OPTIMAL INVESTMENT, VALUATION AND HEDGING

MICHAEL MONOYIOS
MATHEMATICAL INSTITUTE
UNIVERSITY OF OXFORD

Contents

Useful books

Useful textbooks for these lectures include Björk [2, Chapters 19 and 20], Pham [22, Chapters 2 and 3], Rogers [24] and (for the dual approach to portfolio problems) Karatzas [13].

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1. Brief outline

We shall describe the classical Merton optimal investment problem and its solution via stochastic control and dynamic programming methods, involving the Hamilton-Jacobi-Bellman (HJB) equation.

We shall then describe the so-called dual approach (or martingale approach, or convex duality approach) to solving utility maximisation problems, which is an alternative to dynamic programming and the HJB equation, and which works for systems which do not necessarily have Markovian dynamics. We give the main ideas in a complete market (the incomplete market case is considerably more difficult).

We shall then distinguish partial and full information models. In the former, the value of the drift process of the underlying stock price is not known, because the agent is not assumed to observe the Brownian motion driving the stock price. That is, the agent does not have access to the Brownian filtration $\mathbb{F}$, and only has access to the stock price filtration $\widehat{\mathbb{F}}$, the so-called observation filtration. We shall use the theory of linear filtering, in particular the celebrated Kalman-Bucy filter, to re-write the model under the observation filtration, and we shall then use the convex duality method developed earlier to solve the Merton optimal investment problem under $\widehat{\mathbb{F}}$.

If time allows we shall then describe a simple example of an *incomplete* market, sometimes called a *basis risk model*, in which a claim on a non-traded asset is hedged using a correlated traded asset. Because the market is incomplete, the risk from selling the claim cannot be completely eliminated, so any valuation and hedging scheme has to take into account the risk preferences of the agent, and we shall do this via the exponential utility function. We shall describe the method of utility-based valuation and hedging of the claim, derive closed form expressions for the claim value and the optimal hedging strategy, and finally we shall describe the dual approach to optimal investment with the random endowment of the claim payoff, and revisit the basis risk model via duality methods.

2. The Merton problems

A *stochastic optimal control problem* involves a system whose *state*, X, is a stochastic process which, as well as having inherent deterministic and random time evolution, can also have its evolution affected by an agent exerting some influence, or *control*, to optimise some *performance criterion.*

The classical financial example is the so-called *Merton problem* to maximise expected utility of terminal wealth and intermediate consumption, which we shall now describe briefly, by way of introduction. In Section 3 shall describe the technique of *dynamic programming* to handle control problems subject to Markovian state dynamics, and apply this approach to the Merton problems.

In continuous time, suppose a stock price $S = (S_t)_{t\geqslant 0}$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$, evolves according to the classical Black-Scholes-Merton (BSM) model. The stochastic differential equation (SDE) for the stock price is

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \quad \mu \in \mathbb{R}, \quad \sigma \in \mathbb{R}_+,$$

where W is an $\mathbb{F}$-Brownian motion (BM), and $\mu \in \mathbb{R}$, $\sigma > 0$ are known constants. There is a constant interest rate $r \geqslant 0$. We can take the filtration $\mathbb{F}$ to be the $\mathbb{P}$-augmentation of the natural filtration of the Brownian motion W (so augmented with the null sets of $\mathcal{F}$, and then the filtration $\mathbb{F}$ satisfies the so-called usual conditions of right-continuity and completeness).

An agent with initial capital $x > 0$ trades the stock and cash, and may also consume wealth, using a non-negative $\mathbb{F}$-adapted consumption rate process $c = (c_t)_{t\geqslant 0}$. Denote by $X = (X_t)_{t \geqslant 0}$ the wealth process of a self-financing portfolio containing H_t shares of stock at $t \geqslant 0$. The cash in the portfolio at time t is $X_t - H_t S_t$, so the wealth dynamics are

$$\begin{aligned} \mathrm{d}X_t &= H_t \,\mathrm{d}S_t + r(X_t - H_t S_t)\,\mathrm{d}t - c_t\,\mathrm{d}t \\ &= rX_t\,\mathrm{d}t + \sigma\pi_t(\lambda\,\mathrm{d}t + \mathrm{d}W_t) - c_t\,\mathrm{d}t, \end{aligned} \tag{2.1}$$

where $\pi_t := H_t S_t$ is an $\mathbb{F}$-adapted strategy describing the wealth held in the stock, and $\lambda := (\mu - r)/\sigma$ is the market price of risk of the stock.

