by term*. Then we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z} e^z &= \frac{\mathrm{d}}{\mathrm{d}z}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots\right) \\ &= 0 + 1 + \frac{2z}{2!} + \frac{3z^2}{3!} + \cdots + \frac{nz^{n-1}}{n!} + \cdots \\ &= 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^{n-1}}{(n-1)!} + \cdots \\ &= e^z. \end{aligned}$$

We give two proofs of property (ii)

Proof. Method One: Let x be a complex variable and let y be a *constant* (but arbitrary) complex number. Consider the function

$$F(x) = e^{y+x}e^{y-x}.$$

If we differentiate F by the product and chain rules, and knowing that e^x differentiates to itself we have

$$F'(x) = e^{y+x}e^{y-x} + e^{y+x}\left(-e^{y-x}\right) = 0$$

and so F is a constant function. But note that $F(y) = e^{2y}e^0 = e^{2y}$. Hence we have

$$e^{y+x}e^{y-x} = e^{2y}.$$

Now set

$$x = \frac{1}{2}(z-w) \quad \text{and} \quad y = \frac{1}{2}(z+w)$$

and we arrive at required identity: $e^z e^w = e^{z+w}$.

Method Two: If we multiply two convergent power series

$$\sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n t^n$$

we get another convergent power series

$$\sum_{n=0}^{\infty} c_n t^n \quad \text{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Consider

$$\begin{aligned} e^{zt} &= \sum_{n=0}^{\infty} \frac{z^n}{n!} t^n \quad \text{so that} \quad a_n = \frac{z^n}{n!}, \\ e^{wt} &= \sum_{n=0}^{\infty} \frac{w^n}{n!} t^n \quad \text{so that} \quad b_n = \frac{w^n}{n!}. \end{aligned}$$

Then

$$\begin{aligned} c_n &= \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} \\ &= \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} z^k w^{n-k} \\ &= \frac{1}{n!}(z+w)^n, \end{aligned}$$

by the binomial theorem. So

$$e^{zt}e^{wt} = \sum_{n=0}^{\infty}\frac{z^n}{n!}t^n\sum_{n=0}^{\infty}\frac{w^n}{n!}t^n = \sum_{n=0}^{\infty}\frac{(w+z)^n}{n!}t^n = e^{(w+z)t}.$$

If we set $t = 1$ then we have the required result. ■

Property (iii), that $e^z \neq 0$ for all complex z, follows from the fact that $e^z e^{-z} = 1$.

1.4.2 Appendix 2 – Power Series

We have assumed many properties of power series throughout this chapter which we state here, though it is beyond our aims to prove these facts rigorously.

As we have only been considering the power series of exponential, trigonometric and hyperbolic functions it would be reasonable, but incorrect, to think that all power series converge everywhere. This is far from the case.

Given a power series $\sum_{n=0}^{\infty} a_n z^n$, where the coefficients a_n are complex, there is a real or infinite number R in the range $0 \leq R \leq \infty$ such that

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{converges to some complex value when } |z| < R,$$
$$\sum_{n=0}^{\infty} a_n z^n \quad \text{does not converge to a complex value when } |z| > R.$$

What happens to the power series when $|z| = R$ depends very much on the individual power series.

The number R is called the *radius of convergence* of the power series.

For the exponential, trigonometric and hyperbolic power series we have already seen that $R = \infty$.

For the geometric progression $\sum_{n=0}^{\infty} z^n$ this converges to $(1-z)^{-1}$ when $|z| < 1$ and does not converge when $|z| \geq 1$. So for this power series $R = 1$.

An important fact that we assumed in the previous appendix is that a power series can be differentiated term by term to give the derivative of the power series. So the derivative of $\sum_{n=0}^{\infty} a_n z^n$ equals $\sum_{n=1}^{\infty} n a_n z^{n-1}$ and this will have the same radius of convergence as the original power series.

1.5 Exercises

1.5.1 Basic Algebra

Exercise 1 *Put each of the following numbers into the form* $a + bi$.

$$(1+2i)(3-i), \quad \frac{1+2i}{3-i}, \quad (1+i)^4.$$

Exercise 2 *Let* $z_1 = 1 + i$ *and let* $z_2 = 2 - 3i$. *Put each of the following into the form* $a + bi$.

$$z_1 + z_2, \quad z_1 - z_2, \quad z_1 z_2, \quad z_1/z_2, \quad \bar{z}_1 \bar{z}_2.$$

Exercise 3 *Find the modulus and argument of each of the following numbers.*

$$1+\sqrt{3}i, \quad (2+i)(3-i), \quad (1+i)^5, \quad \frac{(1+2i)^3}{(2-i)^3}.$$

Exercise 4 *Let* α *be a real number in the range* $0 < \alpha < \pi/2$. *Find the modulus and argument of the following numbers.*

$$\cos\alpha - i\sin\alpha, \quad \sin\alpha - i\cos\alpha, \quad 1 + i\tan\alpha, \quad 1 + \cos\alpha + i\sin\alpha.$$

Exercise 5 *Let* z *and* w *be two complex numbers such that* $zw = 0$. *Show either* $z = 0$ *or* $w = 0$.

Exercise 6 *Suppose that the complex number* α *is a square root of* z, *that is* $\alpha^2 = z$. *Show that the only other square root of* z *is* $-\alpha$. *Suppose now that the complex numbers* z_1 *and* z_2 *have square roots* $\pm\alpha_1$ *and* $\pm\alpha_2$ *respectively. Show that the square roots of* $z_1 z_2$ *are* $\pm\alpha_1\alpha_2$.

Exercise 7 *Prove that every non-zero complex number has two square roots.*

Exercise 8 *Let* ω *be a cube root of unity (i.e.* $\omega^3 = 1$*) such that* $\omega \neq 1$. *Show that*

$$1 + \omega + \omega^2 = 0.$$

Hence determine the three cube roots of unity in the form $a + bi$.

Exercise 9 *Prove the remaining identities from Proposition 11.*

Exercise 10 *Which of the following quadratic equations require the use of complex numbers to solve them?*

$$3x^2 + 2x - 1 = 0, \quad 2x^2 - 6x + 9 = 0, \quad -4x^2 + 7x - 9 = 0.$$

Exercise 11 *Find the square roots of* $-5 - 12i$, *and hence solve the quadratic equation*

$$z^2 - (4 + i)\, z + (5 + 5i) = 0.$$

Exercise 12 *Show that the complex number* $1+i$ *is a root of the cubic equation*

$$z^3 + z^2 + (5 - 7i)\, z - (10 + 2i) = 0,$$

and hence find the other two roots.

Exercise 13 *Show that the complex number* $2 + 3i$ *is a root of the quartic equation*

$$z^4 - 4z^3 + 17z^2 - 16z + 52 = 0,$$

and hence find the other three roots.

Exercise 14 *On separate axes, sketch the graphs of the following cubics, being sure to carefully label any turning points. In each case state how many of the cubic's roots are real.*

$$\begin{aligned} y_1(x) &= x^3 - x^2 - x + 1; \\ y_2(x) &= 3x^3 + 5x^2 + x + 1; \\ y_3(x) &= -2x^3 + x^2 - x + 1. \end{aligned}$$

Exercise 15 *Let* p *and* q *be real numbers with* $p \leq 0$. *Find the co-ordinates of the turning points of the cubic* $y = x^3 + px + q$. *Show that the cubic equation* $x^3 + px + q = 0$ *has three real roots, with two or more repeated, precisely when*

$$4p^3 + 27q^2 = 0.$$

Under what conditions on p *and* q *does* $x^3+px+q = 0$ *have (i) three distinct real roots, (ii) just one real root? How many real roots does the equation* $x^3+px+q = 0$ *have when* $p > 0$*?*

Exercise 16 *By making a substitution of the form* $X = x - \alpha$ *for a certain choice of* α, *transform the equation* $X^3 + aX^2 + bX + c = 0$ *into one of the form* $x^3 + px + q = 0$. *Hence find conditions under which the equation*

$$X^3 + aX^2 + bX + c = 0$$

has (i) three distinct real roots, (ii) three real roots involving repetitions, (iii) just one real root.

Exercise 17 *The cubic equation* $x^3+ax^2+bx+c=0$ *has roots* α,β,γ *so that*

$$x^3+ax^2+bx+c=(x-\alpha)(x-\beta)(x-\gamma).$$

By equating the coefficients of powers of x *in the previous equation, find expressions for* a,b *and* c *in terms of* α,β *and* γ.

Given that α,β,γ *are real, what can you deduce about their signs if (i)* $c<0$, *(ii)* $b<0$ *and* $c<0$, *(iii)* $b<0$ *and* $c=0$.

Exercise 18 *With* a,b,c *and* α,β,γ *as in the previous exercise, let* $S_n=\alpha^n+\beta^n+\gamma^n$. *Find expressions* S_0,S_1 *and* S_2 *in terms of* a,b *and* c. *Show further that*

$$S_{n+3}+aS_{n+2}+bS_{n+1}+cS_n=0$$

for $n\geq 0$ *and hence find expressions for* S_3 *and* S_4 *in terms of* a,b *and* c.

Exercise 19 *Consider the cubic equation* $z^3+mz+n=0$ *where* m *and* n *are real numbers. Let* Δ *be a square root of* $(n/2)^2+(m/3)^3$. *We then define* t *and* u *by*

$$t=-n/2+\Delta \quad \text{and} \quad u=n/2+\Delta,$$

and let T *and* U *respectively be cube roots of* t *and* u. *Show that* tu *is real, and that if* T *and* U *are chosen appropriately, then* $z=T-U$ *is a solution of the original cubic equation.*

Use this method to completely solve the equation $z^3+6z=20$. *By making a substitution of the form* $w=z-a$ *for a suitable choice of* a, *find all three roots of the equation* $8w^3+12w^2+54w=135$.

1.5.3 De Moivre's Theorem and Roots of Unity

Exercise 20 *Use De Moivre's Theorem to show that*

$$\cos 6\theta=32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1,$$

and that

$$\sin 5\theta=16\sin^5\theta-20\sin^3\theta+5\sin\theta.$$

Exercise 21 *Let* $z=\cos\theta+i\sin\theta$ *and let* n *be an integer. Show that*

$$2\cos\theta=z+\frac{1}{z} \quad \text{and that} \quad 2i\sin\theta=z-\frac{1}{z}.$$

Find expressions for $\cos n\theta$ *and* $\sin n\theta$ *in terms of* z.

Exercise 22 *Show that*

$$\cos^5\theta=\frac{1}{16}(\cos 5\theta+5\cos 3\theta+10\cos\theta)$$

and hence find $\int_0^{\pi/2}\cos^5\theta\ d\theta$.

Exercise 23 *Let*

$$\zeta = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}.$$

Show that $\zeta^5 = 1$, *and deduce that* $1+\zeta+\zeta^2+\zeta^3+\zeta^4 = 0$.

Find the quadratic equation with roots $\zeta+\zeta^4$ *and* $\zeta^2+\zeta^3$. *Hence show that*

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}.$$

Exercise 24 *Determine the modulus and argument of the two complex numbers* $1+i$ *and* $\sqrt{3}+i$. *Also write the number*

$$\frac{1+i}{\sqrt{3}+i}$$

in the form $x+iy$. *Deduce that*

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} \quad \textit{and} \quad \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Exercise 25 *By considering the seventh roots of* -1 *show that*

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}.$$

What is the value of

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\,?$$

Exercise 26 *Find all the roots of the equation* $z^8 = -1$. *Hence, write* z^8+1 *as the product of four quadratic factors.*

Exercise 27 *Show that*

$$z^7 - 1 = (z-1)\left(z^2-\alpha z+1\right)\left(z^2-\beta z+1\right)\left(z^2-\gamma z+1\right)$$

where α, β, γ *satisfy*

$$(u-\alpha)(u-\beta)(u-\gamma) = u^3+u^2-2u-1.$$

Exercise 28 *Find all the roots of the following equations.*

1. $1+z^2+z^4+z^6 = 0$,
2. $1+z^3+z^6 = 0$,
3. $(1+z)^5 - z^5 = 0$,
4. $(z+1)^9 + (z-1)^9 = 0$.

Exercise 29 *Express* $\tan 7\theta$ *in terms of* $\tan\theta$ *and its powers. Hence solve the equation*

$$x^6 - 21x^4 + 35x^2 - 7 = 0.$$

Exercise 30 *Show for any complex number* z*, and any positive integer* n*, that*

$$z^{2n} - 1 = \left(z^2 - 1\right) \prod_{k=1}^{n-1} \left\{ z^2 - 2z\cos\frac{k\pi}{n} + 1 \right\}.$$

By setting $z = \cos\theta + i\sin\theta$ *show that*

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1} \prod_{k=1}^{n-1} \left\{ \cos\theta - \cos\frac{k\pi}{n} \right\}.$$

1.5.4 Geometry and the Argand Diagram

Exercise 31 *On separate Argand diagrams sketch the following sets:*

1. $|z| < 1$;
2. $\operatorname{Re} z = 3$;
3. $|z - 1| = |z + i|$;
4. $-\pi/4 < \arg z < \pi/4$;
5. $\operatorname{Re}(z + 1) = |z - 1|$;
6. $\arg(z - i) = \pi/2$;
7. $|z - 3 - 4i| = 5$;
8. $\operatorname{Re}((1 + i) z) = 1$.
9. $\operatorname{Im}(z^3) > 0$.

Exercise 32 *Multiplication by* i *takes the point* $x + iy$ *to the point* $-y + ix$*. What transformation of the Argand diagram does this represent? What is the effect of multiplying a complex number by* $(1 + i)/\sqrt{2}$*? [Hint: recall that this is square root of* i*.]*

Exercise 33 *Let* ABC *be a triangle in* $\mathbb{C}$ *with vertices* $A = 1 + i, B = 2 + 3i, C = 5 + 2i$*. Write, in the form* $a + bi$*, the images of the three vertices when the plane is rotated about* 0 *through* $\pi/3$ *radians anti-clockwise.*

Exercise 34 *Let* $a, b \in \mathbb{C}$ *with* $|a| = 1$*. Show directly that the map* $f : \mathbb{C} \to \mathbb{C}$ *given by* $f(z) = az + b$ *preserves distances and angles.*

Exercise 35 *Write in the form* $z \mapsto az + b$ *the rotation through* $\pi/3$ *radians anti-clockwise about the point* $2 + i$.

Exercise 36 *Write in the form* $z \mapsto a\bar{z}+b$ *the reflection in the line* $3x+2y = 6$.

Exercise 37 *Let* $f : \mathbb{C} \to \mathbb{C}$ *be given by* $f(z) = iz + 3 - i$. *Find a map* $g : \mathbb{C} \to \mathbb{C}$ *of the form* $g(z) = az + b$ *where* $|a| = 1$ *such that*

$$g(g(z)) = f(z).$$

How many such maps g *are there? Geometrically what transformations do these maps* g *and the map* f *represent?*

Exercise 38 *Find two reflections* $h : \mathbb{C} \to \mathbb{C}$ *and* $k : \mathbb{C} \to \mathbb{C}$ *such that* $k(h(z)) = iz$ *for all* z.

Exercise 39 *What is the centre of rotation of the map* $z \mapsto az + b$ *where* $|a| = 1, a \neq 1$? *What is the invariant line of the reflection* $z \mapsto a\bar{z} + b$ *where* $|a| = 1$?

Exercise 40 *Let* t *be a real number. Find expressions for*

$$x = \operatorname{Re}\frac{1}{2+ti}, \quad y = \operatorname{Im}\frac{1}{2+ti}.$$

Find an equation relating x *and* y *by eliminating* t. *Deduce that the image of the line* $\operatorname{Re} z = 2$ *under the map* $z \mapsto 1/z$ *is contained in a circle. Is the image of the line all of the circle?*

Exercise 41 *Find the image of the line* $\operatorname{Re} z = 2$ *under the maps*

$$z \mapsto iz, \qquad z \mapsto z^2, \qquad z \mapsto e^z, \qquad z \mapsto \sin z, \qquad z \mapsto \frac{1}{z-1}.$$

Exercise 42 *Draw the following parametrised curves in* $\mathbb{C}$.

$$\begin{aligned} z(t) &= e^{it}, \qquad (0 \leqslant t \leqslant \pi); \\ z(t) &= 3 + 4i + 5e^{it}, \qquad (0 \leqslant t \leqslant 2\pi); \\ z(t) &= t + i\cosh t, \qquad (-1 \leqslant t \leqslant 1); \\ z(t) &= \cosh t + i\sinh t, \qquad (t \in \mathbb{R}); \end{aligned}$$

Exercise 43 *Prove, using complex numbers, that the midpoints of the sides of an arbitrary quadrilateral are the vertices of a parallelogram.*

Exercise 44 *Let* z_1 *and* z_2 *be two complex numbers. Show that*

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right).$$

This fact is called the Parallelogram Law — how does this relate the lengths of the diagonals and sides of the parallelogram? [Hint: consider the parallelogram in $\mathbb{C}$ *with vertices* $0, z_1, z_2, z_1 + z_2$.*]*

Exercise 45 *Consider a quadrilateral $OABC$ in the complex plane whose vertices are at the complex numbers $0, a, b, c$. Show that the equation*

$$|b|^2 + |a - c|^2 = |a|^2 + |c|^2 + |a - b|^2 + |b - c|^2$$

can be rearranged as

$$|b - a - c|^2 = 0.$$

Hence show that the only quadrilaterals to satisfy the Parallelogram Law are parallelograms.

Exercise 46 *Let $A = 1 + i$ and $B = 1 - i$. Find the two numbers C and D such that ABC and ABD are equilateral triangles in the Argand diagram. Show that if $C < D$ then*

$$A + \omega C + \omega^2 B = 0 = A + \omega B + \omega^2 D,$$

where $\omega = \left(-1 + \sqrt{3}i\right)/2$ is a cube root of unity other than 1.

Exercise 47 *Let $\omega = e^{2\pi i/3} = \left(-1 + \sqrt{3}i\right)/2$. Show that a triangle ABC, where the vertices are read anti-clockwise, is equilateral if and only if*

$$A + \omega B + \omega^2 C = 0.$$

Exercise 48 *(Napoleon's Theorem) Let ABC be an arbitrary triangle. Place three equilateral triangles ABD, BCE, CAF, one on each face and pointing outwards. Show that the centroids of these three new triangles define a fourth equilateral triangle. [The centroid of a triangle whose vertices are represented by the complex numbers a, b, c is the point represented by $(a + b + c)/3$.]*

Exercise 49 *Let A, C be real numbers and B be a complex number. Consider the equation*

$$Az\bar{z} + \bar{B}z + B\bar{z} + C = 0. \tag{1.17}$$

Show that if $A = 0$, then equation (1.17) defines a line. Conversely show that any line can be put in this form with $A = 0$.

Show that if $A \neq 0$ then equation (1.17) defines a circle, a single point or has no solutions. Under what conditions on A, B, C do the solutions form a circle and, assuming the condition holds, determine the radius and centre of the circle.

Exercise 50 *Determine the equation of the following circles and lines in the form of (1.17):*

1. *The circle with centre $3 + 4i$ and radius 5.*

2. *The circle which passes through $1, 3$ and i*

3. *The line through $1 + 3i$ and $2 - i$.*

4. *The line through 2 and making an angle θ with the real-axis.*

Exercise 51 *Find the image under the map $z \mapsto 1.z$ of the two circles and two lines in the previous exercise. Ensure that your answers are all in the same form as the equation (1.17).*

Exercise 52 *Find the real and imaginary parts, and the magnitude and argument of the following.*

$$e^{3+2i}, \quad \sin(4+2i), \quad \cosh(2-i), \quad \tanh(1+2i).$$

Exercise 53 *Find all the solutions of the following questions.*

$$\begin{aligned} e^z &= 1; \\ \cosh z &= -2; \\ \sin z &= 3. \end{aligned}$$

Exercise 54 *Let* $z \in \mathbb{C}$ *and* $t = \tanh \frac{1}{2} z$*. Show that*

$$\sinh z = \frac{2t}{1-t^2}, \qquad \cosh z = \frac{1+t^2}{1-t^2}, \qquad \tanh z = \frac{2t}{1+t^2}.$$

Exercise 55 *Show that*

$$\overline{\cos z} = \cos \bar{z}, \quad \text{and} \quad \overline{\sin z} = \sin \bar{z}.$$

Show further that, if $z = x + iy$*, then*

$$\begin{aligned} |\sin z|^2 &= \frac{1}{2}(\cosh 2y - \cos 2x); \\ |\cos z|^2 &= \frac{1}{2}(\cosh 2y + \cos 2x). \end{aligned}$$

Sketch the regions $|\sin z| \leqslant 1$ *and* $|\cos z| \leqslant 1$.

Exercise 56 *Use the identities of the previous exercise to show that*

$$|\cos z|^2 + |\sin z|^2 = 1$$

if and only if z *is real.*

Exercise 57 *Let* x, y *be real numbers and assume that* $x \geqslant 1$*. Show that*

$$\cosh^{-1} x = \pm \ln\left(x + \sqrt{x^2-1}\right),$$

and

$$\sinh^{-1} y = \ln\left(y + \sqrt{y^2+1}\right).$$

Exercise 58 *Show that*

$$\begin{aligned} \cosh^2 z &= \frac{1}{2}(1 + \cosh 2z); \\ \sinh^3 z &= \frac{1}{4}(\sinh 3z - 3\sinh z). \end{aligned}$$

Hence find the power series of $\cosh^2 z$ *and* $\sinh^3 z$.

Exercise 59 *Show that*

$$\begin{aligned}\sinh(x+y) &= \sinh x\cosh y+\cosh x\sinh y\\ \cosh(x+y) &= \cosh x\cosh y+\sinh x\sinh y\end{aligned}$$

and

$$\tanh(x+y)=\frac{\tanh x+\tanh y}{1+\tanh x\tanh y}.$$

Exercise 60 *Let* $\omega = e^{2\pi i/3} = \left(-1+\sqrt{3}\right)/2$ *and let* k *be an integer Show that*

$$1+\omega^k+\omega^{2k}=1+2\cos\frac{2\pi k}{3}=\begin{cases}3 & \text{if } k \text{ is a multiple of 3;}\\ 0 & \text{otherwise.}\end{cases}$$

Deduce that

$$\frac{1}{3}\left(e^z+e^{\omega z}+e^{w^2 z}\right)=\sum_{n=0}^{\infty}\frac{z^{3n}}{(3n)!}.$$

Determine

$$\sum_{n=0}^{\infty}\frac{8^n}{(3n)!}$$

ensuring that your answer is in a form that is evidently a real number.

Exercise 61 *Adapt the method of the previous exercise to determine*

$$\sum_{n=0}^{\infty}\frac{z^{5n}}{(5n)!}$$

2. INDUCTION AND RECURSION

Notation 36 *The symbol* $\mathbb{N}$ *denotes the set of natural numbers* $\{0, 1, 2, \ldots\}$. *For* $n \in \mathbb{N}$ *the symbol* $n!$ *read "n factorial" denotes the product* $1 \times 2 \times 3 \times \cdots \times n$ *when* $n \geqslant 1$ *and with the convention that* $0! = 1$.

2.1 Introduction

Mathematical statements can come in the form of a single proposition such as

$$3 < \pi \quad \text{or as} \quad 0 < x < y \implies x^2 < y^2,$$

but often they come as a family of statements such as

A	$e^x > 0$ for all real numbers x;
B	$0 + 1 + 2 + \cdots + n = \frac{1}{2} n\,(n+1)$ for $n \in \mathbb{N}$;
C	$\int_0^\pi \sin^{2n} \theta \,\mathrm{d}\theta = \frac{(2n)!}{(n!)^2} \frac{\pi}{2^{2n}}$ for $n \in \mathbb{N}$;
D	$2n + 4$ can be written as the sum of two primes for all $n \in \mathbb{N}$.

Induction, or more exactly mathematical induction, is a particularly useful method of proof for dealing with families of statements which are indexed by the natural numbers, such as the last three statements above. We shall prove both statements B and C using induction (see below and Example 41). Statement B (and likewise statement C) can be approached with induction because in each case knowing that the nth statement is true helps enormously in showing that the $(n+1)$th statement is true — this is the crucial idea behind induction. Statement D, on the other hand, is a famous problem known as *Goldbach's Conjecture* (Christian Goldbach (1690–1764), who was a professor of mathematics at St. Petersburg, made this conjecture in a letter to Euler in 1742.and it is still an open problem). If we let $D\,(n)$ be the statement that $2n+4$ can be written as the sum of two primes, then it is currently known that $D\,(n)$ is true for $n < 4 \times 10^{14}$. What makes statement D different, and more intractable to induction, is that in trying to verify $D\,(n+1)$ we can't generally make much use of knowledge of $D\,(n)$ and so we can't build towards a proof. For example, we can verify $D(17)$ and $D\,(18)$ by noting that

$$38 = 7 + 31 = 19 + 19, \quad \text{and} \quad 40 = 3 + 37 = 11 + 29 = 17 + 23.$$

Here, knowing that 38 can be written as a sum of two primes, is no help in verifying that 40 can be, as none of the primes we might use for the latter was previously used in splitting 38.

By way of an example we shall prove statement B by induction before giving a formal definition of just what induction is. For any $n \in \mathbb{N}$, let $B(n)$ be the statement

$$0+1+2+\cdots+n=\frac{1}{2}n(n+1).$$

We shall prove two facts:

(i) $B(0)$ is true, and

(ii) for any $n \in \mathbb{N}$, if $B(n)$ is true then $B(n+1)$ is also true.

The first fact is the easy part as we just need to note that

$$\text{LHS of } B(0)=0=\frac{1}{2}\times 0\times 1=\text{RHS of } B(0).$$

To verify (ii) we need to prove *for each* n that $B(n+1)$ is true *assuming* $B(n)$ to be true. Now

$$\text{LHS of } B(n+1)=0+1+\cdots+n+(n+1).$$

But, assuming $B(n)$ to be true, we know that the terms from 0 through to n add up to $n(n+1)/2$ and so

$$\begin{aligned}\text{LHS of } B(n+1) &= \frac{1}{2}n(n+1)+(n+1)\\ &= (n+1)\left(\frac{n}{2}+1\right)\\ &= \frac{1}{2}(n+1)(n+2)=\text{RHS of } B(n+1).\end{aligned}$$

This verifies (ii).

Be sure that you understand the above calculation: it contains the important steps common to any proof by induction. Note in the final step that we have retrieved our original formula of $n(n+1)/2$, but with $n+1$ now replacing n everywhere; this was the expression that we always had to be working towards.

With induction we now know that B is true, i.e. that $B(n)$ is true for any $n \in \mathbb{N}$. How does this work? Well, suppose we want to be sure $B(2)$ is correct — above we have just verified the following three statements:

$B(0)$ is true;

if $B(0)$ is true then $B(1)$ is true;

if $B(1)$ is true then $B(2)$ is true;

and so putting the three together, we see that $B(2)$ is true: the first statement tells us that $B(0)$ is true and the second two are stepping stones, first to the truth about $B(1)$, and then on to proving $B(2)$.

Formally then the Principle of Induction is as follows:

Theorem 37 *(THE PRINCIPLE OF INDUCTION) Let $P(n)$ be a family of statements indexed by the natural numbers. Suppose that*

$$P(0) \text{ is true, and}$$
$$\text{for any } n \in \mathbb{N}, \text{ if } P(n) \text{ is true then } P(n+1) \text{ is also true.}$$

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let S denote the subset of $\mathbb{N}$ consisting of all those n for which $P(n)$ is *false*. We aim to show that S is empty, i.e. that no $P(n)$ is false.

Suppose for a contradiction that S is non-empty. Any non-empty subset of $\mathbb{N}$ has a minimum element; let's write m for the minimum element of S. As $P(0)$ is true then $0 \notin S$, and so m is at least 1.

Consider now $m-1$. As $m \geq 1$ then $m-1 \in \mathbb{N}$ and further, as $m-1$ is smaller than the minimum element of S, then $m-1 \notin S$, i.e. $P(m-1)$ is true. But, as $P(m-1)$ is true, then induction tells us that $P(m)$ is also true. This means $m \notin S$, which contradicts m being the minimum element of S. This is our required contradiction, an absurd conclusion. If S being non-empty leads to a contradiction, the only alternative is that S is empty. ■

It is not hard to see how we might amend the hypotheses of the theorem above to show

Corollary 38 *Let $N \in \mathbb{N}$ and let $P(n)$ be a family of statements for $n = N, N+1, N+2, \ldots$ Suppose that*

$$P(N) \text{ is true, and}$$
$$\text{for any } n \geq N, \text{ if } P(n) \text{ is true then } P(n+1) \text{ is also true.}$$

Then $P(n)$ is true for all $n \geq N$.