The agent chooses a consumption-investment strategy (π, c) from some admissible set $\mathcal{A}(x)$ given initial wealth x. The agent has an objective to maximise the expected utility from consumption and investment, over some given horizon $[0, T]$. We define his *value function* u by

$$u(x) := \sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^T e^{-\delta t} U_1(c_t)\,\mathrm{d}t + U_2(X_T)\right], \tag{2.2}$$

where $U_i, i = 1, 2$ are increasing concave utility functions, encapsulating the agent's preferences, and $\delta > 0$ is some subjective discount rate for consumption (and measures impatience, the desire to spend sooner rather than later).

Remark 2.1 (Notation). We shall use the symbol u for the value function of a variety of problems, so be aware that u will represent different quantities as we proceed through these lectures.

The *stochastic control problem* is to find an optimal strategy (π^*, c^*) achieving the supremum in (2.2). The state variable is the wealth process X, and the control process is (π, c).

The problem (2.2) is known as the finite horizon *Merton problem* for utility from consumption and terminal wealth. There are some natural variations to this objective:

- The finite horizon problem for utility from terminal wealth only, which has value function (for some utility function $U(\cdot)$)

$$u(x) := \sup_{\pi\in\mathcal{A}(x)} \mathbb{E}\left[U(X_T)\right], \tag{2.3}$$

  where we do not consume any wealth, so the wealth dynamics are those in (2.1) but with $c \equiv 0$:

$$\mathrm{d}X_t = rX_t\,\mathrm{d}t + \sigma\pi_t(\lambda\,\mathrm{d}t + \mathrm{d}W_t).$$

  One can in fact include possible consumption in these dynamics, and then show (the intuitively clear fact) that, since the objective in (2.3) features no utility from consumption, the optimal consumption process is indeed null. We won't show this, and just take consumption to be zero from the outset in the terminal wealth problem.
- The finite horizon problem for utility from consumption only, which has value function

$$u(x) := \sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^T \mathrm{e}^{-\delta t} U(c_t)\,\mathrm{d}t\right], \tag{2.4}$$

  where the wealth dynamics are again those in (2.1).
- The infinite horizon problem for utility from consumption only, which has value function

$$u(x) := \sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\delta t} U(c_t)\,\mathrm{d}t\right], \tag{2.5}$$

  where the wealth dynamics are again those in (2.1).

In these lectures we shall focus on terminal wealth problems of the type in (2.3).

2.1. **Utility functions and convex conjugates.**

Definition 2.2 (Utility function)**.** A utility function will be a continuous, strictly increasing, strictly concave, differentiable function $U : (0, \infty) \to \mathbb{R}$ (or sometimes $U : \mathbb{R} \to \mathbb{R}$) with

$$U'(\infty) := \lim_{x \to \infty} U'(x) = 0, \quad U'(0+) := \lim_{x \downarrow 0} U'(x) = \infty,$$

and in the case that U is defined over all of $\mathbb{R}$, the second condition is replaced by

$$U'(-\infty) := \lim_{x \to -\infty} U'(x) = \infty.$$

The classical utility functions are:

Logarithmic utility:

$$U(x) = \log x, \quad x \in \mathbb{R}^+,$$

Power utility:

$$U(x) = \frac{x^p}{p}, \quad p < 1, \quad p \neq 0, \quad x \in \mathbb{R}^+,$$

Exponential utility:

$$U(x) = -\exp(-\alpha x), \quad \alpha > 0, \quad x \in \mathbb{R}.$$

The coefficient of absolute risk aversion associated with a utility function is

$$R_A(x) := -\frac{U''(x)}{U'(x)},$$

and the coefficient of relative risk aversion is

$$R_R(x) := -\frac{xU''(x)}{U'(x)}.$$

For the power and logarithmic utilities we have that $R_A(x)$ is proportional to $1/x$, so these are sometimes called Hyperbolic Absolute Risk Aversion (HARA) utilities. The relative risk aversion for these utility functions is constant. For the exponential utility the absolute risk aversion is the constant $\alpha > 0$. For this reason the exponential utility function is sometimes referred to as a Constant Absolute Risk Aversion (CARA) utility.