This is really just induction again, but we have started the ball rolling at a later stage. Here is another version of induction, which is usually referred to a the *Strong Form Of Induction*:

Theorem 39 *(STRONG FORM OF INDUCTION) Let $P(n)$ be a family of statements for $n \in \mathbb{N}$. Suppose that*

$$P(0) \text{ is true, and}$$
$$\text{for any } n \in \mathbb{N}, \text{ if } P(0), P(1), \ldots, P(n) \text{ are all true then so is } P(n+1).$$

Then $P(n)$ is true for all $n \in \mathbb{N}$.

To reinforce the need for proof, and to show how patterns can at first glance delude us, consider the following example. Take two points on the circumference of a circle and take a line joining them; this line then divides the circle's interior into two regions. If we take three points on the perimeter then the lines joining them will divide the disc into four regions. Four points can result in a maximum of eight regions — surely then, we can confidently predict that n points will maximally result in 2^{n-1} regions. Further investigation shows our conjecture to be true for $n = 5$, but to our surprise, however we take six points on the circle, the maximum number of regions attained is 31. Indeed the maximum number of regions attained from n points on the perimeter is given by the formula [2, p.18]

$$\frac{1}{24}\left(n^4 - 6n^3 + 23n^2 - 18n + 24\right).$$

Our original guess was way out!

There are other well-known 'patterns' that go awry in mathematics: for example, the number

$$n^2 - n + 41$$

is a prime number for $n = 1, 2, 3, \ldots, 40$ (though this takes some tedious verifying), but it is easy to see when $n = 41$ that $n^2 - n + 41 = 41^2$ is not prime. A more amazing example comes from the study of Pell's equation $x^2 = py^2 + 1$ in number theory, where p is a prime number and x and y are natural numbers. If $P(n)$ is the statement that

$991n^2 + 1$ is not a perfect square (i.e. the square of a natural number),

then the first counter-example to $P(n)$ is staggeringly found at [1, pp. 2–3]

$$n = 12, 055, 735, 790, 331, 359, 447, 442, 538, 767.$$

2.2 Examples

On a more positive note though, many of the patterns found in mathematics won't trip us at some later stage and here are some further examples of proof by induction.

Example 40 *Show that n lines in the plane, no two of which are parallel and no three meeting in a point, divide the plane into $n(n+1)/2 + 1$ regions.*

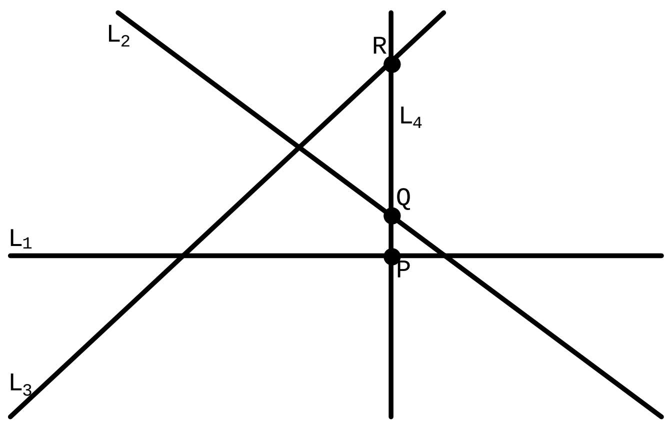
Proof. When we have no lines in the plane then clearly we have just one region, as expected from putting $n = 0$ into the formula $n(n+1)/2 + 1$.

Suppose now that we have n lines dividing the plane into $n(n+1)/2 + 1$ regions and we will add a $(n+1)$th line. This extra line will meet each of the previous n lines because we have assumed it to be parallel with none of them. Also, it meets each of these n lines in a distinct point, as we have assumed that no three lines are concurrent.

These n points of intersection divide the new line into $n+1$ segments. For each of these $n+1$ segments there are now two regions, one on either side of the segment, where previously there had been only one region. So by adding this $(n+1)$th line we have created $n+1$ new regions. In total the number of regions we now have is

$$\frac{n(n+1)}{2}+1+(n+1)=\frac{(n+1)(n+2)}{2}+1.$$

This is the correct formula when we replace n with $n+1$, and so the result follows by induction. ■



An example when $n=3$.

Here the four segments, 'below P', PQ, QR and 'above R' on the fourth line L_4, divide what were four regions previously, into eight new ones.

Example 41 *Prove for $n \in \mathbb{N}$ that*

$$\int_0^{\pi} \sin^{2n}\theta \; d\theta = \frac{(2n)!}{(n!)^2}\frac{\pi}{2^{2n}}. \tag{2.1}$$

Proof. Let's denote the integral on the LHS of equation (2.1) as I_n. The value of I_0 is easy to calculate, because the integrand is just 1, and so $I_0=\pi$. We also see

$$\text{RHS}\,(n=0)=\frac{0!}{(0!)^2}\frac{\pi}{2^0}=\pi,$$

verifying the initial case.

We now prove a *reduction formula* connecting I_n and I_{n+1}, so that we can use this in our induction.

$$\begin{aligned}
I_{n+1} &= \int_0^\pi \sin^{2(n+1)}\theta \ \mathrm{d}\theta \\
&= \int_0^\pi \sin^{2n+1}\theta \times \sin\theta \ \mathrm{d}\theta \\
&= \left[\sin^{2n+1}\theta \times (-\cos\theta)\right]_0^\pi - \int_0^\pi (2n+1)\sin^{2n}\theta\cos\theta \times (-\cos\theta) \ \mathrm{d}\theta \\
&= 0 + (2n+1)\int_0^\pi \sin^{2n}\theta\left(1-\sin^2\theta\right) \ \mathrm{d}\theta \\
&= (2n+1)\int_0^\pi \left(\sin^{2n}\theta - \sin^{2(n+1)}\theta\right) \ \mathrm{d}\theta \\
&= (2n+1)\left(I_n - I_{n+1}\right).
\end{aligned}$$

Rearranging gives

$$I_{n+1} = \frac{2n+1}{2n+2} I_n.$$

Suppose now that equation (2.1) gives the right value of I_k for some natural number k. Then, turning to equation (2.1) with $n = k+1$, and using our assumption and the reduction formula, we see:

$$\begin{aligned}
\text{LHS} &= I_{k+1} = \frac{2k+1}{2(k+1)} \times I_k \\
&= \frac{2k+1}{2(k+1)} \times \frac{(2k)!}{(k!)^2} \times \frac{\pi}{2^{2k}} \\
&= \frac{2k+2}{2(k+1)} \times \frac{2k+1}{2(k+1)} \times \frac{(2k)!}{(k!)^2} \times \frac{\pi}{2^{2k}} \\
&= \frac{(2k+2)!}{((k+1)!)^2} \times \frac{\pi}{2^{2(k+1)}},
\end{aligned}$$

which equals the RHS of equation (2.1) with $n = k+1$. The result follows by induction. ■

Example 42 *Show for* $n = 1, 2, 3\ldots$ *and* $k = 1, 2, 3, \ldots$ *that*

$$\sum_{r=1}^{n} r(r+1)(r+2)\cdots(r+k-1) = \frac{n(n+1)(n+2)\cdots(n+k)}{k+1}. \qquad (2.2)$$

Remark 43 *This problem differs from our earlier examples in that our family of statements now involves two variables n and k, rather than just the one variable. If we write $P(n,k)$ for the statement in equation (2.2) then we can use induction to prove all of the statements $P(n,k)$ in various ways:*

- *we could prove $P(1,1)$ and show how $P(n+1,k)$ and $P(n,k+1)$ both follow from $P(n,k)$ for $n, k = 1, 2, 3, \ldots$;*

- *we could prove* $P(1,k)$ *for all* $k = 1,2,3,\dots$ *and show how knowledge of* $P(n,k)$ *for all* k*, leads to proving* $P(n+1,k)$ *for all* k *— effectively this reduces the problem to one application of induction, but to a family of statements at a time*
- *we could prove* $P(n,1)$ *for all* $n = 1,2,3,\dots$ *and show how knowing* $P(n,k)$ *for all* n*, leads to proving* $P(n,k+1)$ *for all* n *— in a similar fashion to the previous method, now inducting through* k *and treating* n *as arbitrary.*

What these different approaches rely on, is that all the possible pairs (n,k) *are somehow linked to our initial pair (or pairs). Let*

$$S = \{(n,k) : n,k \geq 1\}$$

be the set of all possible pairs (n,k)*.*

The first method of proof uses the fact that the only subset T *of* S *satisfying the properties*

$$\begin{array}{rcl} (1,1) & \in & T, \\ \text{if} \quad (n,k) & \in & T \quad \text{then} \quad (n,k+1) \in T, \\ \text{if} \quad (n,k) & \in & T \quad \text{then} \quad (n+1,k) \in T, \end{array}$$

is S *itself. Starting from the truth of* $P(1,1)$*, and deducing further truths as the second and third properties allow, then every* $P(n,k)$ *must be true. The second and third methods of proof rely on the fact that the whole of* S *is the only subset having similar properties.*

Proof. In this case the second method of proof seems easiest, that is we will prove that $P(1,k)$ holds for each $k = 1,2,3,\dots$ and show that assuming the statements $P(N,k)$, for a particular N and all k, is sufficient to prove the statements $P(N+1,k)$ for all k. Firstly we note

$$\begin{aligned} \text{LHS of } P(1,k) &= 1 \times 2 \times 3 \times \cdots \times k, \text{ and} \\ \text{RHS of } P(1,k) &= \frac{1 \times 2 \times 3 \times \cdots \times (k+1)}{k+1} = 1 \times 2 \times 3 \times \cdots \times k, \end{aligned}$$

are equal, proving $P(1,k)$ for all $k \geq 1$. Then assuming $P(N,k)$ for $k \geqslant 1$ we have

$$\begin{aligned} \text{LHS of } P(N+1,k) &= \sum_{r=1}^{N+1} r(r+1)(r+2)\cdots(r+k-1) \\ &= \frac{N(N+1)\dots(N+k)}{k+1} + (N+1)(N+2)\cdots(N+k) \\ &= (N+1)(N+2)\cdots(N+k)\left(\frac{N}{k+1}+1\right) \\ &= \frac{(N+1)(N+2)\cdots(N+k)(N+k+1)}{k+1} \\ &= \text{RHS of } P(N+1,k), \end{aligned}$$

proving $P(N+1,k)$ simultaneously for each k. This verifies all that is required for the second method. ■

We end with one example which makes use of the Strong Form of Induction.

Recall that a natural number $n \geq 2$ is called *prime* if the only natural numbers which divide it are 1 and n. (Note that 1 is not considered prime.) The list of prime numbers begins $2, 3, 5, 7, 11, 13, \ldots$ and has been known to be infinite since the time of Euclid. (Euclid was an Alexandrian Greek living c. 300 B.C. His most famous work is *The Elements*, thirteen books which present much of the mathematics discovered by the ancient Greeks, and which was a hugely influential text on the teaching of mathematics even into the twentieth century. The work presents its results in a rigorous fashion, laying down basic assumptions, called *axioms,* and carefully proving his theorems from these axioms.) The prime numbers are in a sense the atoms of the natural numbers under multiplication, as every natural number $n \geq 2$ can be written as a product of primes in what is essentially a unique way — this fact is known as the *Fundamental Theorem of Arithmetic.* Here we just prove the existence of such a product.

Example 44 *Every natural number $n \geq 2$ can be written as a product of prime numbers.*

Proof. We begin at $n = 2$ which is prime.

As our inductive hypothesis we assume that every number $2 \leq k \leq N$ is a prime number or can be written as a product of prime numbers. Consider then $N+1$; we need to show this is a prime number, or else a product of prime numbers. Either $N+1$ is prime or it is not. If $N+1$ is prime then we are done. If $N+1$ is not prime, then it has a factor m in the range $2 \leq m < N+1$ which divides $N+1$. Note that $m \leq N$ and $(N+1)/m \leq N$, as m is at least 2. So, by our inductive hypothesis, we know m and $(N+1)/m$ are both either prime or the product of prime numbers. Hence we can write

$$\begin{aligned} m &= p_1 \times p_2 \times \cdots \times p_k, \quad \text{and} \\ \frac{N+1}{m} &= P_1 \times P_2 \times \cdots \times P_K, \end{aligned}$$

where $p_1, \ldots, p_k$ and $P_1, \ldots, P_K$ are prime numbers. Finally we have that

$$N+1 = m \times \frac{N+1}{m} = p_1 \times p_2 \times \cdots \times p_k \times P_1 \times P_2 \times \cdots \times P_K,$$

showing $N+1$ to be a product of primes. The result follows using the strong form of induction. ■

2.3 The Binomial Theorem

All of you will have met the identity

$$(x+y)^2 = x^2 + 2xy + y^2$$

and may even have met identities like

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

It may even have been pointed out to you that these coefficients $1, 2, 1$ and $1, 3, 3, 1$ are simply the numbers that appear in *Pascal's Triangle.* This is the infinite triangle of numbers that has 1s down both sides and a number internal to some row of the triangle is calculated by adding the two numbers above it in the previous row. So the triangle grows as follows:

$$\begin{array}{lccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

From the triangle we could say read off the identity

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

Of course we haven't *proved* this identity yet — these identities, for general n other than just $n = 6$, are the subject of the Binomial Theorem. We introduce now the binomial coefficients; their connection with Pascal's triangle will become clear soon.

Definition 45 *The (n,k)th binomial coefficient is the number*

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

where $n = 0, 1, 2, 3, \ldots$ and $0 \leq k \leq n$. It is read as 'n choose k' and in some books is denoted as ${}^nC_k$. As a convention we set $\binom{n}{k}$ to be zero when $n < 0$ or when $k < 0$ or $k > n$.

Note some basic identities concerning the binomial coefficients

$$\binom{n}{k} = \binom{n}{n-k}, \quad \binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = \binom{n}{n-1} = n.$$

The following lemma demonstrates that the binomial coefficients are precisely the numbers that appear in Pascal's triangle.

Lemma 46 *Let $n \in \mathbb{N}$ and $1 \leq k \leq n$. Then*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Proof. Putting the LHS over a common denominator

$$\begin{aligned}
\text{LHS} &= \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!} \\
&= \frac{n!}{k!\,(n-k+1)!}\{k + (n-k+1)\} \\
&= \frac{n! \times (n+1)}{k!\,(n-k+1)!} \\
&= \frac{(n+1)!}{k!\,(n+1-k)!} \\
&= \binom{n+1}{k} \\
&= \text{RHS}.
\end{aligned}$$

■

Corollary 47 *The kth number in the nth row of Pascal's triangle is $\binom{n}{k}$ (remembering to count from $n = 0$ and $k = 0$). In particular the binomial coefficients are whole numbers.*

Proof. We shall prove this by induction. Note that $\binom{0}{0} = 1$ gives the 1 at the apex of Pascal's triangle, proving the initial step.

Suppose now that the numbers $\binom{N}{k}$ are the numbers that appear in the Nth row of Pascal's triangle. The first and last entries of the next, $(N+1)$th, row (associated with $k = 0$ and $k = N+1$) are

$$1 = \binom{N+1}{0} \quad \text{and} \quad 1 = \binom{N+1}{N+1}$$

as required. For $1 \leq k \leq N$, then the kth entry on the $(N+1)$th row is formed by adding the $(k-1)$th and kth entries from the Nth row. By our hypothesis about the Nth row their sum is

$$\binom{N}{k-1} + \binom{N}{k} = \binom{N+1}{k},$$

using the previous lemma, and this verifies that the $(N+1)$th row also consists of binomial coefficients. So the $(N+1)$th row checks out, and the result follows by induction. ■

Finally, we come to the binomial theorem:

Theorem 48 *(THE BINOMIAL THEOREM): Let $n \in \mathbb{N}$ and x, y be real numbers. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. Let's check the binomial theorem first for $n = 0$. We can verify this by noting

$$\text{LHS} = (x+y)^0 = 1, \qquad \text{RHS} = \binom{0}{0} x^0 y^0 = 1.$$

We aim now to show the theorem holds for $n = N+1$ assuming it to be true for $n = N$. In this case

$$\text{LHS} = (x+y)^{N+1} = (x+y)(x+y)^N = (x+y)\left(\sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k}\right)$$

writing in our assumed expression for $(x+y)^N$. Expanding the brackets gives

$$\sum_{k=0}^{N} \binom{N}{k} x^{k+1} y^{N-k} + \sum_{k=0}^{N} \binom{N}{k} x^k y^{N+1-k},$$

which we can rearrange to

$$x^{N+1} + \sum_{k=0}^{N-1} \binom{N}{k} x^{k+1} y^{N-k} + \sum_{k=1}^{N} \binom{N}{k} x^k y^{N+1-k} + y^{N+1}$$

by taking out the last term from the first sum and the first term from the second sum. In the first sum we now make a change of variable. We set $k = l - 1$, noting that as k ranges over $0, 1, ..., N-1$, then l ranges over $1, 2, ..., N$. So the above equals

$$x^{N+1} + \sum_{l=1}^{N} \binom{N}{l-1} x^l y^{N+1-l} + \sum_{k=1}^{N} \binom{N}{k} x^k y^{N+1-k} + y^{N+1}.$$

We may combine the sums as they are over the same range, obtaining

$$x^{N+1} + \sum_{k=1}^{N} \left\{ \binom{N}{k-1} + \binom{N}{k} \right\} x^k y^{N+1-k} + y^{N+1}$$

which, using Lemma 46, equals

$$x^{N+1} + \sum_{k=1}^{N} \binom{N+1}{k} x^k y^{N+1-k} + y^{N+1} = \sum_{k=0}^{N+1} \binom{N+1}{k} x^k y^{N+1-k} = \text{RHS}.$$

The result follows by induction. ∎

There is good reason why $\binom{n}{k}$ is read as 'n choose k' — there are $\binom{n}{k}$ ways of choosing k elements from the set $\{1, 2, \ldots, n\}$ (when showing no interest in the order that the k elements are to be chosen). Put another way, there are $\binom{n}{k}$ subsets of $\{1, 2, \ldots, n\}$ with k elements in them. To show this, let's think about how we might go about choosing k elements.

For our 'first' element we can choose any of the n elements, but once this has been chosen it can't be put into the subset again. So for our second element any of the remaining $n-1$ elements may be chosen, for our third any of the $n-2$ that are left, and so on. So choosing a set of k elements from $\{1, 2, \ldots, n\}$ in a particular order can be done in

$$n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!} \text{ ways.}$$

But there are lots of different orders of choice that could have produced this same subset. Given a set of k elements there are $k!$ ways of ordering them — that is to say, for each subset with k elements there are $k!$ different orders of choice that will each lead to that same subset. So the number $n!/(n-k)!$ is an 'overcount' by a factor of $k!$. Hence the number of subsets of size k equals

$$\frac{n!}{k!\,(n-k)!}$$

as required.

Remark 49 *There is a Trinomial Theorem and further generalisations of the binomial theorem to greater numbers of variables. Given three real numbers x, y, z and a natural number n we can apply the binomial theorem twice to obtain*

$$\begin{aligned}(x+y+z)^n &= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} x^k (y+z)^{n-k} \\ &= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \sum_{l=0}^{n-k} \frac{(n-k)!}{l!\,(n-k-l)!} x^k y^l z^{n-k-l} \\ &= \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n!}{k!l!\,(n-k-l)!} x^k y^l z^{n-k-l}.\end{aligned}$$

This is a somewhat cumbersome expression; it's easier on the eye, and has a nicer symmetry, if we write $m = n-k-l$ and then we can rewrite the above as

$$(x+y+z)^n = \sum_{\substack{k+l+m=n \\ k,l,m \geq 0}} \frac{n!}{k!\,l!\,m!} x^k\, y^l\, z^m.$$

Again the number $n!/(k!\,l!\,m!)$, where $k+l+m=n$ and $k, l, m \geq 0$, is the number of ways that n elements can be apportioned into three subsets associated with the numbers x, y and z.

2.4 Difference Equations

This final section on induction is mainly concerned with solving linear difference equations with constant coefficients — that is finding an expression for numbers x_n defined recursively by a relation such as

$$x_{n+2} = 2x_{n+1} - x_n + 2 \quad \text{for } n \geq 0, \quad \text{with } x_0 = 1, x_1 = 1.$$

We see that the x_n can be determined by applying this relation sufficiently many times from our initial values of $x_0 = 1$ and $x_1 = 1$. So for example to find x_7 we'd calculate

$$\begin{aligned} x_2 &= 2x_1 - x_0 + 1 = 2 - 1 + 2 = 3; \\ x_3 &= 2x_2 - x_1 + 1 = 6 - 1 + 2 = 7; \\ x_4 &= 2x_3 - x_2 + 1 = 14 - 3 + 2 = 13; \\ x_5 &= 2x_4 - x_3 + 1 = 26 - 7 + 2 = 21; \\ x_6 &= 2x_5 - x_4 + 1 = 42 - 13 + 2 = 31; \\ x_7 &= 2x_6 - x_5 + 1 = 62 - 21 + 2 = 43. \end{aligned}$$

If this was the first time we had seen such a problem, then we might try pattern spotting or qualitatively analysing the sequence's behaviour, in order to make a guess at a general formula for x_n. Simply looking at the sequence x_n above no obvious pattern is emerging. However we can see that the x_n are growing, roughly at the same speed as n^2 grows. We might note further that the differences between the numbers $0, 2, 4, 6, 8, 10, 12, \ldots$ are going up linearly. Even if we didn't know how to sum an arithmetic progression, it would seem reasonable to *try* a solution of the form

$$x_n = an^2 + bn + c, \tag{2.3}$$

where a, b, c are constants, as yet undetermined. *If* a solution of the form (2.3) exists, we can find a, b, c using the first three cases, so that

$$\begin{aligned} x_0 &= 1 = a0^2 + b0 + c \text{ and so } c = 1; \\ x_1 &= 1 = a1^2 + b1 + 1 \text{ and so } a + b = 0; \\ x_2 &= 3 = a2^2 - a2 + 1 \text{ and so } a = 1. \end{aligned}$$

So the only expression of the form (2.3) which gives the right answer in the $n = 0, 1, 2$ cases is

$$x_n = n^2 - n + 1. \tag{2.4}$$

If we put $n = 3, 4, 5, 6, 7$ into (2.4) then we get the correct values of x_n calculated above. This is, of course, not a proof, but we could prove this formula to be correct for all values of $n \geq 0$ using induction as follows. We have already checked that the formula (2.3) is correct for $n = 0$ and $n = 1$. As our inductive hypothesis let's suppose it was also true for $n = k$ and $n = k + 1$. Then

$$\begin{aligned} x_{k+2} &= 2x_{k+1} - x_k + 2 \\ &= 2\left\{(k+1)^2 - (k+1) + 1\right\} - \left\{k^2 - k + 1\right\} + 2 \\ &= k^2 + 3k + 3 \\ &= (k+2)^2 - (k+2) + 1 \end{aligned}$$

which is the correct formula with $n = k + 2$. Note how at each stage we relied on the formula to hold for *two* consecutive values of n to be able to move on to the next value.

Alternatively having noted the differences go up as $0, 2, 4, 6, 8, \ldots$ we can write, using statement B from the start of this chapter,

$$\begin{aligned} x_n &= 1 + 0 + 2 + 4 + \cdots + (2n-2) \\ &= 1 + \sum_{k=0}^{n-1} 2k \\ &= 1 + 2\frac{1}{2}(n-1)\,n \\ &= n^2 - n + 1. \end{aligned}$$

To make this proof water-tight we need to check that the pattern $0, 2, 4, 6, 8, \ldots$ of the differences carries on forever, and that it wasn't just a fluke. But this follows if we note

$$x_{n+2} - x_{n+1} = x_{n+1} - x_n + 2$$

and so the difference between consecutive terms is increasing by 2 each time.

Of course if a pattern to x_n is difficult to spot then the above methods won't apply. We will show now how to solve a difference equation of the form

$$ax_{n+2} + bx_{n+1} + cx_n = 0$$

where a, b, c are real or complex constants. The theory extends to linear constant coefficient difference equations of any order. We will later treat some inhomogeneous examples where the RHS is non-zero. (As with constant coefficient linear differential equations in the later *Differential Equations* chapter, this involves solving the corresponding homogeneous difference equation and finding a particular solution of the inhomogeneous equation. The reason for the similarity in their solution is because the underlying *linear algebra* of the two problems is the same.)

Theorem 50 *Suppose that the sequence* x_n *satisfies the difference equation*

$$a x_{n+2} + b x_{n+1} + c x_n = 0 \quad \text{for } n \geq 0, \tag{2.5}$$

where $a \neq 0$, *and that* α *and* β *be the roots of the auxiliary equation*

$$a\lambda^2 + b\lambda + c = 0.$$

The general solution of (2.5) has the form

$$x_n = A\alpha^n + B\beta^n \quad (n \geq 0),$$

when α *and* β *are distinct, and has the form*

$$x_n = (An + B)\,\alpha^n \quad (n \geq 0),$$

when $\alpha = \beta \neq 0$. *In each case the values of* A *and* B *are uniquely determined by the values of* x_0 *and* x_1.

Proof. Firstly we note that the sequence x_n defined in (2.5) is uniquely determined by the initial values x_0 and x_1. Knowing these values (2.5) gives us x_2, knowing x_1 and x_2 gives us x_3 etc. So if we can find a solution to (2.5) for certain initial values then we have *the* unique solution; if we can find a solution for arbitrary initial values then we have the general solution.

Note that if $\alpha \neq \beta$ then putting $x_n = A\alpha^n + B\beta^n$ into the LHS of (2.5) gives

$$\begin{aligned} & a x_{n+2} + b x_{n+1} + c x_n \\ = \; & a\left(A\alpha^{n+2} + B\beta^{n+2}\right) + b\left(A\alpha^{n+1} + B\beta^{n+1}\right) + c\left(A\alpha^n + B\beta^n\right) \\ = \; & A\alpha^n\left(a\alpha^2 + b\alpha + c\right) + B\beta^n\left(a\beta^2 + b\beta + c\right) \\ = \; & 0 \end{aligned}$$

as α and β are both roots of the auxiliary equation.

Similarly if $\alpha = \beta$ then putting $x_n = (An + B)\,\alpha^n$ into the LHS of (2.5) gives

$$\begin{aligned} & a x_{n+2} + b x_{n+1} + c x_n \\ = \; & a\left(A(n+2) + B\right)\alpha^{n+2} + b\left(A(n+1) + B\right)\alpha^{n+1} + c\left(An + B\right)\alpha^n \\ = \; & A\alpha^n\left(n\left(a\alpha^2 + b\alpha + c\right) + (2a\alpha + b)\,\alpha\right) + B\alpha^n\left(a\alpha^2 + b\alpha + c\right) \\ = \; & 0 \end{aligned}$$

because α is a root of the auxiliary equation and also of the derivative of the auxiliary equation, being a repeated root (or, if you prefer, you can show that $2a\alpha + b = 0$ by remembering that $ax^2 + bx + c = a(x - \alpha)^2$, comparing coefficients and eliminating c).

So in either case we see that we have a set of solutions. But further the initial equations

$$A + B = x_0, \quad A\alpha + B\beta = x_1,$$

are uniquely solvable for A and B when $\alpha \neq \beta$, whatever the values of x_0 and x_1. Similarly when $\alpha = \beta \neq 0$ then the initial equations

$$B = x_0, \quad (A + B)\,\alpha = x_1,$$

also have a unique solution in A and B whatever the values of x_0 and x_1. So in each case our solutions encompassed the general solution. ■

Remark 51 *When $\alpha = \beta = 0$ then (2.5) clearly has solution x_n given by*

$$x_0, x_1, 0, 0, 0, 0, \ldots$$

Probably the most famous sequence defined by such a difference equation is the sequence of Fibonacci numbers. The Fibonacci numbers F_n are defined recursively

$$F_{n+2} = F_{n+1} + F_n, \quad \text{for} \quad n \geq 0,$$

with initial values $F_0 = 0$ and $F_1 = 1$. So the sequence begins as

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

and continues to grow, always producing whole numbers and increasing by a factor of roughly 1.618 each time.

This sequence was first studied by Leonardo of Pisa (c.1170–c.1250), who called himself Fibonacci. (the meaning of the name 'Fibonacci' is somewhat uncertain; it may have meant 'son of Bonaccio' or may have been a nickname meaning 'lucky son'.) The numbers were based on a model of rabbit reproduction: the model assumes that we begin with a pair of rabbits in the first month, which every month produces a new pair of rabbits, which in turn begin producing when they are one month old. If the rabbits never die, find a formula for F_n, the number of rabbit pairs there are after n months. If we look at the F_n pairs we have at the start of the nth month, then these consist of $F_{n-1} - F_{n-2}$ pairs which have just become mature but were immature the previous month, and F_{n-2} pairs which were already mature and their new F_{n-2} pairs of offspring. In other words

$$F_n = (F_{n-1} - F_{n-2}) + 2F_{n-2}$$

which rearranges to the recursion above.

Proposition 52 *For every integer $n \geq 0$,*

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \tag{2.6}$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \textit{and} \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

Proof. From our previous theorem we know that

$$F_n = A\alpha^n + B\beta^n \quad \text{for } n \geq 0,$$

where α and β are the roots of the auxiliary equation

$$x^2 - x - 1 = 0,$$

that is the α and β given in the statement of the proposition, and where A and B are constants uniquely determined by the equations

$$\begin{aligned} A + B &= 0, \\ A\alpha + B\beta &= 1. \end{aligned}$$

So

$$A = -B = \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}},$$

concluding the proof. ∎

The following relation between the Fibonacci numbers follows easily with induction. Note again that we are faced with an identity indexed by two natural numbers m and n.

Proposition 53 *For* $m, n \in \mathbb{N}$

$$F_{n+m+1} = F_n F_m + F_{n+1} F_{m+1}. \tag{2.7}$$

Proof. For $n \in \mathbb{N}$, we shall take $P(n)$ to be the statement that:

equation (2.7) holds true for all $m \in \mathbb{N}$ in the cases $k = n$ and $k = n + 1$.

So we are using induction to progress through n and dealing with m simultaneously at each stage. To verify $P(0)$, we note that

$$\begin{aligned} F_{m+1} &= F_0 F_m + F_1 F_{m+1}, \\ F_{m+2} &= F_1 F_m + F_2 F_{m+1} \end{aligned}$$

for all m, as $F_0 = 0$, $F_1 = F_2 = 1$. For the inductive step we assume $P(n)$, i.e. that for all $m \in \mathbb{N}$,

$$\begin{aligned} F_{n+m+1} &= F_n F_m + F_{n+1} F_{m+1}, \\ F_{n+m+2} &= F_{n+1} F_m + F_{n+2} F_{m+1.} \end{aligned}$$

To prove $P(n+1)$ it remains to show that for all $m \in \mathbb{N}$,

$$F_{n+m+3} = F_{n+2} F_m + F_{n+3} F_{m+1}. \tag{2.8}$$

From our $P(n)$ assumptions and the definition of the Fibonacci numbers,

$$\begin{aligned}
\text{LHS of (2.8)} &= F_{n+m+3} \\
&= F_{n+m+2} + F_{n+m+1} \\
&= F_n F_m + F_{n+1} F_{m+1} + F_{n+1} F_m + F_{n+2} F_{m+1} \\
&= (F_n + F_{n+1}) F_m + (F_{n+1} + F_{n+2}) F_{m+1} \\
&= F_{n+2} F_m + F_{n+3} F_{m+1} \\
&= \text{RHS of (2.8)}.
\end{aligned}$$

We end with two examples of inhomogeneous difference equations. ■

Example 54 *Find the solution of the following difference equation*

$$x_{n+2} - 4x_{n+1} + 4x_n = 2^n + n, \tag{2.9}$$

with initial values $x_0 = 1$ *and* $x_1 = -1$.

Solution. The auxiliary equation

$$\lambda^2 - 4\lambda + 4 = 0$$

has repeated roots $\lambda = 2, 2$. So the general solution of the homogeneous equation

$$x_{n+2} - 4x_{n+1} + 4x_n = 0 \tag{2.10}$$

we know, from Theorem 50 to be $x_n = (An + B) 2^n$ where A and B are undetermined constants.

In order to find a particular solution of the recurrence relation (2.9) we will try various educated guesses $x_n = f(n)$, looking for a solution $f(n)$ which is in similar in nature to $2^n + n$. We can deal with the n on the RHS by contributing a term $an + b$ to $f(n)$ – what the values of a and b are will become evident later. But trying to deal with the 2^n on the RHS with contributions to $f(n)$ that consist of some multiple of 2^n or $n2^n$ would be useless as 2^n and $n2^n$ are both solutions of the homogeneous equation (2.10), and so trying them would just yield a zero on the RHS — rather we need to try instead a multiple of $n^2 2^n$ to deal with the 2^n. So let's try a particular solution of the form

$$x_n = an + b + cn^2 2^n,$$

where a, b, c are constants, as yet undetermined. Putting this expression for x_n into the LHS of (2.9) we get

$$\begin{aligned}
& a(n+2) + b + c(n+2)^2 2^{n+2} \\
& -4a(n+1) - 4b - 4c(n+1)^2 2^{n+1} \\
& +4an + 4b + 4cn^2 2^n \\
= \; & a(n + 2 - 4n - 4 + 4n) \\
& +b(1 - 4 + 4) \\
& +c2^n \left(4n^2 + 16n + 16 - 8n^2 - 16n - 8 + 4n^2\right) \\
= \; & an + (b - 2a) + 8c2^n.
\end{aligned}$$

This expression we need to equal $2^n + n$ and so we see that $a = 1, b = 2, c = 1/8$. Hence a particular solution is

$$x_n = n + 2 + \frac{n^2}{8} 2^n,$$

and the general solution of (2.9) is

$$x_n = (An + B)\, 2^n + n + 2 + \frac{n^2}{8} 2^n.$$

Recalling the initial conditions $x_0 = 1$ and $x_1 = -1$ we see

$$\begin{aligned} n &= 0: \quad B + 2 = 1; \\ n &= 1: \quad 2\,(A + B) + 1 + 2 + \frac{1}{4} = -1. \end{aligned}$$

The first line gives us $B = -1$ and the second that $A = -9/8$. Finally then the unique solution of (2.9) is

$$x_n = n + 2 + \frac{1}{8}\left(n^2 - 9n - 8\right) 2^n.$$

■

Example 55 *Find the solution of the difference equation*

$$x_{n+3} = 2x_n - x_{n+2} + 1,$$

with initial values $x_0 = x_1 = x_2 = 0$.

Solution. The auxiliary equation here is

$$\lambda^3 + \lambda^2 - 2 = 0,$$

which factorises as

$$\lambda^3 + \lambda^2 - 2 = (\lambda - 1)\left(\lambda^2 + 2\lambda + 2\right) = 0$$

and so has roots

$$\lambda = 1, -1 + i, -1 - i.$$

So the general solution of the homogeneous difference equation is

$$x_n = A + B\,(-1 + i)^n + C\,(-1 - i)^n\,.$$

At this point we know need to find a particular solution of the inhomogeneous equation. Because constant sequences are solutions of the homogeneous equation there is no point trying these as particular solutions; instead we try one of the form $x_n = kn$. Putting this into the difference equation we obtain

$$k\,(n + 3) = 2kn - k\,(n + 2) + 1 \quad \text{which simplifies to } 3k = -2k + 1$$

and so $k = \frac{1}{5}$. The general solution of the inhomogeneous difference equation has the form

$$x_n = \frac{n}{5} + A + B\left(-1+i\right)^n + C\left(-1-i\right)^n.$$

At first glance this solution does not necessarily look like it will be a real sequence, and indeed B and C will need to be complex constants for this to be the case. But if we remember that

$$\begin{aligned}(-1+i)^n &= \left(\sqrt{2}e^{i3\pi/4}\right)^n = 2^{n/2}\left(\cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4}\right)\\ (-1-i)^n &= \left(\sqrt{2}e^{i5\pi/4}\right)^n = 2^{n/2}\left(\cos\frac{5n\pi}{4} + i\sin\frac{5n\pi}{4}\right)\end{aligned}$$

we can rearrange our solution in terms of overtly real sequences. To calculate A, B and C then we use the initial conditions. We see that

$$\begin{aligned}n &= 0: \quad A + B + C = 0;\\ n &= 1: \quad A + B\left(-1+i\right) + C\left(-1-i\right) = -1/5;\\ n &= 2: \quad A + B\left(-2i\right) + C\left(2i\right) = -2/5.\end{aligned}$$

Substituting in $A = -B - C$ from the first equation we have

$$\begin{aligned}B\left(-2+i\right) + C\left(-2-i\right) &= \frac{-1}{5};\\ B\left(-1-2i\right) + C\left(-1+2i\right) &= \frac{-2}{5},\end{aligned}$$

and solving these gives

$$B = \frac{4-3i}{50} \quad \text{and} \quad C = \frac{4+3i}{50}, \quad \text{and} \quad A = -B - C = \frac{-8}{50}.$$

Hence the unique solution is

$$\begin{aligned}x_n &= \frac{n}{5} + \frac{-8}{50} + \frac{4-3i}{50}\left(-1+i\right)^n + \frac{4+3i}{50}\left(-1-i\right)^n\\ &= \frac{1}{50}\left(10n - 8 + (4-3i)\left(-1+i\right)^n + (4+3i)\left(-1-i\right)^n\right).\end{aligned}$$

The last two terms are conjugates of one another and so, recalling that

$$z + \overline{z} = 2\,\mathrm{Re}\,z$$

we have

$$\begin{aligned}x_n &= \frac{1}{50}\left(10n - 8 + 2\,\mathrm{Re}\left[(4-3i)\left(-1+i\right)^n\right]\right)\\ &= \frac{1}{50}\left(10n - 8 + 2\times 2^{n/2}\,\mathrm{Re}\left[(4-3i)\left(\cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4}\right)\right]\right)\\ &= \frac{1}{50}\left(10n - 8 + 2^{n/2+1}\left(4\cos\frac{3n\pi}{4} + 3\sin\frac{3n\pi}{4}\right)\right)\\ &= \frac{1}{25}\left(5n - 4 + 2^{n/2}\left(4\cos\frac{3n\pi}{4} + 3\sin\frac{3n\pi}{4}\right)\right)\end{aligned}$$

■

2.5 Exercises

2.5.1 Application to Series

Exercise 62 *Let a and r be real numbers with $r \neq 1$. Prove by induction, that*

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{a\left(r^n - 1\right)}{r - 1} \quad \text{for } n = 1, 2, 3 \ldots$$

Exercise 63 *Prove that*

$$\sum_{r=n}^{2n-1} 2r + 1 = 3n^2 \quad \text{for} \quad n = 1, 2, 3 \ldots$$

Exercise 64 *Prove for $n \in \mathbb{N}$, that*

$$\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1.$$

Exercise 65 *Show that*

$$\sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n} \quad \text{for } n = 1, 2, 3, \ldots$$

Exercise 66 *Use the formula from statement B to show that the sum of an arithmetic progression with initial value a, common difference d and n terms, is*

$$\frac{n}{2}\left\{2a + (n-1)\, d\right\}.$$

Exercise 67 *Prove for $n \geq 2$ that,*

$$\sum_{r=2}^{n} \frac{1}{r^2 - 1} = \frac{(n-1)(3n+2)}{4n(n+1)}.$$

Exercise 68 *Let*

$$S(n) = \sum_{r=0}^{n} r^2 \quad \text{for} \quad n \in \mathbb{N}.$$

Show that there is a unique cubic $f(n) = an^3 + bn^2 + cn + d$, whose coefficients a, b, c, d you should determine, such that $f(n) = S(n)$ for $n = 0, 1, 2, 3$. Prove by induction that $f(n) = S(n)$ for $n \in \mathbb{N}$.

Exercise 69 *Show that*

$$n + 3\sum_{r=1}^{n} r + 3\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n} \left\{(r+1)^3 - r^3\right\} = (n+1)^3 - 1.$$

Hence, using the formula from statement B, find an expression for $\sum_{r=1}^{n} r^2$.

Exercise 70 *Extend the method of Exercise 69 to find expressions for* $\sum_{r=1}^{n} r^3$ *and* $\sum_{r=1}^{n} r^4$.

Exercise 71 *Use induction to show that*

$$\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin 2nx}{2\sin x}.$$

Exercise 72 *Use induction to show that*

$$\sum_{k=1}^{n} \sin kx = \frac{\sin\left\{\frac{1}{2}(n+1)x\right\}\sin\left\{\frac{1}{2}nx\right\}}{\sin\left\{\frac{1}{2}x\right\}}.$$

Exercise 73 *Let* k *be a natural number. Deduce from Example 42 that*

$$\sum_{r=1}^{n} r^k = \frac{n^{k+1}}{k+1} + E_k(n) \qquad (2.11)$$

where $E_k(n)$ *is a polynomial in* n *of degree at most* k.

Exercise 74 *Use the method of Exercise 69 and the binomial theorem to find an alternative proof of equation (2.11).*

2.5.2 Miscellaneous Examples

Exercise 75 *Prove Bernoulli's Inequality which states that*

$$(1+x)^n \geq 1 + nx \quad \text{for } x \geq -1 \text{ and } n \in \mathbb{N}.$$

Exercise 76 *Show by induction that* $n^2 + n \geq 42$ *when* $n \geq 6$ *and* $n \leq -7$.

Exercise 77 *Show by induction that there are* $n!$ *ways of ordering a set with* n *elements.*

Exercise 78 *Show that there are* 2^n *subsets of the set* $\{1, 2, ..., n\}$. *[Be sure to include the empty set.]*

Exercise 79 *Show for* $n \geq 1$ *and* $0 \leq k \leq n$ *that*

$$\frac{n!}{k!\,(n-k)!} < 2^n.$$

[Hint: you may find it useful to note the symmetry in the LHS which takes the same value at $k = k_0$ *as it does at* $k = n - k_0$.*]*

Exercise 80 *Bertrand's Postulate states that for $n \geq 3$ there is prime number p satisfying*

$$\frac{n}{2} < p < n.$$

Use this postulate and the strong form of induction to show that every positive integer can be written as a sum of prime numbers, all of which are distinct. (For the purposes of this exercise you will need to regard 1 as prime number.)

Exercise 81 *Assuming only the product rule of differentiation, show that*

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \quad \text{for } n = 1, 2, 3, \ldots$$

Exercise 82 *Show that every natural number $n \geq 1$ can be written in the form $n = 2^k l$ where k, l are natural numbers and l is odd.*

Exercise 83 *Show that every integer n can be written as a sum $3a + 5b$ where a and b are integers.*

Exercise 84 *Show that $3^{3n} + 5^{4n+2}$ is divisible by 13 for all natural numbers n.*

Exercise 85 *(a) Show that $7^{m+3} - 7^m$ and $11^{m+3} - 11^m$ are both divisible by 19 for all $m \geq 0$.*

(b) Calculate the remainder when $7^m - 11^n$ is divided by 19, for the cases $0 \leq m \leq 2$ and $0 \leq n \leq 2$.

(c) Deduce that $7^m - 11^n$ is divisible by 19, precisely when $m+n$ is a multiple of 3.

Exercise 86 *By setting up an identity between I_n and I_{n-2} show that*

$$I_n = \int_0^{\pi} \frac{\sin nx}{\sin x}\, dx$$

equals π when n is odd. What value does I_n take when n is even?

Exercise 87 *Show that*

$$\int_0^{\pi/2} \cos^{2n+1} x \; dx = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

Exercise 88 *Euler's Gamma function $\Gamma(a)$ is defined for all $a > 0$ by the integral*

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} \; dx.$$

Show that $\Gamma(a+1) = a\Gamma(a)$ for $a > 0$, and deduce that

$$\Gamma(n+1) = n! \quad \text{for } n \in \mathbb{N}.$$

Exercise 89 *Euler's Beta function $B(a,b)$ is defined for all positive a,b by the integral*

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\ dx.$$

Set up a reduction formula involving B, and deduce that if m and n are natural numbers then

$$B(m+1,n+1)=\frac{m!\,n!}{(m+n+1)!}.$$

Exercise 90 *The Hermite polynomials $H_n(x)$ for $n=0,1,2,\ldots$ are defined recursively as*

$$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x)\ \text{ for } n\geq 1,$$

with $H_0(x)=1$ and $H_1(x)=2x$.

(a) Calculate $H_n(x)$ for $n=2,3,4,5$.

(b) Show by induction that

$$H_{2k}(0)=(-1)^k\frac{(2k)!}{k!}\quad\text{and}\quad H_{2k+1}(0)=0.$$

(c) Show by induction that

$$\frac{dH_n}{dx}=2nH_{n-1}.$$

(d) Deduce the $H_n(x)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2ny=0.$$

(e) Use Leibniz's rule for differentiating a product (see Example 101) to show that the polynomials

$$(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

satisfy the same recursion as $H_n(x)$ with the same initial conditions and deduce that

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)\quad\text{for } n=0,1,2,\ldots$$

Exercise 91 *What is wrong with the following 'proof' that all people are of the same height?*

"Let $P(n)$ be the statement that n persons must be of the same height. Clearly $P(1)$ is true as a person is the same height as him/herself. Suppose now that $P(k)$ is true for some natural number k and we shall prove that $P(k+1)$ is also true. If we have a crowd of $k+1$ people then we can invite one person to briefly leave so that k remain — from $P(k)$ we know that these people must all be equally tall. If we invite back the missing person and someone else leaves, then these k persons are also of the same height. Hence the $k+1$ persons were all of equal height and so $P(k+1)$ follows. By induction everyone is of the same height."

Exercise 92 *Below are certain families of statements $P(n)$ (indexed by $n \in \mathbb{N}$), which satisfy rules that are similar (but not identical) to the hypotheses required for induction. In each case say for which $n \in \mathbb{N}$ the truth of $P(n)$ must follow from the given rules.*

(a) $P(0)$ is true; for $n \in \mathbb{N}$ if $P(n)$ is true then $P(n+2)$ is true;

(b) $P(1)$ is true; for $n \in \mathbb{N}$ if $P(n)$ is true then $P(2n)$ is true;

(c) $P(0)$ and $P(1)$ are true; for $n \in \mathbb{N}$ if $P(n)$ is true then $P(n+2)$ is true;

(d) $P(0)$ and $P(1)$ are true; for $n \in \mathbb{N}$ if $P(n)$ and $P(n+1)$ are true then $P(n+2)$ is true;

(e) $P(0)$ is true; for $n \in \mathbb{N}$ if $P(n)$ is true then $P(n+2)$ and $P(n+3)$ are both true;

(f) $P(0)$ is true; for $n \geq 1$ if $P(n)$ is true then $P(n+1)$ is true.

2.5.3 Binomial Theorem

Exercise 93 *Show that Lemma 46 holds true for general integers n and k, remembering the convention that $\binom{n}{k}$ is zero when $n < 0$ or $k < 0$ or $k > n$.*

Exercise 94 *Interpret Lemma 46 in terms of subsets of $\{1, 2, \ldots, n\}$ and subsets of $\{1, 2, \ldots, n+1\}$ to give a new proof of the lemma. [Hint: consider subsets of $\{1, 2, \ldots, n+1\}$ containing k elements, and whether they do, or do not, contain the final element $n+1$.]*

Exercise 95 *Let n be a natural number. Show that*

$$\text{(a)} \qquad \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n;$$

$$\text{(b)} \qquad \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}.$$

Interpret part (a) in terms of the subsets of $\{1, 2, \ldots, n\}$. [Note that the sums above are not infinite as the binomial coefficients $\binom{n}{k}$ eventually become zero once k becomes sufficiently large.]

Exercise 96 *Let n be a positive integer. Simplify the expression $(1+i)^{2n}$. Use the binomial theorem to show that*

$$\binom{2n}{0} - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} + \cdots + (-1)^n \binom{2n}{2n} = \begin{cases} (-1)^{n/2}\, 2^n & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Show that the right-hand side is equal to $2^n \cos(n\pi/2)$. Similarly, find the value of

$$\binom{2n}{1} - \binom{2n}{3} + \binom{2n}{5} - \binom{2n}{7} + \cdots + (-1)^{n-1} \binom{2n}{2n-1}.$$

Exercise 97 *Let n be a natural number. Show that*

$$\binom{2n}{0}+\binom{2n}{4}+\binom{2n}{8}+\cdots=\begin{cases} 2^{2n-2}+(-1)^{n/2}\,2^{n-1} & \text{if } n \text{ is even;} \\ 2^{2n-2} & \text{if } n \text{ is odd.}\end{cases}$$

Distinguishing cases, find the value of

$$\binom{2n}{1}+\binom{2n}{5}+\binom{2n}{9}+\cdots$$

Exercise 98 *Use the identity $(1+x)^{2n}=(1+x)^n\,(1+x)^n$ to show that*

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}.$$

Exercise 99 *By differentiating the binomial theorem, show that*

$$\sum_{k=1}^{n}k\binom{n}{k}=n2^{n-1}.$$

Exercise 100 *Show that*

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}=\frac{2^{n+1}-1}{n+1}.$$

Exercise 101 *Use induction to prove Leibniz's rule for the differentiation of a product — this says that for functions $u(x)$ and $v(x)$ of a variable x, and $n\in\mathbb{N}$, then*

$$\frac{d^n}{dx^n}\left(u(x)\,v(x)\right)=\sum_{k=0}^{n}\binom{n}{k}\frac{d^k u}{dx^k}\frac{d^{n-k}v}{dx^{n-k}}.$$

2.5.4 Fibonacci and Lucas Numbers

Exercise 102 *Use Proposition 53 to show that*

$$F_{2n+1}=(F_{n+1})^2+(F_n)^2 \qquad (n\in\mathbb{N}).$$

Deduce that

$$F_{2n}=(F_{n+1})^2-(F_{n-1})^2 \qquad (n=1,2,3...).$$

Exercise 103 *Prove by induction the following identities involving the Fibonacci numbers*

(a) $F_1+F_3+F_5+\cdots+F_{2n+1}=F_{2n+2}$,

(b) $F_2+F_4+F_6+\cdots+F_{2n}=F_{2n+1}-1$,

(c) $(F_1)^2+(F_2)^2+\cdots+(F_n)^2=F_nF_{n+1}$.

Exercise 104 *(a) The Lucas numbers L_n $(n \in \mathbb{N})$ are defined by*

$$L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2,$$

and by $L_0 = 2$ and $L_1 = 1$. Prove that

$$L_n = 2F_{n-1} + F_n \text{ for } n \geq 1.$$

More generally, show that if a sequence of numbers G_n, for $n \in \mathbb{N}$, is defined by $G_n = G_{n-1} + G_{n-2}$ for $n \geq 2$,and by $G_0 = a$ and $G_1 = b$, show that $G_n = aF_{n-1} + bF_n$ for $n \geq 1$.

Exercise 105 *Show that $F_{2n} = F_n L_n$ for $n \in \mathbb{N}$, where L_n denotes the nth Lucas number, described in the previous exercise. Deduce the identity*

$$F_{2^n} = L_2 L_4 L_8 \cdots L_{2^{n-1}}.$$

Exercise 106 *Show, for $0 \leq k \leq n$, with k even, that*

$$F_{n-k} + F_{n+k} = L_k F_n.$$

Can you find, and prove, a similar expression for $F_{n-k} + F_{n+k}$ when k is odd?

Exercise 107 *Use Lemma 46 to prove by induction that*

$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots \quad \text{for } n \in \mathbb{N}.$$

[Note that the series is not infinite as the terms in the sum will eventually become zero.]

Exercise 108 *Use Proposition 53 to show that*

$$F_{(m+1)k} = F_{mk+1} F_k + F_{k-1} F_{mk}$$

and deduce that if k divides n then F_k divides F_n.

Exercise 109 *Let $\alpha = \left(1 + \sqrt{5}\right)/2$ and $\beta = \left(1 - \sqrt{5}\right)/2$. Show that α and β are roots of $1 + x = x^2$. Use the identity from Proposition 52 and the binomial theorem to show that*

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}.$$

Exercise 110 *Prove for $m, n \in \mathbb{N}$ that*

$$\sum_{k=0}^{n} \binom{m+k}{m} = \binom{n+m+1}{m+1}.$$

Exercise 111 *By considering the identity* $(1+x)^{m+n} = (1+x)^m (1+x)^n$, *or otherwise, prove that*

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} \quad \textit{for } m,n,r \in \mathbb{N}.$$

Exercise 112 *Show that for* $n = 1, 2, 3, \ldots$

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = \frac{(2n-1)!}{\{(n-1)!\}^2}.$$

Exercise 113 *Show that*

$$\sum_{i=m}^{n} F_i = F_{n+2} - F_{m+1} \quad \textit{for } m,n \in \mathbb{N}.$$

Exercise 114 *Show that*

$$F_{i+j+k} = F_{i+1}F_{j+1}F_{k+1} + F_iF_jF_k - F_{i-1}F_{j-1}F_{k-1} \textit{ for } i,j,k = 1,2,3,\ldots$$

Exercise 115 *Show by induction that* $\alpha = \left(1+\sqrt{5}\right)/2$ *and* $\beta = \left(1-\sqrt{5}\right)/2$ *are roots of the equation*

$$F_{m-1} + F_m x = x^m$$

for $m = 1, 2, 3, \ldots$. *Hence generalise the result of Exercise 109 to show that*

$$\sum_{k=0}^{n} \binom{n}{k} (F_m)^k (F_{m-1})^{n-k} F_k = F_{mn}.$$

Exercise 116 *(For those with some knowledge of infinite geometric progressions). Use Proposition 52 to show that*

$$\sum_{k=0}^{\infty} F_k x^k = \frac{x}{1-x-x^2}.$$

This is the generating function of the Fibonacci numbers. For what values of x *does the infinite series above converge?*

2.5.5 Difference Equations

Exercise 117 *The sequence of numbers* x_n *is defined recursively by*

$$x_n = x_{n-1} + 2x_{n-2} \textit{ for } n \geq 2,$$

and by $x_0 = 1$, *and* $x_1 = -1$. *Calculate* x_n *for* $n \leq 6$, *and make an estimate for the value of* x_n *for general* n. *Use induction to prove your estimate correct.*

Exercise 118 *The sequence of numbers* x_n *is defined recursively by*

$$x_n = 2x_{n-1} - x_{n-2} \text{ for } n \geq 2,$$

and by $x_0 = a$ *and* $x_1 = b$. *Calculate* x_n *for* $n \leq 6$, *and make an estimate for the value of* x_n *for general* n. *Use induction to prove your estimate correct.*

Exercise 119 *The sequence of numbers* x_n *is defined recursively by*

$$x_n = 3x_{n-2} + 2x_{n-3} \text{ for } n \geq 3,$$

and by $x_0 = 1$, $x_1 = 3$, $x_2 = 5$. *Show by induction that*

$$2^n < x_n < 2^{n+1} \text{ for } n \geq 1$$

and that

$$x_{n+1} = 2x_n + (-1)^n .$$

Exercise 120 *The sequence* $x_0, x_1, x_2, x_3, ...$ *is defined recursively by*

$$x_0 = 0, \quad x_1 = 0.8, \quad x_n = 1.2x_{n-1} - x_{n-2} \quad \text{for} \quad n \geq 2.$$

With the aid of a calculator list the values of x_i *for* $0 \leq i \leq 10$. *Prove further, by induction, that*

$$x_n = \operatorname{Im}\{(0.6 + 0.8i)^n\}$$

for each $n = 0, 1, 2, ...$ *Deduce that* $|x_n| \leq 1$ *for all* n. *Show also that* x_n *cannot have the same sign for more than three consecutive* n.

Exercise 121 *The sequences* u_n *and* v_n *are defined recursively by*

$$u_{n+1} = u_n + 2v_n \quad \text{and} \quad v_{n+1} = u_n + v_n,$$

with initial values $u_1 = v_1 = 1$. *Show that*

$$(u_n)^2 - 2(v_n)^2 = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Show further, for $n \geq 1$, *that*

$$u_n = \frac{\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n}{2} \quad \text{and} \quad v_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}.$$

Exercise 122 *Solve the initial-value difference equation*

$$3x_{n+2} - 2x_{n+1} - x_n = 0, \quad x_0 = 2, \quad x_1 = 1.$$

Show that $x_n \to 5/4$ *as* $n \to \infty$.

Exercise 123 *Obtain particular solutions to the difference equation*

$$x_{n+2} + 2x_{n+1} - 3x_n = f(n)$$

when (i) $f(n) = 2^n$*; (ii)* $f(n) = n$*; (iii)* $f(n) = n(-3)^n$.

Exercise 124 *Find the general solution of*

$$x_{n+1} = x_n + \sin n.$$

Exercise 125 *Show that* $x_n = n!$ *is a solution of the second-order difference equation*

$$x_{n+2} = (n+2)(n+1)x_n.$$

By making the substitution $x_n = n!u_n$ *find a second independent solution.*

Now find the unique solution for x_n *given that* $x_0 = 1$ *and* $x_1 = 3$.