For a utility function U we shall denote by I the inverse of the marginal utility U', satisfying

$$U'(I(y)) = I(U'(y)) = y, \quad \text{for any } y > 0.$$

Both U' and I are continuous, strictly decreasing, and map $(0, \infty)$ onto itself with $I(0+) = U'(0+) = \infty$, and $I(\infty) = U'(\infty) = 0$.

Definition 2.3 (Convex conjugate of a utility function)**.** The convex dual (or convex conjugate) $V : \mathbb{R}^+ \to \mathbb{R}$ of U is defined by

$$V(y) := \sup_{x \in \mathrm{dom}(U)} [U(x) - xy] = U(I(y)) - yI(y), \quad y > 0. \tag{2.6}$$

The conjugate function V is a convex, decreasing function, continuously differentiable on $(0, \infty)$, satisfying

$$V(y) \geqslant U(x) - xy, \quad \text{with equality if and only if } x = I(y), \tag{2.7}$$

as well as

$$V'(y) = -I(y), \tag{2.8}$$

and the bi-dual relation

$$U(x) = \inf_{y \in \mathbb{R}^+} [V(y) + xy] = V(U'(x)) + xU'(x), \quad x \in \mathrm{dom}(U). \tag{2.9}$$

For the utility functions listed above, the inverse of marginal utility and convex conjugate are as follows:

Logarithmic utility:

$$U(x) = \log x, \quad x \in \mathbb{R}^+, \quad I(y) = \frac{1}{y}, \quad V(y) = -(1 + \log y), \quad y > 0.$$

Power utility:

$$U(x) = \frac{x^p}{p}, \quad p \in (-\infty, 1)\backslash\{0\}, \quad x \in \mathbb{R}^+, \quad I(y) = y^{-(1-q)}, \quad V(y) = -\frac{y^q}{q}, \quad y > 0.$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Exponential utility: For $x \in \mathbb{R}$ and $y > 0$, we have

$$U(x) = -\exp(-\alpha x), \quad \alpha > 0, \quad I(y) = -\frac{1}{\alpha}\log\left(\frac{y}{\alpha}\right), \quad V(y) = \frac{y}{\alpha}\left(\log\left(\frac{y}{\alpha}\right) - 1\right).$$

2.2. **Merton terminal wealth problem: direct solution.** In this subsection, let us assume that the agent does not consume any wealth: $c \equiv 0$, and let us work on a finite horizon $[0, T]$. The portfolio wealth process follows

$$\mathrm{d}X_t = rX_t\,\mathrm{d}t + \sigma\pi_t(\lambda\,\mathrm{d}t + \mathrm{d}W_t). \tag{2.10}$$

The portfolio optimisation problem is to choose a trading strategy $\pi := (\pi_t)_{0 \leqslant t \leqslant T}$ from some admissible set $\mathcal{A}(x)$ of trading strategies to maximise expected utility of terminal wealth at time T. We write $X \equiv X^{x,\pi} \equiv X^x \equiv X^\pi$ whenever we need to emphasise dependence of the wealth on the initial wealth $X_0 = x$ and/or the trading strategy. The value function starting from time zero is

$$u(x) := \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}[U(X_T^\pi)|X_0^\pi = x], \tag{2.11}$$

where $U(\cdot)$ is an increasing concave utility function.

We shall first solve problems of the form (2.11) using a direct approach (which works because the parameters are constant and the model is one-dimensional). In later sections we shall obtain the same results via dynamic programming.

2.2.1. *Direct solution of logarithmic utility Merton problem.* Take $U(x) = \log x, x \in \mathbb{R}^+$. Let the class $\mathcal{A}(x)$ of admissible strategies given $X_0 = x$ be those with non-negative wealth process. In anticipation of our final result, define

$$\theta_t := \frac{\pi_t}{X_t}, \quad 0 \leqslant t \leqslant T,$$

the fraction of wealth in the risky asset. In terms of θ the wealth dynamics are

$$\mathrm{d}X_t = X_t[(r + \sigma\lambda\theta_t)\,\mathrm{d}t + \sigma\theta_t\,\mathrm{d}W_t].$$

Given $X_0 = x$ this implies

$$\log X_t = \log x + rt + \sigma\int_0^t \theta_s\left(\lambda - \frac{1}{2}\sigma\theta_s\right)\mathrm{d}s + \sigma\int_0^t \theta_s\,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant T. \tag{2.12}$$