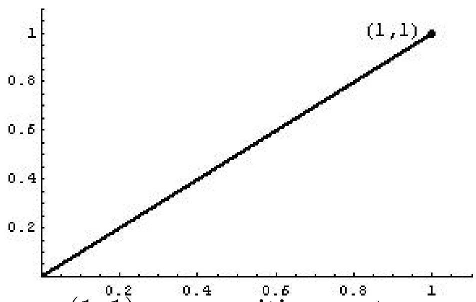
3. VECTORS AND MATRICES

3.1 Vectors

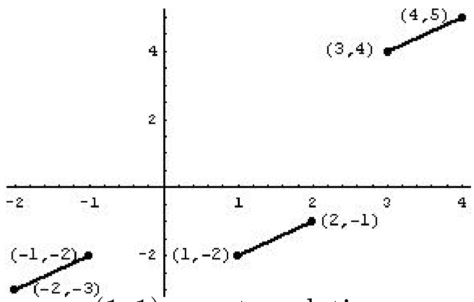
A *vector* can be thought of in two different ways. Let's for the moment concentrate on vectors in $\mathbb{R}^2$, the xy-plane.

- From one point of view a vector is just an ordered pair of numbers (x, y). We can associate this vector with the point in $\mathbb{R}^2$ which has co-ordinates x and y. We call this vector the *position vector* of the point.

- From the second point of view a vector is a 'movement' or translation. For example, to get from the point $(3, 4)$ to the point $(4, 5)$ we need to move 'one to the right and one up'; this is the same movement as is required to move from $(-2, -3)$ to $(-1, -2)$ or from $(1, -2)$ to $(2, -1)$. Thinking about vectors from this second point of view, all three of these movements are the same vector, because the same translation 'one right, one up' achieves each of them, even though the 'start' and 'finish' are different in each case. We would write this vector as $(1, 1)$. Vectors from this second point of view are sometimes called *translation vectors.*



$(1, 1)$ as a position vector



$(1, 1)$ as a translation

Likewise in three (or higher) dimensions the triple (x, y, z) can be thought of as the point in $\mathbb{R}^3$, which is x units along the x-axis, y units along the y-axis and z units along the z-axis, or it can represent the translation which would take the origin to that point.

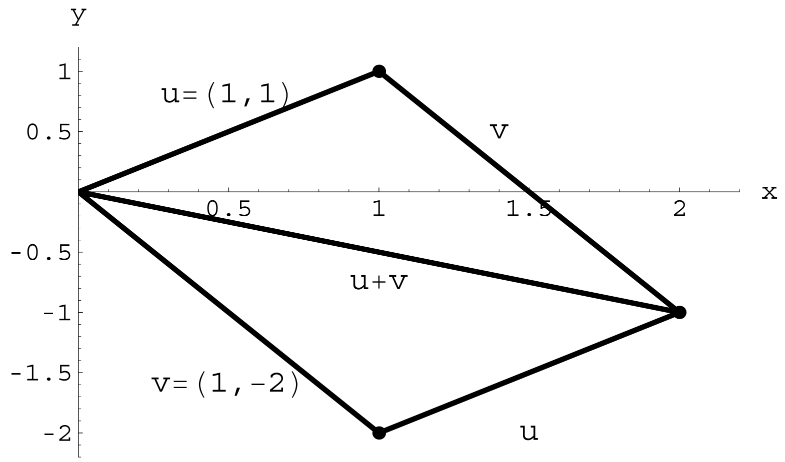
Notation 56 *For ease of notation vectors are often denoted by a single letter, but to show that this is a vector quantity, rather than just a single number, the letter is either underlined such as* $\underline{v}$ *or written in bold as* $\mathbf{v}$.

3.1.1 Algebra of Vectors

Again we return to xy-plane. Given two vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ we can add them, much as you would expect, by adding their co-ordinates. That is:

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) .$$

This is easiest to see from a diagram:



A geometric interpretation of the vector sum

The sum is also easiest to interpret when we consider the vectors as translations. The translation $(u_1 + v_1, u_2 + v_2)$ is the composition of doing the translation (u_1, u_2) first and then doing the translation (v_1, v_2) *or* it can be achieved by doing the translations in the other order — that is, vector addition is *commutative:* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. Note that it makes sense to add two vectors in $\mathbb{R}^2$, or two vectors from $\mathbb{R}^3$ etc., but that we can make no obvious sense of adding a vector in $\mathbb{R}^2$ to one from $\mathbb{R}^3$ — they both need to be of the same type.

Given a vector $\mathbf{v} = (v_1, v_2)$ and a real number (a scalar) k then the *scalar multiple* $k\mathbf{v}$ is defined as

$$k\mathbf{v} = (kv_1, kv_2) .$$

When k is a positive integer then we can think of $k\mathbf{v}$ as the translation achieved when we translate by $\mathbf{v}$ k times. Note that the points $k\mathbf{v}$, as k varies, make up the line which passes through the origin and the point $\mathbf{v}$.

We write $-\mathbf{v}$ for $(-1)\,\mathbf{v} = (-v_1, -v_2)$ and this is the inverse operation of translating by $\mathbf{v}$. And the difference of two vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ is defined as

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = (v_1 - w_1, v_2 - w_2) .$$

Put another way $\mathbf{v} - \mathbf{w}$ is the vector that translates the point with position vector $\mathbf{w}$ to the point with position vector $\mathbf{v}$.

Note that there is also a special vector $\mathbf{0} = (0,0)$ which may be viewed either as a special point *the origin,* where the axes cross, or as the *null translation,* the translation that doesn't move anything.

The vectors $(1,0)$ and $(0,1)$ form the *standard* or *canonical basis* for $\mathbb{R}^2$. They are denoted by the symbols $\mathbf{i}$ and $\mathbf{j}$ respectively. Note that any vector $\mathbf{v} = (x,y)$ can be written uniquely as a linear combination of $\mathbf{i}$ and $\mathbf{j}$: that is

$$(x,y) = x\mathbf{i}+y\mathbf{j}$$

and this is the only way to write (x,y) as a sum of scalar multiples of $\mathbf{i}$ and $\mathbf{j}$. Likewise the vectors $(1,0,0), (0,1,0), (0,0,1)$ form the canonical basis for $\mathbb{R}^3$ and are respectively denoted as $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$.

Proposition 57 *Vector addition and scalar multiplication satisfy the following properties. These properties verify that $\mathbb{R}^2$ is a real vector space (cf. Michaelmas Linear Algebra). Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ and $\lambda, \mu \in \mathbb{R}$. Then*

$$\begin{array}{lll}
\mathbf{u}+\mathbf{0}=\mathbf{u} & \mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u} & 0\mathbf{u}=\mathbf{0} \\
\mathbf{u}+(-\mathbf{u})=\mathbf{0} & (\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}) & 1\mathbf{u}=\mathbf{u} \\
(\lambda+\mu)\mathbf{u}=\lambda\mathbf{u}+\mu\mathbf{u} & \lambda(\mathbf{u}+\mathbf{v})=\lambda\mathbf{u}+\lambda\mathbf{v} & \lambda(\mu\mathbf{u})=(\lambda\mu)\,\mathbf{u}
\end{array}$$

Everything above generalises in an obvious way to the case of $\mathbb{R}^n$ and from now on we will discuss this general case.

3.1.2 Geometry of Vectors

As vectors represent geometric ideas like points and translations, they have important geometric properties as well as algebraic ones. The length of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, which is written $|\mathbf{v}|$, is defined to be

$$|\mathbf{v}| = \sqrt{\sum_{i=1}^{n} (v_i)^2}.$$

This is exactly what you'd expect it to be: from Pythagoras' Theorem we see this is the distance of the point $\mathbf{v}$ from the origin, or equivalently the distance a point moves when it is translated by $\mathbf{v}$. So if $\mathbf{p}$ and $\mathbf{q}$ are the position vectors of two points in the plane, then the vector that will translate $\mathbf{p}$ to $\mathbf{q}$ is $\mathbf{q}-\mathbf{p}$, and the distance between them is $|\mathbf{q}-\mathbf{p}|$ (or equally $|\mathbf{p}-\mathbf{q}|$).

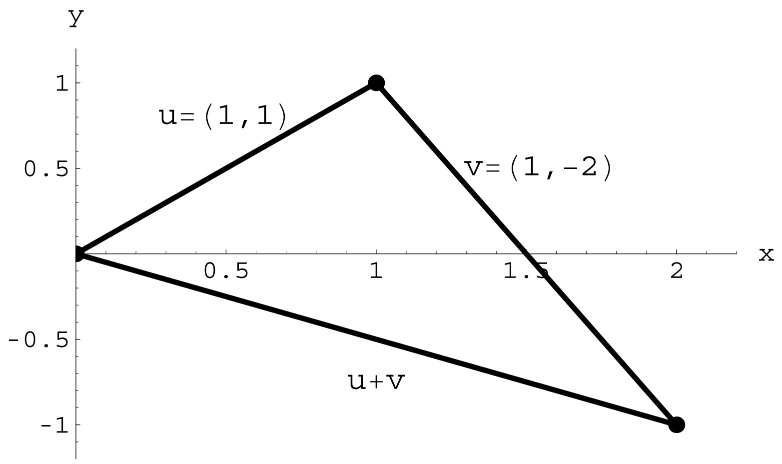
Note that $|\mathbf{v}| \geq 0$ and that $|\mathbf{v}| = 0$ if and only if $\mathbf{v} = \mathbf{0}$. Also $|\lambda\mathbf{v}| = |\lambda|\,|\mathbf{v}|$ for any $\lambda \in \mathbb{R}$.

Proposition 58 *(Triangle Inequality)*

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)\,, \mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. *Then*

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$

with equality when $\mathbf{u} = \lambda \mathbf{v}$ *where* $\lambda > 0$ *or one of* $\mathbf{u}, \mathbf{v}$ *is* $\mathbf{0}$.



Geometric interpretation of the Triangle Inequality

Proof. Assume $\mathbf{u} \neq \mathbf{0}$. Note that for $t \in \mathbb{R}$

$$0 \leq |\mathbf{u} + t\mathbf{v}|^2 = \sum_{i=1}^{n} (u_i + tv_i)^2 = |\mathbf{u}|^2 + 2t \sum_{i=1}^{n} u_i v_i + t^2 |\mathbf{v}|^2 .$$

The RHS of the above inequality is a quadratic in t which is always non-negative and so has non-positive discriminant (i.e. $b^2 \leq 4ac$). Hence

$$4 \left(\sum_{i=1}^{n} u_i v_i \right)^2 \leq 4 |\mathbf{u}|^2 |\mathbf{v}|^2$$

and so

$$\left| \sum_{i=1}^{n} u_i v_i \right| \leq |\mathbf{u}| |\mathbf{v}| . \tag{3.1}$$

Hence

$$|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2 \sum_{i=1}^{n} u_i v_i + |\mathbf{v}|^2 \leq |\mathbf{u}|^2 + 2 |\mathbf{u}| |\mathbf{v}| + |\mathbf{v}|^2 = (|\mathbf{u}| + |\mathbf{v}|)^2$$

to give the required result. The inequality (3.1) is known as the *Cauchy-Schwarz Inequality.*

Note that we have equality in $b^2 \leq 4ac$ if and only if the quadratic $|\mathbf{u} + t\mathbf{v}|^2 = 0$ has a unique real solution, say $t = t_0$. So $\mathbf{u} + t_0 \mathbf{v} = \mathbf{0}$ and we see that $\mathbf{u}$ and $\mathbf{v}$ are multiples of one another. This is for equality to occur in (3.1). With $\mathbf{u} = -t_0 \mathbf{v}$, then equality in

$$\mathbf{u} \cdot \mathbf{v} = -t_0 |\mathbf{u}|^2 \leq t_0 |\mathbf{u}|^2$$

occurs when t_0 is non-positive. ■

3.1.3 The Scalar Product

Given two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n), \mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, the *scalar product* $\mathbf{u} \cdot \mathbf{v}$, also known as the *dot product* or *Euclidean inner product*, is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

The following properties of the scalar product are easy to verify and are left as exercises. (Note (v) has already been proven in the previous section).

Proposition 59 *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$. *Then*

(i) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$*;*

(ii) $(\lambda \mathbf{u}) \cdot \mathbf{v} = \lambda(\mathbf{u} \cdot \mathbf{v})$*;*

(iii) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$*;*

(iv) $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \geq 0$ *and* $\mathbf{u} \cdot \mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$*;*

(v) (Cauchy-Schwarz Inequality)

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}| \, |\mathbf{v}|$$

with equality when $\mathbf{u}$ *and* $\mathbf{v}$ *are multiples of one another.*

We see that the length of a vector $\mathbf{u}$ can be written in terms of the scalar product, namely

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

We can also define *angle* using the scalar product in terms of their scalar product.

Definition 60 *Given two non-zero vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *the angle between them is given by the expression*

$$\cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \, |\mathbf{v}|} \right).$$

Note that the formula makes sense as

$$-1 \leq \mathbf{u} \cdot \mathbf{v} / (|\mathbf{u}| \, |\mathbf{v}|) \leq 1$$

from the Cauchy-Schwarz Inequality. If we take the principles values of $\cos^{-1}$ to be in the range $0 \leq \theta \leq \pi$ then this formula measures the smaller angle between the vectors. Note that two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular precisely when $\mathbf{u} \cdot \mathbf{v} = 0$.

3.2 Matrices

At its simplest a *matrix* is just a two-dimensional array of numbers: for example

$$\begin{pmatrix} 1 & 2 & -3 \\ \sqrt{2} & \pi & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1.2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

are all matrices. The examples above are respectively a 2×3 matrix, a 3×1 matrix and a 2×2 matrix (read '2 by 3' etc.); the first figure refers to the number of *rows* and the second to the number of *columns.* So vectors like (x, y) and (x, y, z) are also matrices, respectively 1×2 and 1×3 matrices.

The 3×1 matrix above could just as easily be thought of as a vector — it is after all just a list of three numbers, but written down rather than across. This is an example of a *column* vector. When we need to differentiate between the two, the vectors we have already met like (x, y) and (x, y, z) are called *row* vectors.

Notation 61 *The set of $1 \times n$ row vectors (or n-tuples) is written as $\mathbb{R}^n$. When we need to differentiate between row and column vectors we write $(\mathbb{R}^n)'$ or $(\mathbb{R}^n)^*$ for the set of $n \times 1$ column vectors. If there is no chance of confusion between the two (because we are only using row vectors, or only column vectors) then $\mathbb{R}^n$ can denote either set.*

Notation 62 *If you are presented with an $m \times n$ matrix $A = (a_{ij})$ then the notation here simply means that a_{ij} is the (i, j)th entry. That is, the entry in the ith row and jth column is a_{ij}. Note that i can vary between 1 and m, and that j can vary between 1 and n. So*

$$i\text{th row} = (a_{i1}, \ldots, a_{in}) \quad \text{and} \quad j\text{th column} = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

Notation 63 *As with vectors there is a zero $m \times n$ matrix, whose every entry is 0, which we denote by 0 unless we need to specify its size.*

Notation 64 *For each n there is another important matrix, the identity matrix I_n. Each diagonal entry of I_n is 1 and all other entries are 0. That is*

$$a_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

The notation δ_{ij} is referred to as the Kronecker Delta.

We shall see though that a matrix is much more than simply an array of numbers. Where vectors can be thought of as points in space, we shall see that a matrix is really a map between spaces: specifically a $m \times n$ matrix can be thought of as a map from $(\mathbb{R}^n)'$ to $(\mathbb{R}^m)'$. To see this we need to first talk about how matrices add and multiply.

1. **Addition:** Let A be an $m \times n$ matrix (recall: m rows and n columns) and B be an $p \times q$ matrix. We would like to add them (as we added vectors) by adding their corresponding entries. So to add them the two matrices have to be the same size, that is $m = p$ and $n = q$. In which case we have
$$(A + B)_{ij} = a_{ij} + b_{ij} \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

2. **Scalar Multiplication:** Let A be an $m \times n$ matrix and k be a constant (a scalar). Then the matrix kA has as its (i, j)th entry
$$(kA)_{ij} = ka_{ij} \quad \text{for } \ 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$
That is, we multiply each of the entries of A by k to get the new matrix kA.

3. **Matrix Multiplication:** Based on how we added matrices then you might think that we multiply matrices in a similar fashion, namely multiplying corresponding entries, but we do not. At first glance the rule for multiplying matrices is going to seem rather odd, but we will soon discover why we multiply them as we do.

   The rule is this: we can multiply an $m \times n$ matrix A with an $p \times q$ matrix B if $n = p$ and we produce an $m \times q$ matrix AB with (i, j)th entry
$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq q. \tag{3.2}$$
   It may help a little to write the rows of A as $\mathbf{r}_1, \ldots, \mathbf{r}_m$ and the columns of B as $\mathbf{c}_1, \ldots, \mathbf{c}_q$ and the above rule says that
$$(AB)_{ij} = \mathbf{r}_i \cdot \mathbf{c}_j \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq q.$$
   We dot the rows of A with the columns of B.

N.B. Here are two important rules of matrix algebra. Let A be an $m \times n$ matrix. Then
$$A0_{np} = 0_{mp} \text{ and } 0_{lm}A = 0_{\text{ln}}$$
and
$$AI_n = A \quad \text{and} \quad I_m A = A.$$

This formula for multiplication will, I am sure, seem pretty bizarre, let alone easy to remember — so here are some examples.

Example 65 *Show that* $2\,(A+B) = 2A + 2B$ *for the following matrices:*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2 \\ 5 & 1 \end{pmatrix}.$$

Solution. Here we are checking the distributive law in a specific example. Generally it is the case that $c\,(A+B) = cA + cB$. But to check it here we note that

$$\begin{aligned} A + B &= \begin{pmatrix} 1 & 0 \\ 8 & 5 \end{pmatrix}, \quad \text{and so } 2\,(A+B) = \begin{pmatrix} 2 & 0 \\ 16 & 10 \end{pmatrix}; \\ 2A &= \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}, \text{ and } 2B = \begin{pmatrix} 0 & -4 \\ 10 & 2 \end{pmatrix}, \text{ so } 2A + 2B = \begin{pmatrix} 2 & 0 \\ 16 & 10 \end{pmatrix}. \end{aligned}$$

■

Example 66 *Where possible, calculate pairwise the products of the following matrices.*

$$\underbrace{A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}}_{2\times 2}, \quad \underbrace{B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}}_{2\times 3}, \quad \underbrace{C = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}}_{2\times 2}.$$

Solution. First up, the products we can form are: AA, AB, AC, CA, CB, CC. Let's slowly go through the product AC.

$$\begin{pmatrix} \boxed{1} & \boxed{2} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \boxed{1} & -1 \\ \boxed{1} & -1 \end{pmatrix} = \begin{pmatrix} 1\times 1 + 2 \times 1 & ?? \\ ?? & ?? \end{pmatrix} = \begin{pmatrix} 3 & ?? \\ ?? & ?? \end{pmatrix}.$$

This is how we calculate the $(1,1)$th entry of AC. We the first row of A and the first column of C and we dot them together. We complete the remainder of the product as follows:

$$\begin{aligned} \begin{pmatrix} \boxed{1} & \boxed{2} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \boxed{-1} \\ 1 & \boxed{-1} \end{pmatrix} &= \begin{pmatrix} 2 & \boxed{1\times(-1) + 2\times -1} \\ ?? & ?? \end{pmatrix} = \begin{pmatrix} 3 & \boxed{-3} \\ ?? & ?? \end{pmatrix}; \\ \begin{pmatrix} 1 & 2 \\ \boxed{-1} & \boxed{0} \end{pmatrix} \begin{pmatrix} \boxed{1} & -1 \\ \boxed{1} & -1 \end{pmatrix} &= \begin{pmatrix} 2 & 4 \\ \boxed{-1\times 1 + 0\times 1} & ?? \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ \boxed{-1} & ?? \end{pmatrix}; \\ \begin{pmatrix} 1 & 2 \\ \boxed{-1} & \boxed{0} \end{pmatrix} \begin{pmatrix} 1 & \boxed{-1} \\ 1 & \boxed{-1} \end{pmatrix} &= \begin{pmatrix} 2 & 4 \\ 0 & \boxed{-1\times -1 + 0\times -1} \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -1 & \boxed{1} \end{pmatrix}. \end{aligned}$$

So finally

$$\begin{pmatrix} 1 & 2 \\ \boxed{-1} & \boxed{0} \end{pmatrix} \begin{pmatrix} 1 & \boxed{-1} \\ 1 & \boxed{-1} \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -1 & \boxed{1} \end{pmatrix}.$$

We complete the remaining examples more quickly but still leaving a middle stage in the calculation to help see the process:

$$
\begin{aligned}
AA &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1-2 & 2+0 \\ -1+0 & -2+0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & -2 \end{pmatrix}; \\
AB &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1+2 & 2+4 & 3+6 \\ -1+0 & -2+0 & -3+0 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 4 & 6 \\ -1 & -2 & -3 \end{pmatrix}; \\
CA &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1+1 & 2-0 \\ 1+1 & 2-0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}; \\
CB &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1-3 & 2-2 & 3-1 \\ 1-3 & 2-2 & 3-1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{pmatrix}; \\
CC &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1-1 & 1-1 \\ 1-1 & 1-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

■

N.B. There are certain important things to note here, which make matrix algebra different from the algebra of numbers.

1. Note the $AC \neq CA$. That is, matrix multiplication is not generally *commutative.*

2. It is, though, *associative,* which means that

   $$(AB)\,C = A\,(BC)$$

   whenever this product makes sense. So we can write down a product like $A_1A_2 \ldots A_n$ without having to specify how we go about multiplying all these matrices or needing to worry about bracketing, but we do have to keep the order in mind.

3. Note that $CC = 0$ even though C is non-zero — not something that happens with numbers.

4. The distributive laws also hold for matrix multiplication, namely

   $$A\,(B+C) = AB + AC \quad \text{and} \quad (A+B)\,C = AC + BC$$

   whenever these products make sense.

Notation 67 *We write A^2 for the product AA and similarly we write A^n for the product $\underbrace{AA\cdots A}_{n \text{ times}}$. Note that A must be a square matrix (same number of rows and columns) for this to make sense.*

3.3 Matrices as Maps

We have seen then how we can multiply an $m \times n$ matrix A and an $n \times p$ matrix B to form a product AB which is an $m \times p$ matrix. Now given the co-ordinates of a point in n-dimensional space, we can put them in a column to form a $n \times 1$ column vector, an element $\mathbf{v} \in (\mathbb{R}^n)'$. If we *pre*multiply this vector $\mathbf{v}$ by the $m \times n$ matrix A we get a $m \times 1$ column vector $A\mathbf{v} \in (\mathbb{R}^m)'$.

So premultiplying by the matrix A gives a map: $(\mathbb{R}^n)' \to (\mathbb{R}^m)' : \mathbf{v} \to A\mathbf{v}$.

Why have we started using column vectors all of a sudden? This is really just a matter of choice: we could just as easily take row vectors and *post*multiply by matrices on the right, and this is the convention some books choose. But, as we are used to writing functions on the left, we will use column vectors and premultiply by matrices.

Importantly though, we can now answer the question of why we choose to multiply matrices as we do. Take an $m \times n$ matrix A and an $n \times p$ matrix B. We have two maps associated with premultiplication by A and B; let's call them α, given by:

$$\alpha : (\mathbb{R}^n)' \to (\mathbb{R}^m)' : \mathbf{v} \mapsto A\mathbf{v}$$

and β, given by

$$\beta : (\mathbb{R}^p)' \to (\mathbb{R}^n)' : \mathbf{v} \mapsto B\mathbf{v}.$$

We also have their composition $\alpha \circ \beta$, that is we do β first and then α, given by

$$\alpha \circ \beta : (\mathbb{R}^p)' \to (\mathbb{R}^m)' : \mathbf{v} \mapsto A\left(B\mathbf{v}\right).$$

But we have already commented that matrix multiplication is associative, so that

$$A\left(B\mathbf{v}\right) = \left(AB\right)\mathbf{v}.$$

That is,

the composition $\alpha \circ \beta$ is premultiplication by the product AB.

So if we think of matrices more as maps, rather than just simple arrays of numbers, matrix multiplication is quite natural and simply represents the composition of the corresponding maps.

3.3.1 Linear Maps

Let A be an $m \times n$ matrix and consider the associated map

$$\alpha : (\mathbb{R}^n)' \to (\mathbb{R}^m)' : \mathbf{v} \mapsto A\mathbf{v}$$

which is just pre-multiplication by A. Because of the distributive laws that hold for matrices, given column vectors $\mathbf{v}_1, \mathbf{v}_2 \in (\mathbb{R}^n)'$ and scalars $c_1, c_2 \in \mathbb{R}$ then

$$\begin{aligned} \alpha (c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= A (c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \\ &= c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 \\ &= c_1\alpha (\mathbf{v}_1) + c_2\alpha (\mathbf{v}_2) . \end{aligned}$$

This means α is what is called a *linear map* — multiplication by a matrix leads to a linear map. The important thing is that the converse also holds true — any linear map $(\mathbb{R}^n)' \to (\mathbb{R}^m)'$ has an associated matrix.

To see this most clearly we return to 2×2 matrices. Let $\alpha : (\mathbb{R}^2)' \to (\mathbb{R}^2)'$ be a linear map — we'll try to work out what its associated matrix is. Let's suppose we're right and α is just premultiplication by some 2×2 matrix; let's write it as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that if we multiply the canonical basis vectors

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

by this matrix we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

So, *if* this matrix exists, the first column is $\alpha (\mathbf{i})$ and the second is $\alpha (\mathbf{j})$. But if we remember that α is linear then we see now that we have the right matrix, let's call it

$$A = \begin{pmatrix} \alpha (\mathbf{i}) & \alpha (\mathbf{j}) \end{pmatrix}.$$

Then

$$\begin{aligned} A\begin{pmatrix} x \\ y \end{pmatrix} &= x\alpha (\mathbf{i}) + y\alpha (\mathbf{j}) \\ &= \alpha (x\mathbf{i}+y\mathbf{j}) \quad \text{[as } \alpha \text{ is linear]} \\ &= \alpha \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

We shall make use of this later when we calculate the matrices of some standard maps.

The calculations we performed above work just as well generally: if $\alpha : (\mathbb{R}^n)' \to (\mathbb{R}^m)'$ is a linear map then it is the same as premultiplying by an $m \times n$ matrix. In the columns of this matrix are the images of the canonical basis of $\mathbb{R}^n$ under α.

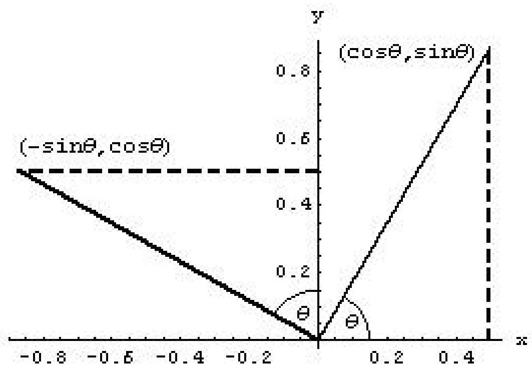
3.3.2 Geometric Aspects

The following are all examples of linear maps of the xy-plane:

- Rotation R_θ anti-clockwise by θ about the origin;
- Reflection in a line through the origin;
- A stretch with an invariant line through the origin;
- A shear with an invariant line through the origin;

though this list is far from comprehensive.

We concentrate on the first example R_θ. Remember to find the matrix for R_θ we need to find the images $R_\theta(\mathbf{i})$ and $R_\theta(\mathbf{j})$. We note from the diagrams



that

$$R_\theta(\mathbf{i}) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad R_\theta(\mathbf{j}) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

So, with a little abuse of notation, we can write

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

- $R_\theta R_\phi = R_{\theta+\phi}$ — rotating by θ and then by ϕ is the same as rotating by $\theta + \phi$. This is one way of calculating the $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$ formulas:

$$\begin{aligned} R_\theta R_\phi &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \cos\theta\sin\phi + \sin\theta\cos\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix}; \end{aligned}$$

- the inverse map of R_θ is $R_{-\theta}$ — rotating by $-\theta$ undoes the effect of rotating by θ — note that this means $R_\theta R_{-\theta} = I_2 = R_{-\theta} R_\theta$.
- $R_\theta(\mathbf{v}) \cdot R_\theta(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ for any two 2×1 vectors $\mathbf{v}$ and $\mathbf{w}$. This perhaps is surprising: this equation says that R_θ is an *orthogonal* matrix (cf. Michaelmas Geometry I course). One consequence of this equation is that R_θ preserves distances and angles.

Definition 68 *An $n \times n$ matrix A is said to be orthogonal if $A^T A = I_n = AA^T$.*

3.4 2 Simultaneous Equations in 2 Variables

Example 69 *Suppose we are given two linear equations in two variables x and y. These might have the form*

$$2x + 3y = 1 \tag{3.3}$$

and

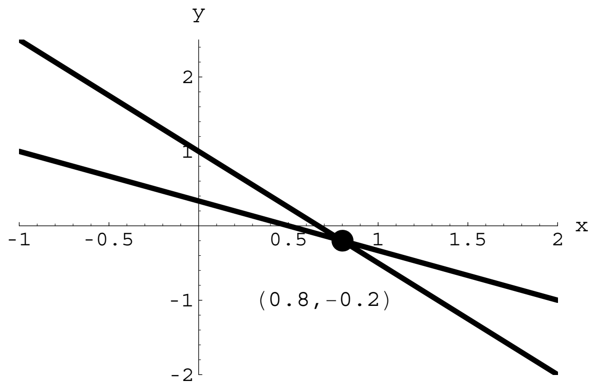
$$3x + 2y = 2. \tag{3.4}$$

Solution. To have solved these in the past, you might have argued along the lines:

$$\begin{aligned} \text{eliminate } x \text{ by using } 3 \times (3.3) - 2 \times (3.4) \ &: \ (6x + 9y) - (6x + 4y) = 3 - 4; \\ \text{simplify:} \ & \ 5y = -1; \\ \text{so} \ &: \ y = -0.2; \\ \text{substitute back in } (3.3) \ &: \ 2x - 0.6 = 1; \\ \text{solving} \ &: \ x = 0.8. \end{aligned}$$

So we see we have a unique solution: $x = 0.8, y = -0.2$.

You may even have seen how this situation could be solved graphically by drawing the lines with equations (3.3) and (3.4); the solution then is their unique intersection.



$2x + 3y = 1$ and $3x + 2y = 2$

■

We can though, put the two equations (3.3) and (3.4) into matrix form. We do this by writing

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The two simultaneous equations in the scalar variables x and y have now been replaced by a single equation in vector quantities — we have in this vector equation two 2×1 vectors (one on each side of the equation), and for the vector equation to be true *both* co-ordinates of the two vectors must agree.

We know that it is possible to *undo* this equation to obtain the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.8 \\ -0.2 \end{pmatrix}$$

because we have already solved this system of equations. In fact, there is a very natural way of unravelling these equations using matrices.

Definition 70 *Let A be a $n \times n$ matrix. We say that B is an inverse of A if*

$$BA = AB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_n.$$

- Recall that the matrix I_n is called the *identity* matrix — or more specifically it is the $n \times n$ identity matrix.
- The identity matrices have the property that

  $$AI_n = A = I_n A$$

  for any $n \times n$ matrix A.
- If an inverse B for A exists then it is unique. This is easy to show: suppose B and C were two inverses then note that

  $$C = I_n C = (BA)\,C = B\,(AC) = BI_n = B.$$
- We write A^{-1} for the inverse of A *if* an inverse exists.
- If $BA = I_n$ then, in fact, it will follow that $AB = I_n$. The converse is also true. We will not prove this here.

Proposition 71 *The matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has an inverse precisely when $ad - bc \neq 0$. If $ad - bc \neq 0$ then

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof. Note for any values of a, b, c, d, that

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad-bc)\, I_2.$$

So if $ad - bc \neq 0$ then we can divide by this scalar and we have found our inverse.

But if $ad - bc = 0$ then we have found a matrix

$$B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

such that

$$BA = 0_2$$

the 2×2 zero matrix. Now if an inverse C for A existed, we'd have that

$$0_2 = 0_2 C = (BA)\, C = B\,(AC) = BI_2 = B.$$

So each of a, b, c and d must be zero. So $A = 0_2$ which makes it impossible for AC to be I_2 — multiplying any C by the 0_2 will always lead to 0_2, not the identity. ■

Let's return now to a pair of simultaneous equations. Say they have the form

$$\begin{aligned} ax + by &= k_1, \\ cx + dy &= k_2. \end{aligned}$$

Then these can be rewritten as a single vector equation:

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

If the matrix A has an inverse A^{-1} then we can premultiply both sides by this and we find

$$\begin{pmatrix} x \\ y \end{pmatrix} = I_2\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

and we have found our unique solution.

What happens though if A doesn't have an inverse? In this case $ad = bc$, or equally that $a : c = b : d$ and we see that

$$ax + by \quad \text{and} \quad cx + dy$$

are multiples of one another.

- If the two equations are entirely multiples of one another, i.e.

$$a : c = b : d = k_1 : k_2$$

  then we essentially just have the one equation and there are infinitely many solutions.

- If the two equations aren't entirely multiples of one another, just the left hand sides i.e.

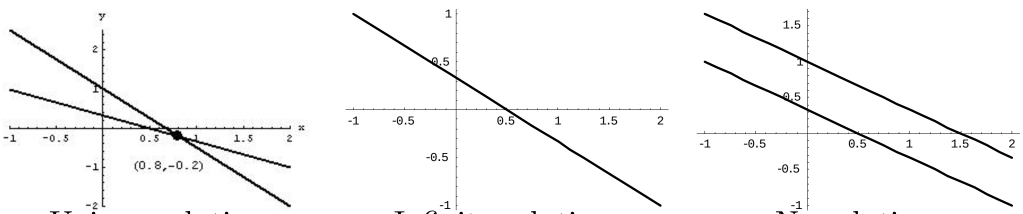
$$a : c = b : d \neq k_1 : k_2$$

  then we have two contradictory equations and there are no solutions.

Geometrically these three cases correspond to the two lines

$$ax + by = k_1 \quad \text{and} \quad cx + dy = k_2$$

being non-parallel and intersecting once, parallel and concurrent, or parallel and distinct as shown below.



Unique solution   Infinite solutions   No solutions

Solution. (Contd.) Returning to our previous linear system of equations

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

we see that the matrix on the left has inverse

$$\frac{1}{2 \times 2 - 3 \times 3} \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -0.4 & 0.6 \\ 0.6 & -0.4 \end{pmatrix}.$$

Hence the system's unique solution is

$$\begin{pmatrix} -0.4 & 0.6 \\ 0.6 & -0.4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -0.4 + 1.2 \\ 0.6 - 0.8 \end{pmatrix} = \begin{pmatrix} 0.8 \\ -0.2 \end{pmatrix},$$

as we previously calculated. ■

3.5 The General Case and EROs

In the general case of a linear system of m equations in n variables such as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1, \cdots\cdots\cdots, a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = c_m,$$

then this can be replaced by a single vector equation

$$A\mathbf{x} = \mathbf{c}$$

where $A = (a_{ij})$ is an $m \times n$ matrix, $\mathbf{x} = (x_1, \ldots, x_n)^T$ is an $n \times 1$ column vector and $\mathbf{c} = (c_1, \ldots, c_m)^T$ is an $m \times 1$ column vector. This system of equations can be reduced using *elementary row operations* (EROs) as described below. These are nothing more sophisticated than the type of manipulations we performed previously in our example of solving two simultaneous equations; we simply aim to be more systematic now in our approach to solving these systems.

Given these m equations in n variables, there are three types of ERO that we can perform on the system:

1. **Swapping two equations.**
2. **Multiplying an equation by a non-zero constant:** note that the constant has to be non-zero, as multiplying by 0 would effectively lose an equation.
3. **Adding a multiple of one equation to another.**

It is important to note that performing any of these EROs to the system preserves the system's set of solutions. The theorem that justifies the merit of this approach, which we shall not prove here, is the following:

Theorem 72 *Given a linear system of equations* $A\mathbf{x} = \mathbf{c}$ *then it is possible, by means of EROs, to reduce this system to an equivalent system*

$$A'\mathbf{x} = \mathbf{c}'$$

where A' *is in row-reduced echelon (RRE) form. To say that a matrix* M *is in RRE form means that*

- *all the zero rows of* M *appear at the bottom of the matrix;*
- *in any non-zero row, the first non-zero entry is a* 1*;*
- *such a leading* 1 *is the only non-zero entry in its column;*
- *the leading* 1 *of a row is to the left of leading* 1*s in successive rows.*

Once a system has been put into RRE form, a general solution can be obtained by assigning a parameter to each variable whose column has no leading 1 *and immediately reading off the values of the other variables in terms of these parameters.*

The application of this theorem is something that is very much better explain with examples and practice than as given above; the important point to note is that this is simply a systematic approach to solving linear systems based on the type of manipulation that we have already been doing.

Example 73 *Find the general solution of the following system of equations in variables x_1, x_2, x_3, x_4:*

$$\begin{aligned} x_1 + x_2 - x_3 + 3x_4 &= 2; \\ 2x_1 + x_2 - x_3 + 2x_4 &= 4; \\ 4x_1 + 3x_2 - 3x_3 + 8x_4 &= 8. \end{aligned}$$

Solution. We will first write this system as an augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 2 \\ 2 & 1 & -1 & 2 & 4 \\ 4 & 3 & -3 & 8 & 8 \end{array} \right),$$

which is simply a more compact presentation of the coefficients in the equations. We can use a type 3 ERO to add a multiple of one row to another. As the first row has a leading 1 this seems a good equation to subtract from the others: let's subtract it twice from the second row, and four times from the third row. This, strictly speaking is doing two EROs at once, but that's ok, and once you become au fait with EROs you'll get used to doing several in one step. The point to what we're doing is to 'clear out' the first column apart from that top 1. The system becomes

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 2 \\ 0 & -1 & 1 & -4 & 0 \\ 0 & -1 & 1 & -4 & 0 \end{array} \right).$$

Note now that the third and second rows are the same. If we subtract the second from the third we will gain a zero row. Why has this happened? Because there was some redundancy in the system; at least one of the equations was telling us something we could have deduced from the other two. Looking back at the original equations, we can spot that the third equation is twice the first one plus the second, and so wasn't necessary — the system really only contained two genuine constraints. The system has now become

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 2 \\ 0 & -1 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Finally it seems sensible to multiply the second row by -1 to get a leading 1. The only thing that remains is to clear out the second column apart from that leading 1; we can do this by taking the new second row from the first row.

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

What we have manipulated our system to is the pair of equations

$$x_1 - x_4 = 2 \quad \text{and} \quad x_2 - x_3 + 4x_4 = 0,$$

which look much more palatable than the original system. As the theorem says, to find the general solution we assign a parameter to each of the columns/variables without a leading 1; in this case these are the third and fourth columns. If we set $x_3 = \lambda$ and $x_4 = \mu$ then we can read off x_1 and x_2 as

$$x_1 = 2 + \mu \quad \text{and} \quad x_2 = \lambda - 4\mu.$$

So we see that

$$(x_1, x_2, x_3, x_4) = (2 + \mu, \lambda - 4\mu, \lambda, \mu).$$

As we let the parameters λ and μ vary over all possible real numbers then we obtain all the system's solutions. The above is the general solution of the equations. ■

Example 74 *Find all the solutions of the equations*

$$\begin{aligned} x_1 + x_2 - x_3 + 3x_4 &= 2; \\ 2x_1 + x_2 - x_3 + 2x_4 &= 4; \\ 4x_1 + 3x_2 - 3x_3 + 8x_4 &= 9. \end{aligned}$$

Solution. With this very similar set of equations, going through the first three EROs as above we would arrive at

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 2 \\ 2 & 1 & -1 & 2 & 4 \\ 4 & 3 & -3 & 8 & 9 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 2 \\ 0 & -1 & 1 & -4 & 0 \\ 0 & -1 & 1 & -4 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 2 \\ 0 & -1 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Whilst the system is still not in RRE form, we might as well stop now because of the third row of the system. This no represents the equation $0 = 1$. There are clearly no solutions to this, let alone any that also satisfy the other two equations. Hence this system is 'inconsistent', there are no solutions. **An inconsistent linear system of equations will always yield a row**

$$\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

when row-reduced. ■

EROs have a further useful application in the determining of invertibility and inverses. Given a linear system, represented by an augmented matrix

$$\left(\begin{array}{c|c} A & \mathbf{c} \end{array} \right),$$

then the effect of any ERO is the same as that of pre-multiplying by a certain *elementary matrix* to get

$$\left(\begin{array}{c|c} EA & E\mathbf{c} \end{array} \right).$$

For example, if we had four equations then premultiplying the system with the following matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

would have the effect of swapping the first two rows, adding the third row to the first, and multiplying the third row by 5, respectively. This process generalises naturally to n equations. There is no neat expression for the inverse of an $n \times n$ matrix in general, though the following method shows how to determine if an $n \times n$ matrix is invertible, and in such cases to find the inverse.

Algorithm 75 *Let A be an $n \times n$ matrix. Place this side by side with I_n in an augmented matrix*

$$\left(\begin{array}{c|c} A & I_n \end{array} \right).$$

Repeatedly perform EROs on this augmented matrix until A has been row-reduced. Let's say these EROs are equivalent to premultiplication by elementary matrices $E_1, E_2, \ldots, E_k$, so that the system has become

$$\left(\begin{array}{c|c} E_k E_{k-1} \cdots E_1 A & E_k E_{k-1} \cdots E_1 \end{array} \right).$$

If A is not invertible then there is some redundancy in the rows of A and the left-hand matrix will contain a zero row. Otherwise A will have reduced to I_n The matrix that has performed this is $E_k \cdots E_1$ and this is what we have on the right-hand side of our augmented matrix.

Example 76 *Calculate the inverse of the following matrix*

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}.$$

Solution. Quickly applying a sequence of EROs to the following augmented matrix we see

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & -2 & -5 & 1 & 3 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & -5 & 1 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2.5 & -0.5 & -1.5 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.5 & 0.5 & -0.5 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2.5 & -0.5 & -1.5 \end{array}\right).$$

Hence

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0.5 & 0.5 & -0.5 \\ -1 & 0 & 1 \\ 2.5 & -0.5 & -1.5 \end{pmatrix}.$$

■

3.6 Determinants

The determinant of a square $n \times n$ matrix is a number which reflects the way a matrix (or rather its associated map α) stretches space. We will define this only for 2×2 matrices.

Definition 77 *Given a* 2×2 *matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\det A = ad - bc.$$

We have already seen that A has an inverse precisely when $\det A \neq 0$. More generally $|\det A|$ is an area-scaling factor for α. So if R is a region of the plane, then

$$\text{area}\,(\alpha\,(R)) = |\det A| \times \text{area}\,(R)\,.$$

The sign of $\det A$ is also important (whether it is positive or negative). If $\det A$ is positive then α will preserve a sense of orientation (such as a rotation does) but if $\det A$ is negative then the sense of orientation will be reversed (such as a reflection does). If $\det A = 0$ then α collapses space: under α then the xy-plane will map to a line (or a single point in the case of $A = 0_2$). Viewed from this geometric point the following multiplicative property of determinants should not seem too surprising, if we recall that AB represents the composition of the maps represented by B then A.

Proposition 78 *Let A and B be two 2×2 matrices. Then*

$$\det(AB) = \det A \det B.$$

Proof. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then

$$\begin{aligned} \det(AB) &= \det \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \\ &= (ae+bg)(cf+dh) - (af+bh)(ce+dg) \\ &= bgcf + aedh - bhce - afdg \quad \text{[after cancelling]} \\ &= (ad-bc)(eh-fg) \\ &= \det A \det B. \end{aligned}$$

■

Example 79 *Recall that the matrix for R_θ, which denoted rotation by θ anti-clockwise about the origin was*

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Note that $\det R_\theta = +1$, unsurprisingly, as such a rotation preserves area and it preserves a sense of orientation.

Reflection in the line $y = x\tan\theta$ is given by the matrix

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Again we see that the determinant has modulus 1 (reflections are area-preserving) but that the determinant is negative as reflections reverse orientation.

3.7 Exercises

3.7.1 Algebra of Vectors

Exercise 126 *Given a subset* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$ *the span of* S *is defined to be*

$$\langle S \rangle = \{\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k : \alpha_i \in \mathbb{R}\}.$$

Let $S = \{(1,0,2), (0,3,1)\} \subseteq \mathbb{R}^3$.*Show that*

$$(x, y, z) \in \langle S \rangle \Leftrightarrow 6x + y - 3z = 0.$$

Let Π *denote the plane* $3x + y + 2z = 0$ *in* $\mathbb{R}^3$. *Find two vectors* $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ *such that*$\langle \{\mathbf{v}_1, \mathbf{v}_2\} \rangle = \Pi$.

Exercise 127 *A non-empty subset* $W \subseteq \mathbb{R}^n$ *is called a subspace if*

$$\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 \in W \text{ whenever } \alpha_1, \alpha_2 \in \mathbb{R} \text{ and } \mathbf{w}_1, \mathbf{w}_2 \in W.$$

Which of the following subsets of $\mathbb{R}^4$ *are subspaces? Justify your answers.*

(i) $W = \{\mathbf{x} \in \mathbb{R}^4 : x_1 + x_2 = x_3 + x_4\}$;
(ii) $W = \{\mathbf{x} \in \mathbb{R}^4 : x_1 = 1\}$;
(iii) $W = \{\mathbf{x} \in \mathbb{R}^4 : x_1 = x_3 = 0\}$;
(iv) $W = \left\{\mathbf{x} \in \mathbb{R}^4 : x_1 = (x_2)^2\right\}$.

Exercise 128 *For which of the following sets* S_i *is it true that* $\langle S_i \rangle = \mathbb{R}^4$*?*

$S_1 = \{(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1)\}$;
$S_2 = \{(1,1,1,1), (0,0,1,1), (1,1,0,0)\}$;
$S_3 = \{(0,1,2,3), (1,2,3,0), (3,0,1,2), (1,2,3,0)\}$;
$S_4 = \{(0,2,3,1), (3,1,0,2)\}$.

3.7.2 Geometry of Vectors

Exercise 129 *Verify properties (i)-(iv) of the scalar product given in Proposition 59.*

Exercise 130 *Find the lengths of the vectors*

$$\mathbf{u} = (1,0,1) \quad \text{and} \quad \mathbf{v} = (3,2,1) \in \mathbb{R}^3$$

and the angle between the two vectors. Find all those vectors in $\mathbb{R}^3$ *which are perpendicular to both* $\mathbf{u}$ *and* $\mathbf{v}$.

Exercise 131 *Show that the following three vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$ *are each of unit length and are mutually perpendicular:*

$$\mathbf{u} = \frac{1}{\sqrt{2}}(1,-1,0), \ \mathbf{v} = \frac{1}{\sqrt{3}}(1,1,1), \ \mathbf{w} = \frac{1}{\sqrt{6}}(1,1,-2).$$

Show that (x, y, z) *can be written as a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$, *i.e. as* $\lambda\mathbf{u} + \mu\mathbf{v}$, *if and only if* $x + y = 2z$.

Exercise 132 *A particle* P *has position vector*

$$\mathbf{r}(t) = (\cos t, \sin t, t) \in \mathbb{R}^3$$

which varies with time t. *Find the velocity vector* $d\mathbf{r}/dt$ *and acceleration vector* $d^2\mathbf{r}/dt^2$ *of the particle.*

The helix's unit tangent vector $\mathbf{t}$ *is the unit vector in the direction* $d\mathbf{r}/dt$. *Find* $\mathbf{t}$ *and show that* $\mathbf{t} \cdot \mathbf{k}$ *is constant. [This show that the particle's path is a helix.]*

Show that the particle is closest to the point $(-1, 0, 0)$ *at* $t = 0$.

Exercise 133 *The two vectors* $\mathbf{e}$ *and* $\mathbf{f}$ *in* $\mathbb{R}^2$ *are given by*

$$\mathbf{e} = (\cos\theta, \sin\theta) \quad \textit{and} \quad \mathbf{f} = (-\sin\theta, \cos\theta)$$

where θ *is a function of time* t. *Show that*

$$\dot{\mathbf{e}} = \dot{\theta}\mathbf{f} \quad \textit{and} \quad \dot{\mathbf{f}} = -\dot{\theta}\mathbf{e}.$$

Let $\mathbf{r} = r\mathbf{e}$. *Show that*

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e} + r\dot{\theta}\mathbf{f} \quad \textit{and} \quad \ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\mathbf{f}.$$

Exercise 134 *Deduce from the triangle inequality that*

$$\left|\sum_{k=1}^{n} \mathbf{v}_k\right| \leq \sum_{k=1}^{n} |\mathbf{v}_k|$$

for n *vectors* $\mathbf{v}_1, \ldots \mathbf{v}_n \in \mathbb{R}^2$. *By choosing appropriate* $\mathbf{v}_k$ *show that*

$$\sum_{k=1}^{n} \sqrt{k^2+1} \geq \frac{1}{2} n\sqrt{n^2+2n+5}.$$

3.7.3 Algebra of Matrices

Exercise 135 *Consider the following matrices, A, B, C, D. Where it makes sense to do so, calculate their sums and products.*

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 3 & 2 \\ 1 & 0 \end{pmatrix}, \\ C &= \begin{pmatrix} -1 & 2 & 3 \\ -2 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} -1 & 12 \\ 6 & 0 \end{pmatrix}. \end{aligned}$$

Exercise 136 *Let*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 3 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}. \tag{3.5}$$

Calculate the products AB, BA, CA, BC.

Exercise 137 *Given that*

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ -1 & 1 \\ -0 & 1 \end{pmatrix}, \tag{3.6}$$

verify the distributive law $A(B + C) = AB + AC$ for the three matrices.

Exercise 138 *Let*

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}. \tag{3.7}$$

Show that $AB = 0$, but that $BA \neq 0$.

Exercise 139 *Use the product formula in equation (3.2) to show that matrix multiplication is associative — that is, for an $k \times l$ matrix A, an $l \times m$ matrix B,and and $m \times n$ matrix C:*

$$A\,(BC) = (AB)\,C.$$

Exercise 140 *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & -1 & 0 \\ b & c & 1 \end{pmatrix}.$$

Find A^2. Under what condition on a, b and c is $A^2 = I$ (the identity matrix)? Given this condition holds find the inverse matrix of A. [Hint: Think first before calculating anything!]

Exercise 141 *A square matrix* $A = (a_{ij})$ *is said to be upper triangular if* $a_{ij} = 0$ *when* $i > j$. *Let* A, B *be* $n \times n$ *upper triangular matrices. Show that* $A + B, AB$ *and* A^{-1}, *if it exists, are all upper triangular.*

Exercise 142 *Let* A *and* B *be be* $k \times m$ *and* $m \times n$ *matrices respectively. Show that* $(AB)^T = B^T A^T$.

Exercise 143 *The trace of* $k \times k$ *matrix equals the sum of its diagonal elements. Let* A *and* B *be* 2×2 *matrices. Show that*

$$\textit{trace}\,(AB) = \textit{trace}\,(BA)\,.$$

This result applies generally when A *is an* $m \times n$ *matrix and* B *a* $n \times m$ *matrix.*

Exercise 144 *Let*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Find all those 2×2 *matrices* X *which commute with* A *— i.e. those satisfying* $XA = AX$.

Find all those 2×2 *matrices* Y *which commute with* B *— i.e. those satisfying* $YB = BY$.

Hence show that the only 2×2 *matrices, which commute with all other* 2×2 *matrices, are scalar multiples of the identity matrix. [This result holds generally for* $n \times n$ *matrices.]*

Exercise 145 *Given a non-zero complex number* $z = x + iy$, *we can associate with* z *a matrix*

$$Z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

Show that if z *and* w *are complex numbers with associated matrices* Z *and* W, *then the matrices associated with* $z + w, zw$ *and* $1/z$ *are* $Z + W, ZW$ *and* Z^{-1} *respectively. Hence, for each of the following matrix equations, find a matrix* Z *which is a solution.*

$$\begin{aligned} Z^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ Z^2 + 2Z &= \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}, \\ Z^2 + \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} Z &= \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}, \\ Z^5 &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

Exercise 146 *Let A denote the 2×2 matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Show that

$$A^2 - (traceA)A + (\det A)I = 0 \tag{3.8}$$

where $traceA = a+d$ is the trace of A, that is the sum of the diagonal elements, $detA = ad - bc$ is the determinant of A, and I is the 2×2 identity matrix.

Suppose now that $A^n = 0$ for some $n \geq 2$. Prove that $\det A = 0$. *Deduce using equation (3.8) that $A^2 = 0$.*

Exercise 147 *Let*

$$A = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}.$$

Show that

$$A^n = 3^{n-1} \begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix}$$

for $n = 1, 2, 3, \ldots$. Can you find a matrix B such that $B^2 = A$?

3.7.4 Simultaneous Equations. Inverses.

Exercise 148 *What are the possible numbers of solutions of three linear equations in two variables? What about of two linear equations in three variables?*

Exercise 149 *Find the solutions (if any) of the following systems of equations:*

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 4, \\ x_1 + 3x_2 + x_3 &= 11, \\ 2x_1 + 5x_2 - 4x_3 &= 13, \\ 2x_1 + 6x_2 + 2x_3 &= 22, \end{aligned}$$

and

$$\begin{aligned} 2x_1 + x_2 - 2x_3 + 3x_4 &= 1, \\ 3x_1 + 2x_2 - x_3 + 2x_4 &= 4, \\ 3x_1 + 3x_2 + 3x_3 - 3x_4 &= 5. \end{aligned}$$

Exercise 150 *Solve the linear system*

$$\begin{aligned} x_1 + 2x_2 - 3x_4 + x_5 &= 2, \\ x_1 + 2x_2 + x_3 - 3x_4 + x_5 + 2x_6 &= 3, \\ x_1 + 2x_2 - 3x_4 + 2x_5 + x_6 &= 4, \\ 3x_1 + 6x_2 + x_3 - 9x_4 + 4x_5 + 3x_6 &= 9, \end{aligned}$$

expressing the solutions in terms of parameters.

Exercise 151 *For what values of a do the simultaneous equations*

$$\begin{aligned} x + 2y + a^2 z &= 0, \\ x + ay + z &= 0, \\ x + ay + a^2 z &= 0, \end{aligned}$$

have a solution other than $x = y = z = 0$. For each such a find the general solution of the above equations.

Exercise 152 *Find all values of a for which the system*

$$\begin{aligned} x_1 + x_2 + x_3 &= a, \\ ax_1 + x_2 + 2x_3 &= 2, \\ x_1 + ax_2 + x_3 &= 4, \end{aligned}$$

has (a) a unique solution, (b) no solution, (c) infinitely many solutions.

Exercise 153 *For which values of a is the system*

$$\begin{aligned} x - y + z - t &= a^2, \\ 2x + y + 5z + 4t &= a, \\ x + 2z + t &= 2, \end{aligned}$$

consistent? For each such a, find the general solution of the system.

Exercise 154 *Consider the system of equations in x, y, z,*

$$\begin{array}{cccccccc} & x & & & + & z & = & 1 \\ & 2x & + & \alpha y & + & 4z & = & 1 \\ - & x & - & \alpha y & + & \alpha z & = & \beta \end{array}$$

where α and β are constants. Determine for which values of α and β the system has: (a) exactly one solution and (b) more than one solution.

Exercise 155 *Let m and n be natural numbers with $1 \leq m < n$. Prove that the linear system of m homogeneous equations in n variables*

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0, \cdots\cdots\cdots \quad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0,$$

has infinitely many solutions.

Exercise 156 *(See Exercise 126 for the definition of the span of a set.) Find the general solution of the equations*

$$x_1 + x_2 = x_3 + x_4 = x_5 + x_6.$$

Hence find four vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^6$ such that

$$\langle \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \rangle = \left\{ \mathbf{x} \in \mathbb{R}^6 : x_1 + x_2 = x_3 + x_4 = x_5 + x_6 \right\}.$$

Exercise 157 *(See Exercise 126 for the definition of the span of a set.) Let*

$$S = \{(1,2,3,0,0,1\}, (2,0,1,0,1,1), (1,0,3,0,2,1)\} \subseteq \mathbb{R}^6.$$

Find three homogeneous linear equations in $x_1, \ldots, x_6$ *the set of solutions of which is* $\langle S \rangle$.

Exercise 158 *Let* A *and* B *be invertible* $n \times n$ *matrices. Show that* $(AB)^{-1} = B^{-1}A^{-1}$. *Give an example to show that* $(A+B)^{-1} \neq A^{-1} + B^{-1}$ *in general.*

Exercise 159 *Determine whether the following matrices are invertible and find those inverses that do exist.*

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 5 & 4 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 3 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

Exercise 160 *Calculate* A^{-1} *when* A *is*

$$\begin{pmatrix} 1 & -3 & 0 \\ 1 & -2 & 4 \\ 2 & -5 & 5 \end{pmatrix}.$$

Express your answer as a product of elementary matrices.