Equivalently,

$$X_t = x\exp\left(rt + \sigma\int_0^t \theta_s\left(\lambda - \frac{1}{2}\sigma\theta_s\right)\mathrm{d}s + \sigma\int_0^t \theta_s\,\mathrm{d}W_s\right), \quad 0 \leqslant t \leqslant T. \tag{2.13}$$

Assume the stochastic integral in (2.12) is a martingale. We can (at least formally) solve the utility maximisation problem (2.11) for $U(x) = \log x$ directly. From (2.12) we have

$$\mathbb{E}[\log X_T] = \log x + rT + \sigma\mathbb{E}\left[\int_0^T \theta_t\left(\lambda - \frac{1}{2}\sigma\theta_t\right) \mathrm{d}t\right]. \tag{2.14}$$

This is maximised if we maximise the integrand on the right-hand-side. So we choose $\theta = \theta^*$, given by

$$\theta_t^* = \frac{\lambda}{\sigma}, \quad 0 \leqslant t \leqslant T, \tag{2.15}$$

which is constant. The optimal strategy to maximise expected logarithmic utility of terminal wealth is to keep a constant proportion of wealth in the risky asset. Note that this requires continuous portfolio rebalancing.

The maximum utility in (2.11) is then

$$u(x) = \mathbb{E}[\log X_T^* | X_0 = x],$$

where X^* denotes the wealth process with optimal strategy θ^*. This gives, using (2.14),

$$u(x) = \log x + \left(r + \frac{1}{2}\lambda^2\right) T. \tag{2.16}$$

Remark 2.4. Notice that if we define the Radon-Nikodym derivative of the unique equivalent martingale measure $\mathbb{Q}$ by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \equiv Z_T := \mathcal{E}(-\lambda W)_T = \exp\left(-\lambda W_T - \frac{1}{2}\lambda^2 T\right),$$

as well as the deflator (or state price density) at time T, $Y_T := \exp(-rT)Z_T$, then the value function in (2.16) can be written as

$$u(x) = \log x + rT - \mathbb{E}[\log Z_T] = \log x - \mathbb{E}[\log(Y_T)]. \tag{2.17}$$

The quantity $-\mathbb{E}[\log Z_T]$ is called the reverse relative entropy between $\mathbb{Q}$ and $\mathbb{P}$. The reason for the structure in (2.17) is to do with the *dual* problem to the utility maximisation problem (2.11), as we shall explore in Section 5.

Exercise 2.5. Repeat the above calculation with $U(x) = x^p/p, p < 0, p \neq 1$, to show that the optimal trading strategy θ^* is given by

$$\theta_t^* = \frac{\lambda}{\sigma(1-p)}, \quad 0 \leqslant t \leqslant T, \tag{2.18}$$

and so keeps a constant proportion of wealth in stock at all times. Note that this the result for logarithmic utility is the $p \to 0$ limit of this result. Show that the maximum utility is given by

$$u(x) = \frac{x^p}{p}\exp\left[\left(rp + \frac{1}{2}\left(\frac{p}{1-p}\right)\lambda^2\right)T\right]. \tag{2.19}$$

[Hint: use (2.13) to compute $\mathbb{E}[X_T^p/p]$.]

Exercise 2.6. Repeat the above calculation with $U(x) = -\exp(-\alpha x), \alpha > 0$, to show that the optimal trading strategy is described in terms of the process π, the wealth in the risky asset, and the optimal such strategy is π^*, given by

$$\pi_t^* = \mathrm{e}^{-r(T-t)}\frac{\lambda}{\alpha\sigma}, \quad 0 \leqslant t \leqslant T. \tag{2.20}$$

Show that the maximum utility is given by

$$u(x) = -\exp\left(-\alpha x\mathrm{e}^{rT} - \frac{1}{2}\lambda^2 T\right). \tag{2.21}$$

[Hint: use (2.10) to find the SDE for $\mathrm{e}^{-rt}X_t$ and hence solve for $(X_t)_{0\leqslant t\leqslant T}$. Then compute $\mathbb{E}[-\exp(-\alpha X_T)|X_0 = x]$, assuming $\pi = (\pi_t)_{0\leqslant t\leqslant T}$ is deterministic. You should find that the solution to (2.10) in terms of π is

$$X_t = \mathrm{e}^{rt}\left[x + \sigma\lambda\int_0^t \mathrm{e}^{-rs}\pi_s\,\mathrm{d}s + \sigma\int_0^t \mathrm{e}^{-rs}\pi_s\,\mathrm{d}W_s\right].$$

Assume the stochastic integral is a martingale.]