Exercise 161 *Find two different right inverses for*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix};$$

i.e. two different 3×2 *matrices* M_1 *and* M_2 *such that* $AM_1 = I_2 = AM_2$.

Exercise 162 *Show that an* $n \times n$ *matrix is orthogonal if and only if its columns are of unit length and mutually orthogonal to one another.*

Exercise 163 *Show there are* 2^n *upper triangular orthogonal* $n \times n$ *matrices.*

Exercise 164 *Show that a linear map* $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ *preserves the scalar product i.e.*

$$A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y} \quad \text{for all} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

if and only if its matrix is orthogonal.

Exercise 165 *We say that two* $n \times n$ *matrices* A *and* B *are similar if there exists an invertible* $n \times n$ *matrix* P *such that* $A = P^{-1}BP$.

(i) Show that if A *and* B *are similar matrices then so are* A^2 *and* B^2 *(with the same choice of* P*).*

(ii) Show that if A *and* B *are similar matrices then* $\det A = \det B$ *and* $\text{trace}A = \text{trace}B$.

Show that if A *and* B *are similar matrices then so are* A^T *and* B^T *(with generally a different choice of* P*).*

Exercise 166 *Let*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Show that $P^{-1}AP$ is diagonal.

Use the identity $(P^{-1}AP)^n = P^{-1}A^nP$ to calculate a general formula for A^n where $n \in \mathbb{Z}$. Hence find a matrix E such that $E^2 = A$.

Exercise 167 *(a) Show that if A is inevitable then so is A^T and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.*

(b) Show that if A and B are symmetric then so are $A + B$ and AB and A^{-1} (provided A is invertible).

(c) Show that if A and B are orthogonal then so are A^{-1} and AB.

Exercise 168 *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the rows of an $n \times n$ matrix A. Show that A is invertible if and only if*

$$\langle \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \rangle = \mathbb{R}^n.$$

3.7.5 Matrices as Maps

Exercise 169 *What transformations of the xy-plane do the following matrices represent:*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Which, if any, of these transformations are invertible?

Exercise 170 *Find all the linear maps $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ which map the x-axis back onto itself. Find all those that satisfy $\alpha^2 = id$. What do these map represent geometrically?*

Exercise 171 *(i) Write down the 2×2 matrix A which represents reflection in the x-axis.*

(ii) Write down the 2×2 matrix B which represents reflection in the $y = x$ line.

Find a 2×2 invertible matrix P such that $A = P^{-1}BP$.

Exercise 172 *In Exercise 145 a matrix Z was sought such that*

$$Z^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By considering which map of $\mathbb{R}^2$ the matrix on the RHS represents, show that there are infinitely many matrices Z which satisfy this equation.

Exercise 173 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by*

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix. For what values of the coefficients a, b, c, d of A is f (a) injective? (b) surjective?

What are the corresponding answers $f : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by multiplying by a 3×2 matrix A, or if $f : \mathbb{R}^3 \to \mathbb{R}^2$ is defined using a 2×3 matrix A?

Exercise 174 *Show that the matrix*

$$A = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

is orthogonal. Find the position vector $(x, y)^T$ all points which are fixed by A. What map of the xy-plane does A represent geometrically?

Exercise 175 *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and let} \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

be a 2×2 matrix and its transpose. Suppose that $\det A = 1$ *and*

$$A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $a^2 + c^2 = 1$, and hence that a and c can be written as

$$a = \cos\theta \quad \text{and} \quad c = \sin\theta.$$

for some θ in the range $0 \le \theta < 2\pi$. Deduce that A has the form

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

3.7.6 Determinants

Exercise 176 *The determinant of a 3×3 matrix (a_{ij}) equals*

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Write down 3×3 matrices E_1, E_2, E_3, which represent the three types of elementary row operation that may be performed on a 3×3 matrix, and calculate their determinants.

Verify that $\det(E_i A) = \det E_i \det A$ *in each case.*

Exercise 177 *Prove that* $\det A^T = \det A$ *for* 3×3 *matrices.*

Exercise 178 *Let*

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}, \quad \textit{and} \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 3 \end{pmatrix}.$$

Calculate $\det A$ *and* $\det B$. *Further calculate* AB *and* BA *and verify*

$$\det AB = \det A \times \det B = \det BA.$$

Exercise 179 *Given two vectors* $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ *their cross product is given by*

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}.$$

Show that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$. *Show also that* $\mathbf{u} \times \mathbf{v}$ *is perpendicular to both* $\mathbf{u}$ *and* $\mathbf{v}$.

Exercise 180 *Show for any* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *that*

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v})\, \mathbf{w} - (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v}.$$

Exercise 181 *Do* 2×2 *matrices exist which exhibit the following properties? Either find such matrices or show that no such exist.*

(i) A *such that* $A^5 = I$ *and* $A^i \neq I$ *for* $1 \leq i \leq 4$*;*

(ii) A *such that* $A^n \neq I$ *for all positive integers* n*;*

(iii) A *and* B *such that* $AB \neq BA$*;*

(iv) A *and* B *such that* AB *is invertible and* BA *is singular (i.e. not invertible);*

(v) A *such that* $A^5 = I$ *and* $A^{11} = 0$.

4. DIFFERENTIAL EQUATIONS

4.1 Introduction

The study of ordinary differential equations (DEs) is as old as The Calculus itself and dates back to the time of Newton (1643-1727) and Leibniz (1646-1716). At that time most of the interest in DEs came from applications in physics and astronomy — one of Newton's greatest achievements, in his *Principia Mathematica* (1687), was to show that a force between a planet and the sun, which is inversely proportional to the square of the distance between them, would lead to an elliptical orbit. The study of differential equations grew as increasingly varied mathematical and physical situations led to differential equations, and as more and more sophisticated techniques were found to solve them — besides astronomy, DEs began appearing naturally in applied areas such as fluid dynamics, heat flow, waves on strings, and equally in pure mathematics, in determining the curve a chain between two points will make under its own weight, the shortest path between two points on a surface, the surface across a given boundary of smallest area (i.e. the shape of a soap film), the largest area a rope of fixed length can bound, etc. (We will be studying here *ordinary differential equations* (ODEs) rather than *partial differential equations* (PDEs). This means that the DEs in question will involve *full* derivatives, such as $\mathrm{d}y/\mathrm{d}x$, rather than *partial* derivatives, such as $\partial y/\partial x$. The latter notation is a measure of how a function y changes with x whilst all other variables (which y depends on) are kept constant.)

We give here, and solve, a simple example which involves some of the key ideas of DEs — the example here is the movement of a particle P under gravity, in one vertical dimension. Suppose that we write $h(t)$ for the height (in metres, say) of P over the ground; if the heights involved are small enough then we can reasonably assume that the gravity acting (denoted as g) is constant, and so

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = -g. \tag{4.1}$$

The *velocity* of the particle is the quantity $\mathrm{d}h/\mathrm{d}t$ — the rate of change of distance (here, height) with time. The rate of change of velocity with time is called *acceleration* and is the quantity $\mathrm{d}^2h/\mathrm{d}t^2$ on the LHS of the above equation. The acceleration here is entirely due to gravity. Note the need for a minus sign here as gravity is acting downwards.

Equation (4.1) is not a difficult DE to solve; we can integrate first once,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -gt + K_1 \tag{4.2}$$

and then again

$$h(t) = \frac{-1}{2}gt^2 + K_1 t + K_2 \tag{4.3}$$

where K_1 and K_2 are constants. Currently we don't know enough about the specific case of the particle P to be able to say anything more about these constants. Note though that whatever the values of K_1 and K_2 the graph of h against t is a parabola.

Remark 80 *Equation (4.1) is a second order DE. A derivative of the form $d^k y/dx^k$ is said to be of order k and we say that a DE has order k if it involves derivatives of order k and less. In some sense solving a DE of order k involves integrating k times, though not usually in such an obvious fashion as in the DE above. So we would expect the solution of an order k DE to have k undetermined constants in it, and this will be the case in most of the simple examples that we look at here. However this is not generally the case and we will see other examples where more, or fewer, than k constants are present in the solution.*

So the *general solution (4.3)* for $h(t)$ is not unique, but rather depends on two constants. And this isn't unreasonable as the particle P could follow many a path; at the moment we don't have enough information to characterise the path uniquely. One way of filling in the missing info would be to say how high P was at $t = 0$ and how fast it was going at that point. For example, suppose P started at a height of 100m and we threw it up into the air at a speed of 10ms^{-1} — that is

$$h(0) = 100 \quad \text{and} \quad \frac{\mathrm{d}h}{\mathrm{d}t}(0) = 10. \tag{4.4}$$

Then putting these values into equations (4.2) and (4.3) we'd get

$$10 = \frac{\mathrm{d}h}{\mathrm{d}t}(0) = -g \times 0 + K_1 \quad \text{giving } K_1 = 10,$$

and

$$100 = h(0) = \frac{-1}{2}g \times 0^2 + K_1 \times 0 + K_2 \quad \text{giving } K_2 = 100.$$

So the height of P at time t has been uniquely determined and is given by

$$h(t) = 100 + 10t - \frac{1}{2}gt^2.$$

The extra bits of information given in equation (4.4) are called *initial conditions* — particles like P can travel along infinitely many paths, but we need extra information to identify this path exactly.

Having solved the DE and found an equation for h then we could easily answer other questions about P's behaviour such as

- what is the greatest height P achieves? The maximum height will be a stationary value for $h(t)$ and so we need to solve the equation $h'(0) = 0$, which has solution $t = 10/g$. At this time the height is

$$h(10/g) = 100 + \frac{100}{g} - \frac{100g}{2g^2} = 100 + \frac{50}{g}.$$

- what time does P hit the ground? To solve this we see that

$$0 = h(t) = 100 + 10t - \frac{1}{2}gt^2$$

has solutions

$$t = \frac{-10 \pm \sqrt{100 + 200g}}{-g}.$$

One of these times is meaningless (being negative, and so before our experiment began) and so we take the other (positive) solution and see that P hits the ground at

$$t = \frac{10 + 10\sqrt{1 + 2g}}{g}.$$

We end this section with an example by way of warning of doing anything too cavalier with DEs.

Example 81 *Find the general solution of the DE*

$$\left(\frac{dy}{dx}\right)^2 = 4y. \tag{4.5}$$

Solution. Given this equation we might argue as follows — taking square roots we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{y}, \tag{4.6}$$

which we would recognise as a *separable* DE. This means we can separate the variables on to either side of the DE to get

$$\frac{\mathrm{d}y}{2\sqrt{y}} = \mathrm{d}x, \tag{4.7}$$

not worrying too much about what this formally means, and then we can integrate this to get

$$\sqrt{y} = \int \frac{\mathrm{d}y}{2\sqrt{y}} = \int \mathrm{d}x = x + K,$$

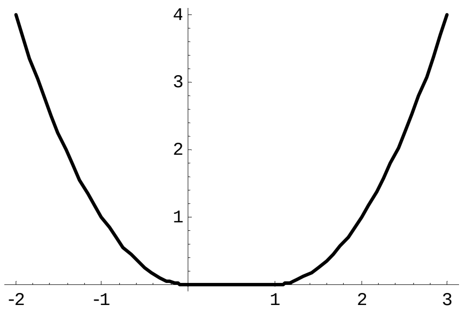
where K is a constant. Squaring this, we might think that the general solution has the form

$$y = (x + K)^2 .$$

What, if anything, could have gone wrong with this argument? We could have been more careful to include positive and negative square roots at the (4.6) stage, but actually we don't lose any solutions by this oversight. Thinking a little more, we might realise that we have missed the most obvious of solutions: the zero function, $y = 0$, which isn't present in our 'general' solution. At this point we might scold ourselves for committing the crime of dividing by zero at stage (4.7), rather than treating $y = 0$ as a separate case. But we have lost many more than just one solution at this point here by being careless. The general solution of (4.5) is in fact

$$y(x) = \begin{cases} (x-a)^2 & x \leq a \\ 0 & a \leq x \leq b \\ (x-b)^2 & b \leq x \end{cases}$$

where a and b are constants satisfying $-\infty \leq a \leq b \leq \infty$. We missed whole families of solutions by being careless — note also that the general solution requires *two* constants in its description even though the DE is only first order.



A missed solution with $a = 0$ and $b = 1$

■

One DE which we can approach rigorously is the following.

Example 82 *Show that the general solution of*

$$\frac{dy}{dx} = ky$$

is $y(x) = Ae^{kx}$ where A is a constant.

Solution. We make a substitution $z(x) = y(x)e^{-kx}$ first. From the product rule

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{-kx}\right) = e^{-kx}\left(\frac{\mathrm{d}y}{\mathrm{d}x} - ky\right) = 0.$$

The only functions with zero derivative are the constant functions and so $z(x) = A$ for some A and hence $y(x) = Ae^{kx}$. ■

4.2 Linear Differential Equations

A *homogeneous linear* differential equation is a DE with the following properties

- if y_1 and y_2 are solutions of the DE then so is $y_1 + y_2$;
- if y is a solution of the DE and c is a constant then cy is also a solution.

So, the following are examples of linear DEs

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + y &= 0, \\ x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x^2 - 1\right) y &= 0, \end{aligned}$$

whilst the following two are not

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = 1, \tag{4.8}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = 0. \tag{4.9}$$

Note that it is easy to check the first two DEs are linear even though solving the second equation is no routine matter. To see that the second two DEs aren't linear we could note that:

- 1 and $e^{-x} + 1$ are solutions of equation (4.8) though their sum $e^{-x} + 2$ is not a solution.
- x^{-1} is a solution of equation (4.9) though $2x^{-1}$ is not a solution.

A homogeneous linear DE, of order k, involving a function y of a single variable x has, as its most general form

$$f_k(x) \frac{\mathrm{d}^k y}{\mathrm{d}x^k} + f_{k-1}(x) \frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + f_1(x) \frac{\mathrm{d}y}{\mathrm{d}x} + f_0(x)\, y = 0. \tag{4.10}$$

The word 'homogeneous' here refers to the fact that there is a zero on the RHS of the above equation. An *inhomogeneous* linear DE of order k, involving a function y of a single variable x is one of the form

$$f_k(x) \frac{\mathrm{d}^k y}{\mathrm{d}x^k} + f_{k-1}(x) \frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + f_1(x) \frac{\mathrm{d}y}{\mathrm{d}x} + f_0(x)\, y = g(x). \tag{4.11}$$

Remark 83 *Solving the two equations are very closely linked because of the linear algebra behind their solution. The solutions of equation (4.10) form a real vector space, (cf. first term linear algebra course) the dimension of which is the number of independent constants present in the general solution of the equation. The solutions of equation (4.11) form a real affine space of the same dimension, and its vector space of translations is just the solution space of (4.10).*

What all this technical verbiage means is that the general solution of the inhomogeneous equation (4.11) is $Y(x) + y(x)$ *where* $Y(x)$ *is one particular solution of (4.11) and* $y(x)$ *is the general solution of the homogeneous equation (4.10). (From a geometrical point of view the solution spaces of (4.10) and (4.11) are parallel; we just need some point* $Y(x)$ *in the second solution space and from there we move around just as before.) The meaning of this remark will become clearer as we go through some examples.*

4.2.1 Homogeneous Equations with Constant Coefficients

We begin by considering homogeneous linear DEs where the functions $f_i(x)$ are all constant.

Theorem 84 *Consider the DE*

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

with auxiliary equation (AE)

$$m^2 + Am + B = 0.$$

The general solution to the DE is:

1. *in the case when the AE has two distinct real solutions* α *and* β*:*

$$Ae^{\alpha x} + Be^{\beta x};$$

2. *in the case when the AE has a repeated real solution* α*:*

$$(Ax + B)\,e^{\alpha x};$$

3. *in the case when the AE has a complex conjugate roots* $\alpha + i\beta$ *and* $\alpha - i\beta$*:*

$$e^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

Proof. Let's call the roots of the AE λ and μ, at the moment allowing for any of the above cases to hold. We can rewrite the original DE as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (\lambda + \mu)\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda\mu y = 0.$$

We will make the substitution

$$z(x) = y(x)\,e^{-\mu x},$$

noting that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(ze^{\mu x}) = \frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x} + \mu ze^{\mu x},$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\mu x} + 2\mu\frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x} + \mu^2 ze^{\mu x}.$$

Hence our original DE, as an new DE involving $z(x)$, has become

$$\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\mu x} + 2\mu\frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x} + \mu^2 ze^{\mu x}\right) - (\lambda+\mu)\left(\frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x} + \mu ze^{\mu x}\right) + \lambda\mu ze^{\mu x} = 0,$$

which simplifies to

$$\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\mu x} + 2\mu\frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x}\right) - (\lambda+\mu)\frac{\mathrm{d}z}{\mathrm{d}x}e^{\mu x} = 0.$$

Dividing through by $e^{\mu x}$ gives

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} + (\mu - \lambda)\frac{\mathrm{d}z}{\mathrm{d}x},$$

and we can simplify this further by substituting

$$w(x) = \frac{\mathrm{d}z}{\mathrm{d}x}$$

to get

$$\frac{\mathrm{d}w}{\mathrm{d}x} = (\lambda - \mu)\,w. \tag{4.12}$$

We now have two cases to consider: when $\lambda = \mu$ and when $\lambda \neq \mu$.

In the case when the roots are equal then (4.12) leads to the following line of argument

$$\begin{aligned} w(x) &= \frac{\mathrm{d}z}{\mathrm{d}x} = A \quad \text{(a constant)}, \\ z(x) &= Ax + B \quad (A \text{ and } B \text{ constants}), \\ y(x) &= z(x)\,e^{\mu x} = (Ax+B)\,e^{\mu x}, \end{aligned}$$

as we stated in case 2 of the theorem.

In the case when the roots are distinct (either real or complex) then (4.12) has solution from Example 82

$$w(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = c_1 e^{(\lambda-\mu)x}$$

(where c_1 is a constant) and so integrating gives

$$z(x) = \frac{c_1}{\lambda - \mu} e^{(\lambda-\mu)x} + c_2$$

(where c_2 is a second constant) to finally find

$$y(x) = z(x)\, e^{\mu x} = \frac{c_1}{\lambda - \mu} e^{\lambda x} + c_2 e^{\mu x}.$$

When λ and μ are real then this solution is in the required form for case 1 of the theorem. When $\lambda = \alpha + i\beta$ and $\mu = \alpha - i\beta$ are complex conjugates then this solution is in the correct form for case 3 of the theorem once we remember that

$$e^{(\alpha \pm i\beta)x} = e^{\alpha x} (\cos \beta x \pm i \sin \beta x).$$

■

Example 85 *Solve the equation*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0,$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0.$$

Solution. This has auxiliary equation

$$0 = m^2 - 3m + 2 = (m-1)(m-2)$$

which has roots $m = 1$ and $m = 2$. So the general solution of the equation is

$$y(x) = Ae^x + Be^{2x}.$$

Now the initial conditions imply

$$\begin{aligned} 1 &= y(0) = A + B, \\ 0 &= y'(0) = A + 2B. \end{aligned}$$

Hence

$$A = 2 \quad \text{and} \quad B = -1.$$

So the *unique* solution of this DE with initial solutions is

$$y(x) = 2e^x - e^{2x}.$$

■

The theory behind the solving of homogeneous linear DEs with constant coefficients extends to all orders, and not to second order DEs, provided suitable adjustments are made.

Example 86 *Write down the general solution of the following DE*

$$\frac{d^6y}{dx^6} + 2\frac{d^5y}{dx^5} + \frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 4y = 0$$

Solution. This has auxiliary equation

$$m^6 + 2m^5 + m^4 - 4m^3 - 4m^2 + 4 = 0.$$

With can see (with a little effort) that this factorises as

$$(m-1)^2 \left(m^2 + 2m + 2\right)^2 = 0$$

which has roots $1, -1+i$ and $-1-i$, all of which are repeated roots. So the general solution of the DE is

$$y(x) = (Ax + B)e^x + (Cx + D)e^{-x}\cos x + (Ex + F)e^{-x}\sin x.$$

■

4.2.2 Inhomogeneous Equations

The examples we discussed in the previous subsection were homogeneous — that is, they had the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + A(x)\frac{\mathrm{d}y}{\mathrm{d}x} + B(x)y = 0, \tag{4.13}$$

and we concentrated on examples where the functions $A(x)$ and $B(x)$ were constants. Here we shall look at *inhomogeneous* examples of second order linear DEs with constant coefficients: that is those of the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + A(x)\frac{\mathrm{d}y}{\mathrm{d}x} + B(x)y = f(x). \tag{4.14}$$

As commented in Remark (83) we have done much of the work already — what remains is just to find *a particular solution* of (4.14).

Proposition 87 *If $Y(x)$ is a solution of the inhomogeneous equation (4.14) and $y(x)$ is a solution of the homogeneous equation (4.13) then $Y(x) + y(x)$ is also a solution of the inhomogeneous equation. Indeed every solution of the inhomogeneous equation is of the form*

$$Y(x) + y(x)$$

as $y(x)$ varies over all the solutions of the homogeneous equation.

So we see that once we have solved (4.13) then solving (4.14) reduces to finding a single solution $y(x)$ of (4.14); such a solution is usually referred to as a *particular solution.* Finding these solutions is usually a matter of trial and error accompanied with educated guess-work — this usually involves looking for a particular solution that is roughly in the same form as the function $f(x)$. To explain here are some examples.

Example 88 *Find the general solution of*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x. \tag{4.15}$$

Solution. As the function on the right is $f(x) = x$ then it would seem sensible to *try* a function of the form

$$Y(x) = Ax + B,$$

where A and B are, as yet, undetermined constants. There is no presumption that such a solution exists, but this seems a sensible range of functions where we may well find a particular solution. Note that

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = A \quad \text{and} \quad \frac{\mathrm{d}^2Y}{\mathrm{d}x^2} = 0.$$

So *if* $Y(x)$ is a solution of (4.15) then substituting it in gives

$$0 - 3A + 2(Ax + B) = x$$

and this is an equation which must hold for all values of x. So comparing the coefficients of x on both sides, and the constant coefficients, gives

$$\begin{aligned} 2A &= 1 \text{ giving } A = \frac{1}{2}, \\ -3A + 2B &= 0 \text{ giving } B = \frac{3}{4}. \end{aligned}$$

What this means is that

$$Y(x) = \frac{x}{2} + \frac{3}{4}$$

is a particular solution of (4.15). Having already found the *complementary function* — that is the general solution of the corresponding homogeneous DE in Example (85) then the above proposition tells us that the general solution of (4.15) is

$$y(x) = Ae^x + Be^{2x} + \frac{x}{2} + \frac{3}{4},$$

for constants A and B. ■

Example 89 *Find particular solutions of the following DE*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = f(x)$$

where

- $f(x) = \sin x$ — *Simply trying* $Y(x) = A\sin x$ *would do no good as* $Y'(x)$ *would contain* $\cos x$ *terms whilst* $Y(x)$ *and* $Y''(x)$ *would contain* $\sin x$ *terms. Instead we need to try the more general* $Y(x) = A\sin x + B\cos x$;
- $f(x) = e^{3x}$ — *This causes few problems and, as we would expect, we can find a solution of the form* $Y(x) = Ae^{3x}$;
- $f(x) = e^x$ — *This is different to the previous case because we know* Ae^x *is part of the general solution to the corresponding homogeneous DE, and simply substituting in* $Y(x) = Ae^x$ *will yield* 0*. Instead we can successfully try a solution of the form* $Y(x) = Axe^x$.
- $f(x) = xe^{2x}$ — *Again* Ae^{2x} *is part of the solution to the homogeneous DE. Also from the previous example we can see that* Axe^{2x} *would only help us with a* e^{2x} *term on the RHS. So we need to 'move up' a further power and try a solution of the form* $Y(x) = (Ax^2 + Bx)e^{2x}$.
- $f(x) = e^x \sin x$ — *Though this may look somewhat more complicated a particular solution of the form* $Y(x) = e^x (A\sin x + B\cos x)$ *can be found.*
- $f(x) = \sin^2 x$ — *Making use of the identity* $\sin^2 x = (1 - \cos 2x)/2$ *we can see that a solution of the form* $Y(x) = A + B\sin 2x + C\cos 2x$ *will work.*

4.3 Integrating Factors

The method of integrating factors can be used with first order DEs of the form

$$P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = R(x). \tag{4.16}$$

The idea behind the method is to rewrite the LHS as the derivative of a product $A(x)y$. In general, the LHS of (4.16) isn't expressible as such, but if we multiply both sides of the DE by an appropriate *integrating factor* $I(x)$ then we can turn the LHS into the derivative of a product.

Let's first of all simplify the equation by dividing through by $P(x)$, and then multiplying by an integrating factor $I(x)$ (which we have yet to determine) to get

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x}+I(x)\frac{Q(x)}{P(x)}y=I(x)\frac{R(x)}{P(x)}. \tag{4.17}$$

We would like the LHS to be the derivative of a product $A(x)y$, which equals

$$A(x)\frac{\mathrm{d}y}{\mathrm{d}x}+A'(x)y. \tag{4.18}$$

So equating the coefficients of y and y' in (4.17) and (4.18), we have

$$A(x)=I(x)\quad\text{and } A'(x)=\frac{I(x)Q(x)}{P(x)}.$$

Rearranging this gives

$$\frac{I'(x)}{I(x)}=\frac{Q(x)}{P(x)}.$$

The LHS is the derivative of $\log I(x)$ and so we see

$$I(x)=\exp\int\frac{Q(x)}{P(x)}\,\mathrm{d}x.$$

For such an integrating factor $I(x)$ then (4.17) now reads as

$$\frac{\mathrm{d}}{\mathrm{d}x}(I(x)y)=\frac{I(x)R(x)}{P(x)}$$

which has the general solution

$$y(x)=\frac{1}{I(x)}\int\frac{I(x)R(x)}{P(x)}\,\mathrm{d}x.$$

Example 90 *Find the general solution of the DE*

$$x\frac{dy}{dx}+(x-1)y=x^2.$$

Solution. If we divide through by x we get

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1-\frac{1}{x}\right)y=x$$

and we see that the integrating factor is

$$\begin{aligned}I(x)&=\exp\int\left(1-\frac{1}{x}\right)\,\mathrm{d}x\\&=\exp(x-\log x)\\&=\frac{1}{x}e^x.\end{aligned}$$

Multiplying through by the integrating factor gives

$$\frac{1}{x}e^{x}\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{1}{x}-\frac{1}{x^{2}}\right)e^{x}y=e^{x},$$

which, by construction rearranges to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}e^{x}y\right)=e^{x}.$$

Integrating gives

$$\frac{1}{x}e^{x}y=e^{x}+K$$

where K is a constant, and rearranging gives

$$y\left(x\right)=x+Kxe^{-x}$$

as our general solution. ■

Example 91 *Solve the initial value problem*

$$\frac{dy}{dx}+2xy=1,\quad y\left(0\right)=0.$$

Solution. The integrating factor here is

$$I\left(x\right)=\exp\int 2x\ \mathrm{d}x=\exp\left(x^{2}\right).$$

Multiplying through we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^{2}}y\right)=e^{x^{2}}\frac{\mathrm{d}y}{\mathrm{d}x}+2xe^{x^{2}}y=e^{x^{2}}.$$

Noting that $y\left(0\right)=0$, when we integrate this we arrive at

$$e^{x^{2}}y=\int_{0}^{x}e^{t^{2}}\ \mathrm{d}t,$$

(this can't be expressed in a closed form involving elementary equations) and rearranging gives

$$y\left(x\right)=e^{-x^{2}}\int_{0}^{x}e^{t^{2}}\ \mathrm{d}t.$$

■

4.4 Homogeneous Polar Equations

By a homogeneous polar differential equation we will mean one of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right). \tag{4.19}$$

These can be solved with a substitution of the form

$$y(x) = v(x)\,x \tag{4.20}$$

to get a new equation in terms of v, x and $\mathrm{d}v/\mathrm{d}x$. Note that the product rule of differentiation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

and so making the substitution (4.20) into the DE (4.19) gives us the new DE

$$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = f(v),$$

which is a *separable* DE.

Example 92 *Find the general solution of the DE*

$$\frac{dy}{dx} = \frac{x-y}{x+y}.$$

Solution. At first glance this may not look like a homogeneous polar DE, but dividing the numerator and denominator in the RHS will quickly dissuade us of this. If we make the substitution $y(x) = xv(x)$ then we have

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x - vx}{x + vx} = \frac{1-v}{1+v}.$$

Rearranging this gives

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1-v}{1+v} - v = \frac{1-2v+v^2}{1+v}$$

and so, separating the variables, we find

$$\int \frac{1+v}{(1-v)^2}\,\mathrm{d}v = \int \frac{\mathrm{d}x}{x}.$$

Using partial fractions gives

$$-\log|1-v| + \frac{2}{1-v} = \int \left(\frac{-1}{1-v} + \frac{2}{(1-v)^2}\right)\mathrm{d}v = \log x + \text{const}.$$

and resubstituting $v = y/x$ leads us to the general solution

$$-\log\left|1 - \frac{y}{x}\right| + \frac{2x}{x-y} = \log x + \text{const}.$$

■

Example 93 *Solve the initial value problem*

$$\frac{dy}{dx} = \frac{y}{x+y+2} \quad y(0) = 1. \tag{4.21}$$

Solution. This DE is not homogeneous polar, but it can easily be made into such a DE with a suitable change of variables. We introduce new variables

$$X = x + a \quad \text{and} \quad Y = y + b.$$

If we make these substitutions then the RHS becomes

$$\frac{Y - b}{X + Y + 2 - a - b}$$

which is homogeneous if $b = 0$ and $a = 2$. With these values of a and b, noting that

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}(y)}{\mathrm{d}(x+2)} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

and that the initial condition has become $Y(X = 2) = Y(x = 0) = y(x = 0) = 1$, our initial value problem now reads as

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{Y}{X+Y}, \quad Y(2) = 1.$$

Substituting in $Y = VX$ gives us

$$V + X\frac{\mathrm{d}V}{\mathrm{d}X} = \frac{VX}{X + VX} = \frac{V}{1+V}, \quad V(2) = \frac{1}{2}.$$

Rearranging the equation gives us

$$X\frac{\mathrm{d}V}{\mathrm{d}X} = \frac{V}{1+V} - V = \frac{-V^2}{1+V},$$

and separating variables gives

$$\frac{1}{V} - \log V = \int \left(-\frac{1}{V^2} - \frac{1}{V}\right) \mathrm{d}V = \int \frac{\mathrm{d}X}{X} = \log X + K.$$

Substituting in our initial condition we see

$$2 - \log\left(\frac{1}{2}\right) = \log 2 + K \quad \text{and hence } K = 2.$$

So

$$\frac{1}{V} - \log V = \log X + 2,$$

becomes, when we remember $V = Y/X$,

$$\frac{X}{Y} - \log\left(\frac{Y}{X}\right) = \log X + 2,$$

which simplifies to

$$X - Y\log Y = 2Y.$$

Further, as $X = x + 2$ and $Y = y$, our solution to the initial value problem (4.21) has become

$$x + 2 = 2y + y\log y.$$

■

4.5 Exercises

Exercise 182 *Find the general solutions of the following separable differential equations.*

$$\frac{dy}{dx} = \frac{x^2}{y}, \qquad \frac{dy}{dx} = \frac{\cos^2 x}{\cos^2 2y}, \qquad \frac{dy}{dx} = e^{x+2y}.$$

Exercise 183 *Find all solutions of the following separable differential equations:*

$$\begin{aligned} \frac{dy}{dx} &= \frac{y - xy}{xy - x}, \\ \frac{dy}{dx} &= \frac{\sin^{-1} x}{y^2\sqrt{1-x^2}}, \qquad y(0) = 0. \\ \frac{d^2y}{dx^2} &= \left(1 + 3x^2\right)\left(\frac{dy}{dx}\right)^2 \quad \textit{where } y(1) = 0 \quad \textit{and} \quad y'(1) = \frac{-1}{2}. \end{aligned}$$

Exercise 184 *Find all the solutions (if any) of the following boundary-value problems*

$$\begin{aligned} \frac{d^2y}{dx^2} &= y, \quad y(0) = 1, y(\pi) = -1; \\ \frac{d^2y}{dx^2} &= -y, \quad y(0) = 1, y(\pi) = -1; \\ \frac{d^2y}{dx^2} &= -y, \quad y(0) = 1, y(\pi) = 1. \end{aligned}$$

Exercise 185 *By means of a substitution transform the following into a separable equation and find its general solution:*

$$\frac{dy}{dx} = \cos(x + y).$$

Exercise 186 *Solve the initial value problem*

$$\frac{dy}{dx} = 1 - |y|, \quad \textit{for } x > 0$$

when (i) $y(0) = 2$, *(ii)* $y(0) = 1/2$, *(iii)* $y(0) = -2$.

Exercise 187 *Find the solution of the following initial value problems. On separate axes sketch the solution to each problem.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{1-2x}{y}, \qquad y(1) = -2,\\ \frac{dy}{dx} &= \frac{x(x^2+1)}{4y^3}, \qquad y(0) = \frac{-1}{\sqrt{2}},\\ \frac{dy}{dx} &= \frac{1+y^2}{1+x^2} \quad \textit{where } y(0) = 1.\end{aligned}$$

Exercise 188 *The equation for Simple Harmonic Motion, with constant frequency* ω*, is*

$$\frac{d^2x}{dt^2} = -\omega^2 x.$$

Show that

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx}$$

where $v = dx/dt$ *denotes velocity. Find and solve a separable differential equation in* v *and* x *given that* $x = a$ *when* $v = 0$*. Hence show that*

$$x(t) = a\sin(\omega t + \varepsilon)$$

for some constant ε*.*

Exercise 189 *Show that the solution of the differential equation*

$$\frac{d^2x}{dt^2} + \omega^2 x = \cos\Omega t$$

are bounded when $\Omega \neq \omega$*, but become unbounded when* $\Omega = \omega$*.*

Exercise 190 *Find the most general solution of the following homogeneous constant coefficient differential equations:*

$$\begin{aligned}\frac{d^2y}{dx^2} - y &= 0,\\ \frac{d^2y}{dx^2} + 4y &= 0, \quad \textit{where } y(0) = y'(0) = 1,\\ \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y &= 0,\\ \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y &= 0, \quad \textit{where } y(0) = y'(0) = 1.\end{aligned}$$

Exercise 191 *Find the most general solution of the following higher order homogeneous constant coefficient differential equations:*

$$\begin{aligned}\frac{d^4y}{dx^4}-y &= 0,\\ \frac{d^3y}{dx^3}-y &= 0,\\ \frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\frac{dy}{dx}+y &= 0,\\ \frac{d^4y}{dx^4}+\frac{d^2y}{dx^2} &= 0.\end{aligned}$$

Exercise 192 *By means of the substitution $z=\ln x$ find the general solution of the differential equation*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$

Exercise 193 *Write the left hand side of the differential equation*

$$(2x+y)+(x+2y)\frac{dy}{dx}=0,$$

in the form

$$\frac{d}{dx}(F(x,y))=0,$$

where $F(x,y)$ is a polynomial in x and y. Hence find the general solution of the equation.

Exercise 194 *Use the method of the previous exercise to solve*

$$(y\cos x+2xe^y)+\left(\sin x+x^2e^y-1\right)\frac{dy}{dx}=0.$$

Exercise 195 *Use the method of integrating factors to solve the following equations with initial conditions*

$$\begin{aligned}\frac{dy}{dx}+xy &= x \quad \textit{where} \quad y(0)=0,\\ 2x^3\frac{dy}{dx}-3x^2y &= 1 \quad \textit{where} \quad y(1)=0,\\ \frac{dy}{dx}-y\tan x &= 1 \quad \textit{where} \quad y(0)=1.\end{aligned}$$

Exercise 196 *Find the most general solution of the following inhomogeneous constant coefficient differential equations:*

$$\begin{aligned}
\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y &= x, \\
\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y &= \sin x, \\
\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y &= e^x, \\
\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y &= e^{-x}.
\end{aligned}$$

Exercise 197 *Write down a family of trial functions $y(x)$ which will contain a particular solution of*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = f(x),$$

for each of the following different choices of $f(x)$:

- $f(x) = x^2$
- $f(x) = xe^x$
- $f(x) = xe^{-x}$
- $f(x) = x^2 \sin x$
- $f(x) = \sin^3 x.$

Exercise 198 *By making the substitution $y(x) = xv(x)$ in the following homogeneous polar equations, convert them into separable differential equations involving v and x, which you should then solve*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x^2 + y^2}{xy}, \\
x\frac{dy}{dx} &= y + \sqrt{x^2 + y^2}.
\end{aligned}$$

Exercise 199 *Make substitutions of the form $x = X + a$, $y = Y + b$, to turn the differential equation*

$$\frac{dy}{dx} = \frac{x + y - 3}{x - y - 1}$$

into a homogeneous polar differential equation in X and Y. Hence find the general solution of the above equation.

Exercise 200 *Show that the differential equation*

$$\frac{dy}{dx} = \frac{x+y-1}{2x+2y-1}$$

cannot be transformed into a homogeneous polar differential equation by means of substitutions $x = X + a$, $y = Y + b$. *By means of the substitution* $z = x + y$ *find the general solution of the equation.*

Exercise 201 *A particle* P *moves in the* xy*-plane. Its co-ordinates* $x(t)$ *and* $y(t)$ *satisfy the equations*

$$\frac{dy}{dt} = x + y \quad \text{and} \quad \frac{dx}{dt} = x - y,$$

and at time $t = 0$ *the particle is at* $(1, 0)$. *Find, and solve, a homogeneous polar equation relating* x *and* y.

By changing to polar co-ordinates $(r^2 = x^2 + y^2, \ \tan\theta = y/x)$, *sketch the particle's journey for* $t \geq 0$.

Exercise 202 *Show that the function* $y(x)$ *given by the power series*

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

satisfies the initial value problem

$$\frac{dy}{dx} = y, \quad y(0) = 1,$$

if $a_0 = 1$ *and* $na_n = a_{n-1}$ *for* $n \geqslant 1$. *Determine* a_n *for each* n *and hence find* $y(x)$.

Exercise 203 *Use the power series approach of the previous exercise to solve the following initial value problem:*

$$\frac{d^2y}{dx^2} + y = 0, \quad y(0) = 1, y'(0) = 0.$$

Exercise 204 *Suppose that* $y = u(x)$ *is a solution of the differential equation*

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0.$$

Show that the substitution $y = u(x)\,v(x)$ *reduces the above equation into a first order differential equation in* dv/dx.

Exercise 205 *Show that* $u(x) = x$ *is a solution of*

$$x\frac{d^2y}{dx^2} - (x+1)\frac{dy}{dx} + 2y = 0.$$

Use the substitution $v(x) = y/x^2$ *to find the equation's general solution.*

5. TECHNIQUES OF INTEGRATION

Remark 94 *We will demonstrate each of the techniques here by way of examples, but concentrating each time on what general aspects are present. Integration, though, is not something that should be learnt as a table of formulae, for at least two reasons: one is that most of the formula would be far from memorable, and the second is that each technique is more flexible and general than any memorised formula ever could be. If you can approach an integral with a range of techniques at hand you will find the subject less confusing and not be fazed by new and different functions.*

Remark 95 *When it comes to checking your answer there are various quick rules you can apply. If you have been asked to calculate an indefinite integral then, if it's not too complicated, you can always differentiate your answer to see if you get the original integrand back. With a definite integral it is also possible to apply some simple estimation rules: if your integrand is positive (or negative) then so should your answer be; if your integrand is less than a well-known function, then its integral will be less than the integral of the well-known function. These can be useful checks to quickly apply at the end of the calculation.*

5.1 Integration by Parts

Integration by parts (IBP) can be used to tackle products of functions, but not just any product. Suppose we have an integral

$$\int f(x)\, g(x)\ \mathrm{d}x$$

in mind. This will be approachable with IBP if one of these functions integrates, or differentiates, perhaps repeatedly, to something simpler, whilst the other function differentiates and integrates to something of the same kind. Typically then $f(x)$ might be a polynomial which, after differentiating enough times, will become a constant; $g(x)$ on the other hand could be something like e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, all of which are functions which continually integrate to something similar. This remark reflects the nature of the formula for IBP which is:

Proposition 96 *(Integration by Parts) Let F and G be functions with derivatives f and g. Then*

$$\int F(x)\, g(x)\ dx = F(x)\, G(x) - \int f(x)\, G(x)\ dx.$$

IBP takes the integral of a product and leaves us with another integral of a product — but as we commented above, the point is that $f(x)$ should be a simpler function than $F(x)$ was whilst $G(x)$ should be no worse a function than $g(x)$ was.

Proof. The proof is simple — we just integrate the product rule of differentiation below, and rearrange.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F(x)\,G(x)\right) = F(x)\,g(x) + f(x)\,G(x)$$

■

Example 97 *Determine*

$$\int x^2 \sin x \; dx \quad \textit{and} \quad \int_0^1 x^3 e^{2x} \; dx.$$

Solution. Clearly x^2 will be the function that we need to differentiate down, and $\sin x$ is the function that will integrate *in house.* So we have, with *two* applications of IBP:

$$\begin{aligned}
\int x^2 \sin x \; \mathrm{d}x &= x^2(-\cos x) - \int 2x(-\cos x) \; \mathrm{d}x \quad \text{[IBP]} \\
&= -x^2 \cos x + \int 2x \cos x \; \mathrm{d}x \quad \text{[Rearranging]} \\
&= -x^2 \cos x + 2x \sin x - \int 2 \sin x \; \mathrm{d}x \quad \text{[IBP]} \\
&= -x^2 \cos x + 2x \sin x - 2(-\cos x) + \text{const.} \\
&= \left(2 - x^2\right) \cos x + 2x \sin x + \text{const.} \quad \text{[Rearranging]}
\end{aligned}$$

■

Solution. In a similar fashion

$$\begin{aligned}
\int_0^1 x^3 e^{2x} \; \mathrm{d}x &= \left[x^3 \frac{e^{2x}}{2}\right]_0^1 - \int_0^1 3x^2 \frac{e^{2x}}{2} \; \mathrm{d}x \quad \text{[IBP]} \\
&= \frac{e^2}{2} - \left(\left[3x^2 \frac{e^{2x}}{4}\right]_0^1 - \int_0^1 6x \frac{e^{2x}}{4} \; \mathrm{d}x\right) \quad \text{[IBP]} \\
&= \frac{e^2}{2} - \frac{3e^2}{4} + \left[6x \frac{e^{2x}}{8}\right]_0^1 - \int_0^1 6 \frac{e^{2x}}{8} \; \mathrm{d}x \quad \text{[IBP]} \\
&= \frac{-e^2}{4} + \frac{3e^2}{4} - \left[\frac{6e^{2x}}{16}\right]_0^1 \\
&= \frac{e^2}{8} + \frac{3}{8}.
\end{aligned}$$

■

This is by far the main use of IBP, the idea of eventually differentiating out one of the two functions. There are other important uses of IBP which don't quite fit into this type. These next two examples fall into the original class, but are a little unusual : in these cases we choose to integrate the polynomial factor instead as it is easier to differentiate the other factor. This is the case when we have a logarithm or an inverse trigonometric function as the second factor.

Example 98 *Evaluate*

$$\int (2x-1)\ln\left(x^2+1\right)\ dx \quad \textit{and} \quad \int \left(x^2-4\right)\tan^{-1}x\ dx.$$

Solution. In both cases integrating the second factor looks rather daunting, certainly to integrate, but each factor differentiates nicely; recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x=\frac{1}{x} \quad \text{and that} \quad \frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x=\frac{1}{1+x^2}.$$

So if we apply IBP to the above examples then we get

$$\int (2x-1)\ln\left(x^2+1\right)\ \mathrm{d}x=\left(x^2-x\right)\ln\left(x^2+1\right)-\int\left(x^2-x\right)\frac{2x}{x^2+1}\ \mathrm{d}x,$$

and

$$\int\left(3x^2-4\right)\tan^{-1}x\ \mathrm{d}x=\left(x^3-4x\right)\tan^{-1}x-\int\left(x^3-4x\right)\frac{1}{x^2+1}\ \mathrm{d}x.$$

Here we will stop for the moment — we will see how to determine these integrals, the integrands of which are known as rational functions, in section 5.3.

■

In the same vein as this we can use IBP to integrate functions which, at first glance, don't seem to be a product — this is done by treating a function $F(x)$ as the product $F(x)\times 1$.

Example 99 *Evaluate*

$$\int \ln x\ dx \ \textit{and} \int \tan^{-1}x\ dx.$$

Solution. With IBP we see (integrating the 1 and differentiating the $\ln x$)

$$\begin{aligned}\int \ln x\ \mathrm{d}x &= \int 1\times\ln x\ \mathrm{d}x\\ &= x\ln x-\int x\frac{1}{x}\ \mathrm{d}x\\ &= x\ln x-\int \mathrm{d}x\\ &= x\ln x-x+\text{const.}\end{aligned}$$

and similarly

$$\begin{aligned}\int \tan^{-1} x \; \mathrm{d}x &= \int 1 \times \tan^{-1} x \; \mathrm{d}x \\ &= x \tan^{-1} x - \int x \frac{1}{1+x^2} \; \mathrm{d}x \\ &= x \tan^{-1} x - \frac{1}{2} \ln\left(1+x^2\right) + \text{const.}\end{aligned}$$

spotting this by *inspection* or by using *substitution* (see the next section). ■

Sometimes both functions remain *in house,* but we eventually return to our original integrand.

Example 100 *Determine*

$$\int e^x \sin x \; dx.$$

Solution. Both of these functions now remain in house, but if we apply IBP twice, integrating the e^x and differentiating the $\sin x$, then we see

$$\begin{aligned}\int e^x \sin x \; \mathrm{d}x &= e^x \sin x - \int e^x \cos x \; \mathrm{d}x \quad \text{[IBP]} \\ &= e^x \sin x - \left(e^x \cos x - \int e^x \left(-\sin x\right) \; \mathrm{d}x\right) \\ &= e^x \left(\sin x - \cos x\right) - \int e^x \sin x \; \mathrm{d}x.\end{aligned}$$

We see that we have returned to our original integral, and so we can rearrange this equality to get

$$\int e^x \sin x \; \mathrm{d}x = \frac{1}{2} e^x \left(\sin x - \cos x\right) + \text{const.}$$

■

5.2 Substitution

In many ways the hardest aspect of integration to teach, a technique that can become almost an art form, is substitution. Substitution is such a varied and flexible approach that it is impossible to classify (and hence limit) its uses, and quite difficult even to find general themes within. We shall discuss later some standard trigonometric substitutions useful in integrating rational functions. For now we will simply state what substitution involves and highlight one difficulty than can occur (and cause errors) unless substitution is done carefully.

Proposition 101 *Let* $g:[c,d]\to[a,b]$ *be an increasing function, such that* $g(c)=a$ *and* $g(d)=b$*, and which has derivative* g'*. Then*

$$\int_a^b f(x)\ dx=\int_c^d f(g(t))\,g'(t)\ dt.$$

Similarly if $g:[c,d]\to[a,b]$ *is a decreasing function, such that* $g(c)=b$ *and* $g(d)=a$*, then*

$$\int_a^b f(x)\ dx=\int_d^c f(g(t))\,g'(t)\ dt.$$

The important point here is that the function g be increasing or decreasing so that it is a *bijection* from $[c,d]$ to $[a,b]$ — what this technical term simply means is that to each value of x in the range $[a,b]$ there should be *exactly one* value of t in the range $[c,d]$ such that $g(t)=x$, and as we vary x over $[a,b]$ each $t\in[c,d]$ appears in this way. Here is an example of what might go wrong if substitution is incorrectly applied.

Example 102 *Evaluate*

$$\int_{-1}^2 x^2\ dx.$$

Solution. This is not a difficult integral and we would typically not think of using substitution to do this; we would just proceed and find

$$\int_{-1}^2 x^2\ \mathrm{d}x=\left[\frac{x^3}{3}\right]_{-1}^2=\frac{1}{3}\left(2^3-(-1)^3\right)=\frac{9}{3}=3.$$

But suppose that we'd tried to use (in a less than rigorous fashion) the substitution $u=x^2$ here. We'd see that

$$\begin{aligned}\mathrm{d}u&=2x\,\mathrm{d}x=2\sqrt{u}\,\mathrm{d}x\ \text{ so that } \mathrm{d}x=\frac{\mathrm{d}u}{2\sqrt{u}}\\ \text{and when } x&=-1, u=1 \text{ and when } x=2,\ u=4.\end{aligned}$$

So surely we'd find

$$\int_{-1}^2 x^2\ \mathrm{d}x=\int_1^4 u\frac{\mathrm{d}u}{2\sqrt{u}}=\frac{1}{2}\int_1^4\sqrt{u}\ \mathrm{d}u=\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^4=\frac{1}{3}(8-1)=\frac{7}{3}.$$

What's gone wrong is that the assignment $u=x^2$ doesn't provide a bijection between $[-1,2]$ and $[1,4]$ as the values in $[-1,0]$ square to the same values as those in $[0,1]$. The missing $2/3$ error in the answer is in fact the integral $\int_{-1}^1 x^2\ \mathrm{d}x$. If we'd particularly wished to use this substitution then it could have been correctly made by splitting our integral as

$$\int_{-1}^2 x^2\ \mathrm{d}x=\int_{-1}^0 x^2\ \mathrm{d}x+\int_0^2 x^2\ \mathrm{d}x$$

and using the substitution $u = x^2$ separately on each integral; this would work because $u = x^2$ gives a bijection between $[-1, 0]$ and $[0, 1]$, and between $[0, 2]$ and $[0, 4]$. ■

Here are some examples where substitution can be applied, provided some care is taken.

Example 103 *Evaluate the following integrals*

$$\int_0^1 \frac{1}{1+e^x}\,dx, \qquad \int_{-\pi/2}^{\pi} \frac{\sin x}{1+\cos x}\,dx.$$

Solution. In the first integral a substitution that might suggest itself is $u = 1 + e^x$ or $u = e^x$; let's try the first of these $u = 1 + e^x$. As x varies from $x = 0$ to $x = 1$ then u varies from $u = 2$ to $u = 1 + e$. Moreover u is increasing with x so that the rule $u = 1 + e^x$ is a bijection from the x-values in $[0, 1]$ to the u-values in the range $[2, 1 + e]$. We also have that

$$\mathrm{d}u = e^x \mathrm{d}x = (u-1)\,\mathrm{d}x.$$

So

$$\begin{aligned}
\int_0^1 \frac{1}{1+e^x}\,\mathrm{d}x &= \int_2^{1+e} \frac{1}{u}\frac{\mathrm{d}u}{u-1} \quad \text{[substitution]} \\
&= \int_2^{1+e} \left(\frac{1}{u-1} - \frac{1}{u}\right) \mathrm{d}u \text{ [using partial fractions]} \\
&= [\ln|u-1| - \ln|u|]_2^{1+e} \\
&= \ln(e) - \ln(1+e) - \ln 1 + \ln 2 \\
&= 1 + \ln\left(\frac{2}{1+e}\right).
\end{aligned}$$

For the second integral, it would seem sensible to use $u = 2 + \cos x$ or $u = \cos x$ here. Let's try the second one: $u = \cos x$. Firstly note that u is not a bijection on the range $[-\pi/2, \pi]$, it takes the same values in the range $[-\pi/2, 0]$ as it does in the range $[0, \pi/2]$. In fact the integrand is *odd* (that is $f(-x) = -f(x)$) and so its integral between $x = -\pi/2$ and $\pi/2$ will be zero automatically. So we can write

$$\begin{aligned}
\int_{-\pi/2}^{\pi} \frac{\sin x}{2+\cos x}\,\mathrm{d}x &= \int_{-\pi/2}^{\pi/2} \frac{\sin x}{2+\cos x}\,\mathrm{d}x + \int_{\pi/2}^{\pi} \frac{\sin x}{2+\cos x}\,\mathrm{d}x \\
&= \int_{\pi/2}^{\pi} \frac{\sin x}{2+\cos x}\,\mathrm{d}x.
\end{aligned}$$

Now we can use the substitution $u = \cos x$ noticing that $\mathrm{d}u = -\sin x \, \mathrm{d}x$, when $x = \pi/2, u = 0$ and when $x = \pi, u = -1$, so that

$$\begin{aligned}\int_{-\pi/2}^{\pi} \frac{\sin x}{2 + \cos x} \, \mathrm{d}x &= \int_{0}^{-1} \frac{-\mathrm{d}u}{2+u} \\ &= -\left[\ln|2+u|\right]_0^{-1} \\ &= -(\ln 1 - \ln 2) \\ &= \ln 2.\end{aligned}$$

■

5.3 Rational Functions

A *rational function* is one of the form

$$\frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0},$$

where the a_i and b_i are constants — that is, the quotient of two polynomials. In principle, (because of the Fundamental Theorem of Algebra which says that the roots of the denominator can all be found in the complex numbers), it is possible to rewrite the denominator as

$$b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0 = p_1(x) \, p_2(x) \cdots p_k(x)$$

where the polynomials $p_i(x)$ are either linear factors (of the form $Ax + B$) or quadratic factors $(Ax^2 + Bx + C)$ with $B^2 < 4AC$ and complex conjugates for roots. From here we can use partial fractions to simplify the function.

5.3.1 Partial Fractions

Given a rational function

$$\frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{p_1(x) \, p_2(x) \cdots p_k(x)}$$

where the factors in the denominator are linear or quadratic terms, we follow several simple steps to put it into a form we can integrate.

1. if the numerator has greater degree than the denominator, then we divide the denominator into the numerator (using polynomial long division) till we have an expression of the form

$$P(x) + \frac{A_i x^i + A_{i-1} x^{i-1} + \cdots + A_0}{p_1(x) \, p_2(x) \cdots p_k(x)}$$

where $P(x)$ is a polynomial, and the numerator $A_i x^i + A_{i-1}x^{i-1} + \cdots + A_0$ now has a strictly smaller degree than the denominator $p_1(x)\cdots p_k(x)$. Of course, integrating the polynomial part $P(x)$ will not cause us any difficulty so we will ignore it from now on.

2. Let's suppose, for now, that none of the factors in the denominator are the same. In this case we can use partial fractions to rewrite this new rational function as
$$\frac{A_i x^i + A_{i-1}x^{i-1} + \cdots + A_0}{p_1(x)\,p_2(x)\cdots p_k(x)} = \frac{\alpha_1(x)}{p_1(x)} + \frac{\alpha_2(x)}{p_2(x)} + \cdots + \frac{\alpha_k(x)}{p_k(x)}$$
where each polynomial $\alpha_i(x)$ is of smaller degree than $p_i(x)$. This means that we have rewritten the rational function in terms of rational functions of the form
$$\frac{A}{Bx+C} \quad \text{and} \quad \frac{Ax+B}{Cx^2+Dx+E}$$
which we will see how to integrate in the next subsection.

3. If however a factor, say $p_1(x)$, is repeated N times say, then rather than the $\alpha_1(x)/p_1(x)$ term in the equation above, the best we can do with partial fractions is to reduce it to an expression of the form
$$\frac{\beta_1(x)}{p_1(x)} + \frac{\beta_2(x)}{(p_1(x))^2} + \cdots + \frac{\beta_N(x)}{(p_1(x))^N}$$
where the polynomials $\beta_i(x)$ have smaller degree than $p_1(x)$. This means the final expression may include functions of the form
$$\frac{A}{(Bx+C)^n} \quad \text{and} \quad \frac{Ax+B}{(Cx^2+Dx+E)^n} \quad \text{where } D^2 < 4CE.$$

Example 104 *Use the method of partial fractions to write the following rational function in simpler form*
$$\frac{x^5}{(x-1)^2(x^2+1)}.$$

Solution. The numerator has degree 5 whilst the denominator has degree 4, so we will need to divide the denominator into the numerator first. The denominator expands out to
$$(x-1)^2\left(x^2+1\right) = x^4 - 2x^3 + 2x^2 - 2x + 1.$$

Using polynomial long-division we see that

$$\begin{array}{r|rrrrrr}
 & & & & & x & +2 \\
\hline
x^4-2x^3+2x^2-2x+1 & x^5 & +0x^4 & +0x^3 & +0x^2 & +0x & +0 \\
 & x^5 & -2x^4 & +2x^3 & -2x^2 & +x & \\
\hline
 & & 2x^4 & -2x^3 & +2x^2 & -x & +0 \\
 & & 2x^4 & -4x^3 & +4x^2 & -4x & +2 \\
\hline
 & & & 2x^3 & -2x^2 & +3x & -2
\end{array}$$

So we have that

$$\frac{x^5}{(x-1)^2(x^2+1)} \equiv x+2+\frac{2x^3-2x^2+3x-2}{(x-1)^2(x^2+1)},$$

which leaves us to find the constants A, B, C, D, in the identity

$$\frac{2x^3-2x^2+3x-2}{(x-1)^2(x^2+1)} \equiv \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+1}.$$

Multiplying through by the denominator, we find

$$2x^3-2x^2+3x-2 \equiv A(x-1)\left(x^2+1\right)+B\left(x^2+1\right)+(Cx+D)(x-1)^2.$$

As this holds for all values of x, then we can set $x=1$ to deduce

$$2-2+3-2=1=2B \quad \text{and so } B=\frac{1}{2}.$$

If we set $x=0$ then we also get that

$$-2=-A+\frac{1}{2}+D. \tag{5.1}$$

Other things we can do are to compare the coefficients of x^3 on either side which gives

$$2=A+C \tag{5.2}$$

and to compare the coefficients of x which gives

$$3=A+C-2D. \tag{5.3}$$

Substituting (5.2) into (5.3) yields $3=2-2D$ and so $D=-1/2$. From equation (5.1) this means that $A=2$ and so $C=0$. Finally then we have

$$\frac{2x^3-2x^2+3x-2}{(x-1)^2(x^2+1)} \equiv \frac{2}{x-1}+\frac{1/2}{(x-1)^2}-\frac{1/2}{x^2+1}$$

and

$$\frac{x^5}{(x-1)^2(x^2+1)} \equiv x+2+\frac{2}{x-1}+\frac{1/2}{(x-1)^2}-\frac{1/2}{x^2+1}.$$

■

5.3.2 Trigonometric Substitutions

Solution. (Contd.) If we look now at the function we are faced with, namely

$$x+2+\frac{2}{x-1}+\frac{1/2}{(x-1)^2}-\frac{1/2}{x^2+1},$$

then only the final term is something that would cause trouble from an integrating point. To deal with such functions we recall the trigonometric identities

$$\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta. \qquad (5.4)$$

So a substitution of the form $x = \tan\theta$ into an expression like $1 + x^2$ simplifies it to $\sec^2\theta$. Noting

$$\mathrm{d}x = \sec^2\theta \ \mathrm{d}\theta$$

we find

$$\begin{aligned} \int \frac{\mathrm{d}x}{1+x^2} &= \int \frac{\sec^2\theta \ \mathrm{d}\theta}{1+\tan^2\theta} \\ &= \int \frac{\sec^2\theta \ \mathrm{d}\theta}{\sec^2\theta} \\ &= \int \mathrm{d}\theta \\ &= \theta + \text{const.} \\ &= \tan^{-1} x + \text{const.} \end{aligned}$$

So returning to our example we see

$$\begin{aligned} \int \frac{x^5 \ \mathrm{d}x}{(x-1)^2 (x^2+1)} &\equiv \int \left(x + 2 + \frac{2}{x-1} + \frac{1/2}{(x-1)^2} - \frac{1/2}{x^2+1} \right) \mathrm{d}x \\ &= \frac{x^2}{2} + 2x + 2\ln|x-1| - \frac{1/2}{x-1} - \frac{1}{2}\tan^{-1} x + \text{const.} \end{aligned}$$

■

Returning to the most general form of a rational function, we were able to reduce (using partial fractions) the problem to integrands of the form

$$\frac{A}{(Bx+C)^n} \quad \text{and} \quad \frac{Ax+B}{(Cx^2+Dx+E)^n} \quad \text{where } D^2 < 4CE.$$

Integrating functions of the first type causes us no difficulty as

$$\int \frac{A \ \mathrm{d}x}{(Bx+C)^n} = \begin{cases} \frac{A}{B(1-n)} (Bx+C)^{1-n} + \text{const.} & n \neq 1; \\ \frac{A}{B} \ln|Bx+C| + \text{const.} & n = 1. \end{cases}$$

The second integrand can be simplified, firstly by completing the square and then with a trigonometric substitution. Note that

$$Cx^2 + Dx + E = C\left(x + \frac{D}{2C}\right)^2 + \left(E - \frac{D^2}{4C}\right).$$

If we make a substitution of the form $u = x + D/2C$ then we can simplify this integral to something of the form

$$\int \frac{(au+b) \ \mathrm{d}u}{(u^2+k^2)^n} \quad \text{for new constants } a, b \text{ and } k > 0.$$

Part of this we can integrate directly:

$$\int \frac{u\ \mathrm{d}u}{\left(u^2+k^2\right)^n} = \begin{cases} \frac{1}{2(1-n)}\left(u^2+k^2\right)^{1-n} + \text{const.} & n \neq 1; \\ \frac{1}{2}\ln\left(u^2+k^2\right) + \text{const.} & n = 1. \end{cases}$$

The other integral

$$\int \frac{\mathrm{d}u}{\left(u^2+k^2\right)^n}$$

can be simplified with a trigonometric substitution $u = k\tan\theta$, the integral becoming

$$\begin{aligned} \int \frac{\mathrm{d}u}{\left(u^2+k^2\right)^n} &= \int \frac{k\sec^2\theta\ \mathrm{d}\theta}{\left(k^2\tan^2\theta+k^2\right)^n} \\ &= \frac{1}{k^{2n-1}}\int \frac{\sec^2\theta\ \mathrm{d}\theta}{\left(\sec^2\theta\right)^n} \\ &= \frac{1}{k^{2n-1}}\int \cos^{2n-2}\theta\ \mathrm{d}\theta. \end{aligned}$$

The $n = 0, 1, 2$ cases can all easily be integrated. We will see in the next section on Reduction Formulae how to deal generally with integrals of this form. For now we will simply give an example where $n = 2$.

Example 105 *Determine*

$$I = \int \frac{dx}{\left(3x^2+2x+1\right)^2}$$

Solution. Remember that the first step is to complete the square:

$$\begin{aligned} I &= \int \frac{\mathrm{d}x}{\left(3x^2+2x+1\right)^2} \\ &= \frac{1}{9}\int \frac{\mathrm{d}x}{\left(x^2+\frac{2}{3}x+\frac{1}{3}\right)^2} \\ &= \frac{1}{9}\int \frac{\mathrm{d}x}{\left(\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right)^2} \end{aligned}$$

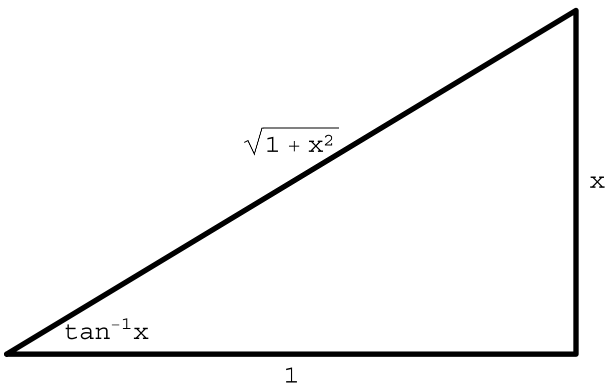
Our first substitution is simply a translation — let $u = x + 1/3$ noting that $\mathrm{d}u = \mathrm{d}x$:

$$I = \frac{1}{9}\int \frac{\mathrm{d}x}{\left(\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right)^2} = \frac{1}{9}\int \frac{\mathrm{d}u}{\left(u^2+2/9\right)^2}.$$

Then we set $u = \frac{\sqrt{2}}{3} \tan\theta$ to further simplify the integral. So

$$\begin{aligned}
I &= \frac{1}{9}\int \frac{\left(2/\sqrt{3}\right)\sec^2\theta \ \mathrm{d}\theta}{\left(2\sec^2\theta/9\right)^2} \\
&= \frac{1}{9}\times\frac{2}{\sqrt{3}}\times\left(\frac{9}{2}\right)^2\int\cos^2\theta \ \mathrm{d}\theta \\
&= \frac{9}{2\sqrt{3}}\int\frac{1}{2}\left(1+\cos 2\theta\right) \ \mathrm{d}\theta \quad [\text{using } \cos 2\theta = 2\cos^2\theta - 1] \\
&= \frac{9}{4\sqrt{3}}\left(\theta + \frac{1}{2}\sin 2\theta\right) + \text{const.} \\
&= \frac{9}{4\sqrt{3}}\left(\theta + \sin\theta\cos\theta\right) + \text{const.} \quad [\text{using } \sin 2\theta = 2\sin\theta\cos\theta] \\
&= \frac{9}{4\sqrt{3}}\left(\tan^{-1}\frac{3u}{\sqrt{2}} + \sin\tan^{-1}\frac{3u}{\sqrt{2}}\cos\tan^{-1}\frac{3u}{\sqrt{2}}\right) + \text{const.}
\end{aligned}$$

by undoing the substitution $u = \frac{\sqrt{2}}{3}\tan\theta$. From the right-angled triangle



we see that

$$\sin\tan^{-1}x = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad \cos\tan^{-1}x = \frac{1}{\sqrt{1+x^2}}.$$

So

$$\begin{aligned}
I &= \frac{9}{4\sqrt{3}}\left(\tan^{-1}\frac{3u}{\sqrt{2}} + \frac{3u/\sqrt{2}}{\sqrt{1+9u^2/2}}\times\frac{1}{\sqrt{1+9u^2/2}}\right) + \text{const.} \\
&= \frac{9}{4\sqrt{3}}\left(\tan^{-1}\frac{3u}{\sqrt{2}} + \frac{6u}{\sqrt{2}\left(2+9u^2\right)}\right) + \text{const.} \\
&= \frac{9}{4\sqrt{3}}\left(\tan^{-1}\left(\frac{3}{\sqrt{2}}\left(x+\frac{1}{3}\right)\right) + \frac{6x+2}{\sqrt{2}\left(9x^2+6x+3\right)}\right) + \text{const.} \\
&= \frac{9}{4\sqrt{3}}\left(\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + \frac{2(3x+1)}{3\sqrt{2}\left(3x^2+2x+1\right)}\right) + \text{const.}
\end{aligned}$$

This example surely demonstrates the importance of remembering the method and not the formula! ■

5.3.3 Further Trigonometric Substitutions

The trigonometric identities in equation (5.4) can be applied in more general cases than those above used for integrating rational functions above. A similar standard trigonometric integral is

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \sin^{-1} x + \text{const.}$$

This can be deduced in exactly the same way: this time we make use of the trigonometric identity

$$1 - \sin^2\theta = \cos^2\theta$$

and make a substitution $x = \sin\theta$ to begin this calculation. Likewise the integral

$$\int \frac{\mathrm{d}x}{\sqrt{3x^2+2x+1}}$$

could be tackled with the substitutions we used in the previous example.

Multiple angle trigonometric identities can also be very useful: we have already made use of the formula

$$\cos 2\theta = 2\cos^2\theta - 1$$

to determine the integral of $\cos^2\theta$. Likewise, in principle ,we could integrate $\cos^n\theta$ by first writing it in terms of $\cos k\theta$ (for various k); alternatively, as in the next section, we can approach this integral using reduction formulae.

We close this section with a look at the t-substitution, which makes use of the *half-angle tangent formulas.* Faced with the integral

$$\int_0^{\pi} \frac{\mathrm{d}\theta}{2+\cos\theta},$$

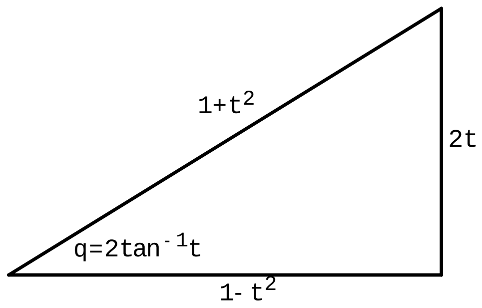
we make a substitution of the form

$$t = \tan\frac{\theta}{2}.$$

Each of the trigonometric functions $\sin, \cos, \tan$ can be written in terms of t. The formulae are

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad \tan\theta = \frac{2t}{1-t^2}.$$

An easy way to remember these formulae is probably by means of the right-angled triangle:



If we make this substitution in the above integral then firstly we need to note that $t = \tan(\theta/2)$ is a bijection from the range $[0, \pi)$ to the range $[0, \infty)$. Also

$$\mathrm{d}\theta = \mathrm{d}\left(2\tan^{-1} t\right) = \frac{2\ \mathrm{d}t}{1+t^2}.$$

So

$$\begin{aligned}
\int_0^{\pi} \frac{\mathrm{d}\theta}{2+\cos\theta} &= \int_0^{\infty} \frac{1}{2+\frac{1-t^2}{1+t^2}} \frac{2\ \mathrm{d}t}{1+t^2} \\
&= \int_0^{\infty} \frac{2\ \mathrm{d}t}{2+2t^2+1-t^2} \\
&= \int_0^{\infty} \frac{2\ \mathrm{d}t}{3+t^2} \\
&= \left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}}\right]_0^{\infty} \\
&= \frac{2}{\sqrt{3}} \left(\frac{\pi}{2} - 0\right) \\
&= \frac{\pi}{\sqrt{3}}.
\end{aligned}$$

5.4 Reduction Formulae

In the previous section on integrating rational functions we were left with the problem of calculating

$$I_n = \int \cos^n \theta\ \mathrm{d}\theta,$$

and we will approach such integrals using *reduction formulae*. The idea is to write I_n in terms of other I_k where $k < n$, eventually reducing the problem to calculating I_0, or I_1 say, which are simple integrals.

Using IBP we see

$$\begin{aligned}
I_n &= \int \cos^{n-1}\theta \times \cos\theta \; \mathrm{d}\theta \\
&= \cos^{n-1}\theta \sin\theta - \int (n-1)\cos^{n-2}\theta\,(-\sin\theta)\sin\theta \; \mathrm{d}\theta \\
&= \cos^{n-1}\theta \sin\theta + (n-1)\int \cos^{n-2}\theta \left(1-\cos^2\theta\right) \; \mathrm{d}\theta \\
&= \cos^{n-1}\theta \sin\theta + (n-1)\left(I_{n-2} - I_n\right).
\end{aligned}$$

Rearranging this we see

$$I_n = \frac{\cos^{n-1}\theta\sin\theta}{n} + \frac{n-1}{n} I_{n-2}.$$

With this reduction formula I_n can be rewritten in terms of simpler and simpler integrals until we are left only needing to calculate I_0, if n is even, or I_1, if n is odd — both these integrals are easy to calculate.

Example 106 *Calculate*

$$I_7 = \int \cos^7\theta \; d\theta.$$

Solution. Using the reduction formula above

$$\begin{aligned}
I_7 &= \frac{\cos^6\theta\sin\theta}{7} + \frac{6}{7} I_5 \\
&= \frac{\cos^6\theta\sin\theta}{7} + \frac{6}{7}\left(\frac{\cos^4\theta\sin\theta}{5} + \frac{4}{5} I_3\right) \\
&= \frac{\cos^6\theta\sin\theta}{7} + \frac{6\cos^4\theta\sin\theta}{35} + \frac{24}{35}\left(\frac{\cos^2\theta\sin\theta}{3} + \frac{2}{3} I_1\right) \\
&= \frac{\cos^6\theta\sin\theta}{7} + \frac{6\cos^4\theta\sin\theta}{35} + \frac{24\cos^2\theta\sin\theta}{105} + \frac{48}{105}\sin\theta + \text{const.}
\end{aligned}$$

■

Example 107 *Calculate*

$$\int_0^1 x^3 e^{2x} \; dx$$

Solution. This is an integral we previously calculated in Example 97. We can approach this in a simpler, yet more general, fashion by setting up a reduction formula. For a natural number n, let

$$J_n = \int_0^1 x^n e^{2x} \; \mathrm{d}x.$$

We can use then integration by parts to show

$$\begin{aligned} J_n &= \left[x^n \frac{e^{2x}}{2}\right]_0^1 - \int_0^1 n x^{n-1} \frac{e^{2x}}{2} \text{ d}x \\ &= \frac{e^2}{2} - \frac{n}{2} J_{n-1} \quad \text{if } n \geq 1. \end{aligned}$$

and so the calculation in Example 97 simplifies enormously (at least on the eye). We first note

$$J_0 = \int_0^1 e^{2x} \text{ d}x = \left[\frac{e^{2x}}{2}\right]_0^1 = \frac{e^2 - 1}{2},$$

and then applying the reduction formula:

$$\begin{aligned} J_3 &= \frac{e^2}{2} - \frac{3}{2} J_2 \\ &= \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^2}{2} - \frac{2}{2} J_1\right) \\ &= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3}{2}\left(\frac{e^2}{2} - \frac{1}{2} J_0\right) \\ &= \frac{e^2}{8} + \frac{3}{8}. \end{aligned}$$

■

Some integrands may involve two variables, such as:

Example 108 *Calculate for positive integers m, n the integral*

$$B(m, n) = \int_0^1 x^{m-1} (1 - x)^{n-1} \ dx.$$

Solution. Calculating either $B(m, 1)$ or $B(1, n)$ is easy; for example

$$B(m, 1) = \int_0^1 x^{m-1} \text{ d}x = \frac{1}{m}. \tag{5.5}$$

So it would seem best to find a reduction formula that moves us towards either of these integrals. Using integration by parts, if $n \geq 2$ we have

$$\begin{aligned} B(m, n) &= \left[\frac{x^m}{m}(1 - x)^{n-1}\right]_0^1 - \int_0^1 \frac{x^m}{m} \times (n-1) \times (-1)(1 - x)^{n-2} \text{ d}x \\ &= 0 + \frac{n-1}{m} \int_0^1 x^m (1 - x)^{n-2} \text{ d}x \\ &= \frac{n-1}{m} B(m+1, n-1). \end{aligned}$$

So if $n \geq 2$ we can apply this to see

$$\begin{aligned} B(m,n) &= \frac{n-1}{m} B(m+1, n-1) \\ &= \frac{n-1}{m} \times \frac{n-2}{m+1} B(m+2, n-2) \\ &= \left(\frac{n-1}{m}\right)\left(\frac{n-2}{m+1}\right)\cdots\left(\frac{1}{m+n-2}\right) B(m+n-1, 1) \\ &= \left(\frac{n-1}{m}\right)\left(\frac{n-2}{m+1}\right)\cdots\left(\frac{1}{m+n-2}\right)\frac{1}{m+n-1} \\ &= \frac{(n-1)!}{(m+n-1)!/(m-1)!} \\ &= \frac{(m-1)!\,(n-1)!}{(m+n-1)!}. \end{aligned}$$

Equation (5.5) shows this formula also holds for $n = 1$. ■

5.5 Numerical Methods

Of course it's not always possible to calculate integrals exactly and there are numerical rules that will provide approximate values for integrals — approximate values, which by 'sampling' the function more and more times, can be made better and better.

Suppose that $f : [a, b] \to \mathbb{R}$ is the function we are wishing to integrate. Our idea will be to sample the function at $n+1$ evenly spread points through the interval:

$$x_k = a + k\left(\frac{b-a}{n}\right) \quad \text{for } k = 0, 1, 2, \ldots, n,$$

so that $x_0 = a$ and $x_n = b$. The corresponding y-value we will denote as

$$y_k = f(x_k).$$

For ease of notation the width between each sample we will denote as

$$h = \frac{b-a}{n}.$$

There are various rules for making an estimate for the integrals of the function based on this data. We will consider the *Trapezium Rule* and *Simpson's Rule.*

- *Trapezium Rule.* This estimates the area as:

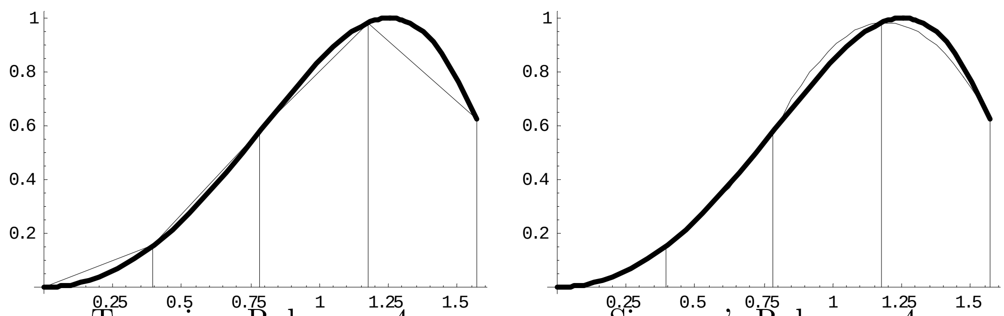
$$h\left(\frac{y_0}{2}+y_1+y_2+\cdots+y_{n-1}+\frac{y_n}{2}\right).$$

  This estimate is arrived at (as you might guess from the name) by approximating the area under the graph with trapezia. We presume that the graph behaves linearly between (x_k, y_k) and (x_{k+1}, y_{k+1}) and take the area under the line segment connecting these points as our contribution.

- *Simpson's Rule.* This requires that n be even and estimates the area as:

$$\frac{h}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+\cdots+2y_{n-2}+4y_{n-1}+y_n\right).$$

  The more sophisticated Simpson's Rule works on the presumption that between the three points $(x_k, y_k), (x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2})$ (where k is even) the function f will change quadratically and it calculates the area contributed beneath each of these quadratic curves.



Trapezium Rule: $n = 4$ Simpson's Rule: $n = 4$

The above two graphs show applications of the trapezium rule and Simpson's rule in calculating

$$\int_0^{\pi/2} \sin\left(x^2\right)\ \mathrm{d}x$$

with $n = 4$ subintervals.

Example 109 *Estimate the integral*

$$\int_0^1 x^3\ dx$$

using both the trapezium rule and Simpson's rule using $2n$ *intervals.*

Solution. This is, of course, an integral we can calculate exactly as $1/4$. The two rules above give us:

$$\begin{aligned}
\text{Trapezium Approx}^{\text{n}} &= \frac{1}{2n}\left(\frac{0^3}{2}+\left(\frac{1}{2n}\right)^3+\cdots+\left(\frac{2n-1}{2n}\right)^3+\frac{1^3}{2}\right)\\
&= \frac{1}{2n}\left(\frac{1}{8n^3}\sum_{k=1}^{2n-1}k^3+\frac{1}{2}\right)\\
&= \frac{1}{2n}\left(\frac{1}{8n^3}\times\frac{1}{4}(2n-1)^2(2n)^2+\frac{1}{2}\right)\\
&= \frac{4n^2+1}{16n^2}\\
&= \frac{1}{4}+\frac{1}{16n^2}.
\end{aligned}$$

and we also have

$$\begin{aligned}
\text{Simpson's Approx}^{\text{n}} &= \frac{1}{6n}\left(0^3+4\left(\frac{1}{2n}\right)^3+2\left(\frac{2}{2n}\right)^3+\cdots+1^3\right)\\
&= \frac{1}{6n}\left(0+\frac{4}{(2n)^3}\sum_{k=1}^{2n-1}k^3-\frac{2}{(2n)^3}\sum_{k=1}^{n-1}(2k)^3+1\right)\\
&= \frac{1}{6n}\left(\frac{4}{8n^3}\frac{1}{4}(2n-1)^2(2n)^2-\frac{2}{8n^3}\frac{8}{4}(n-1)^2n^2+1\right)\\
&= \frac{3n^2}{12n^2}\\
&= \frac{1}{4}.
\end{aligned}$$

■

Remark 110 *Note in these calculations we make use of the formula*

$$\sum_{k=1}^{n}k^3=\frac{1}{4}n^2(n+1)^2.$$

We see then that the error from the Trapezium Rule is $1/(16n^2)$ and so decreases very quickly. Amazingly Simpson's Rule does even better here and gets the answer spot on — the overestimates and underestimates of area from under these quadratics actually cancel out. In general Simpson's Rule is an improvement on the Trapezium Rule with the two errors (associated with $2n$ intervals) being given by:

$$|E_{\text{Trapezium}}|\leq\frac{(b-a)^3}{48n^2}\max\{|f''(x)|:a\leq x\leq b\},$$

and Simpson's Rule with $2n$ steps

$$|E_{\text{Simpson}}| \leq \frac{(b-a)^5}{2880n^4} \max\left\{\left|f^{(4)}(x)\right| : a \leq x \leq b\right\}.$$

Note that the error is $O(n^{-4})$ for the Simpson Rule but only $O(n^{-2})$ for the Trapezium Rule.

5.6 Exercises

Exercise 206 *Evaluate*

$$\int \frac{\ln x}{x}\, dx, \qquad \int x \sec^2 x\, dx, \qquad \int \frac{dx}{4\cos x + 3\sin x}$$

Exercise 207 *Evaluate*

$$\int x^6 \ln x\, dx, \qquad \int \frac{dx}{1+\sqrt{x}}, \qquad \int \frac{dx}{\sinh x}.$$

Exercise 208 *Let*

$$I_1 = \int \frac{\sin x\, dx}{\sin x + \cos x}, \quad I_2 = \int \frac{\cos x\, dx}{\sin x + \cos x}.$$

By considering $I_1 + I_2$ and $I_2 - I_1$ find I_1 and I_2. Generalize your method to calculate

$$\int \frac{\sin x\, dx}{a\sin x + b\cos x}, \quad \text{and} \quad \int \frac{\cos x\, dx}{a\sin x + b\cos x}.$$

Exercise 209 *Evaluate*

$$\int_3^\infty \frac{dx}{(x-1)(x-2)}, \qquad \int_0^{\pi/2} \cos x\sqrt{\sin x}\, dx, \qquad \int_0^1 \tan^{-1} x\, dx.$$

Exercise 210

$$\int_2^\infty \frac{1}{x\sqrt{x-1}}, \qquad \int_0^1 \ln x\, dx, \qquad \int_0^1 \frac{dx}{e^x+1}.$$

Exercise 211 *For what values of α does the integral*

$$\int_\varepsilon^1 x^\alpha\, dx$$

remain bounded as ε becomes arbitrarily small? For what values of β does the integral

$$\int_1^R x^\beta\, dx$$

remain bounded as R becomes arbitrarily large?

Exercise 212 *Determine*

$$\int \frac{x^5}{x^3-1}\ dx.$$

Exercise 213 *Evaluate, using trigonometric and/or hyperbolic substitutions,*

$$\int \frac{dx}{x^2+1}, \qquad \int_1^2 \frac{dx}{\sqrt{x^2-1}}, \qquad \int \frac{dx}{\sqrt{4-x^2}}, \qquad \int_2^\infty \frac{dx}{(x^2-1)^{3/2}}$$

Exercise 214 *By completing the square in the denominator, and using the substitution*

$$x = \frac{\sqrt{2}}{3}\tan\theta - \frac{1}{3}$$

evaluate

$$\int \frac{dx}{3x^2+2x+1}..$$

Exercise 215 *Evaluate*

$$\int \frac{dx}{\sqrt{x^2+2x+5}}, \qquad \int_0^\infty \frac{dx}{4x^2+4x+5}.$$

Exercise 216 *Let* $t = \tan\frac{1}{2}\theta$. *Show that*

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}, \qquad \tan\theta = \frac{2t}{1-t^2}$$

and that

$$d\theta = \frac{2\ dt}{1+t^2}.$$

Use the substitution $t = \tan\frac{1}{2}\theta$ *to evaluate*

$$\int_0^{\pi/2} \frac{d\theta}{(1+\sin\theta)^2}.$$

Exercise 217 *Let*

$$I_n = \int_0^{\pi/2} x^n \sin x\ dx.$$

Evaluate I_0 *and* I_1.

Show, using integration by parts, that

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\,I_{n-2}.$$

Hence, evaluate I_5 *and* I_6.

Exercise 218 *Let*

$$I_n = \int_0^\infty x^n e^{-x^2} \ dx.$$

Show that

$$I_n = \frac{n-1}{2} I_{n-2}$$

for $n \geqslant 2$. Find I_5. Given that $I_0 = \sqrt{\pi}/2$, calculate I_6.

Exercise 219 *Show that*

$$\begin{aligned} \int \cos^5 x \ dx &= \frac{5}{8}\sin x + \frac{5}{48}\sin 3x + \frac{1}{80}\sin 5x + \textit{const.} \\ &= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + \textit{const.} \end{aligned}$$

Exercise 220 *Show that, for any polynomial $P(x)$,*

$$\int P(x)\, e^{ax} \ dx = \frac{e^{ax}}{a} \sum_{k=0}^{m} (-1)^k \frac{P^{(k)}(x)}{a^k} + \textit{const.}$$

Exercise 221 *Show that*

$$\int x^n (\ln x)^m \ dx = \frac{x^{n+1} (\ln x)^m}{n+1} - \frac{m}{n+1} \int x^n (\ln x)^{m-1} \ dx.$$

Hence find $\int x^3 (\ln x)^2 \ dx$.

Exercise 222 *Estimate*

$$\int_0^1 \sin\left(x^2\right) \ dx$$

with four steps.

Exercise 223 *Find an upper bound for the error in calculating the integral in the previous exercise with the n-step Trapezium Rule. How large does n need to be to guarantee that the estimate is accurate to within 10^{-4}?*

Exercise 224 *Calculate*

$$\int_0^1 e^x \ dx$$

using Trapezium Rule and Simpson's Rule, both with $2n$ steps.

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