3. Dynamic programming and the HJB equation

Consider a *controlled diffusion process* with state $X = (X_s)_{s\geqslant 0}$, for $X_s \in \mathbb{R}^n$, satisfying

$$\mathrm{d}X_s = b(X_s, \alpha_s)\,\mathrm{d}s + \sigma(X_s, \alpha_s)\,\mathrm{d}W_s. \tag{3.1}$$

Here, W is d-dimensional BM on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, the control is $\alpha = (\alpha_s)_{s\geqslant 0}$, an $\mathbb{F}$-adapted process, with $\alpha_s \in A \subset \mathbb{R}^m$, and the coefficient functions are $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times A \to \mathbb{R}^{n\times d}$.

3.1. **Finite horizon problem.** Fix $T \in (0, \infty)$. Let $\mathcal{A}(x)$ denote the set of admissible controls given $X_0 = x$. We are interested in the problem

$$u(x) := \sup_{\alpha\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^T f(s, X_s, \alpha_s)\,\mathrm{d}s + F(X_T)\,\middle|\, X_0 = x\right]. \tag{3.2}$$

The method of dynamic programming tackles this problem by considering a starting state $(t, x) \in [0, T] \times \mathbb{R}^n$. Let $\mathcal{A}(t, x)$ denote the set of admissible controls given this starting state. We will often write $(X_s^{t,x})_{s\in[t,T]}$ to denote the solution to (3.1) starting at $X_t = x$, for any $t \in [0, T]$. Define the objective functional

$$J(t, x; \alpha) := \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)\,\mathrm{d}s + F(X_T^{t,x})\right].$$

The value function is

$$u(t, x) := \sup_{\alpha\in\mathcal{A}(t,x)} J(t, x; \alpha)$$

Thus, with this notation, the time zero value function in (3.2) is $u(x) \equiv u(0, x)$.

Given an initial state $(t, x) \in [0, T]\times\mathbb{R}^n$, we say that $\alpha^* \in \mathcal{A}(t, x)$ is an *optimal control* if $u(t, x) = J(t, x; \alpha^*)$.

A control process of the form $\alpha_s = a(s, X_s^{t,x})$ for $s \in [t, T]$ and some function $a : [0, T] \times \mathbb{R}^n \to A$ is called a *Markov control*, or *feedback control*. We shall always assume that the controls are of this type.

3.1.1. *Dynamic programming principle.* Compare two strategies:

- **I:** Using the optimal control $(\alpha_s^*)_{s\in[t,T]}$ over the interval $[t, T]$, versus:
- **II:** using an arbitrary control $(\alpha_s)_{s\in[t,t+h)}$ over the interval $[t, t + h)$ (where h represents a small time interval) and then using the optimal control $(\alpha_s^*)_{s\in[t+h,T]}$ over the interval $[t + h, T]$.

Strategy I is at least as good as strategy II, leading to

$$u(t, x) \geqslant \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, \alpha_s)\,\mathrm{d}s + u(t + h, X_{t+h}^{t,x})\right]. \tag{3.3}$$

We suppose we get equality if we maximise the RHS over α, leading to the *Bellman equation*

$$u(t, x) = \sup_{(\alpha_s)_{s\in[t,t+h)}} \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, \alpha_s)\,\mathrm{d}s + u(t + h, X_{t+h}^{t,x})\right].$$

Assuming that u is smooth enough, we apply Itô to write

$$u(t+h, X^{t,x}_{t+h}) = u(t,x) + \int_t^{t+h} \left(\frac{\partial u}{\partial t} + \mathcal{L}^\alpha u\right)(s, X^{t,x}_s)\,\mathrm{d}s + \text{local martingale},$$

where $\mathcal{L}^\alpha$ denotes the generator of the diffusion (3.1). Assuming the local martingale is a martingale, dividing by h and letting $h \to 0$ in the Bellman equation leads us to expect that the value function solves the Hamilton-Jacobi-Bellman (HJB) equation

$$\frac{\partial u}{\partial t}(t,x) + \sup_{\alpha \in A}\left[\mathcal{L}^\alpha u(t,x) + f(t,x,\alpha)\right] = 0, \quad u(T,x) = F(x). \tag{3.4}$$

3.1.2. *Martingale optimality principle.* Another way to state the dynamic programming principle and so arrive at the HJB equation is to invoke the idea that:

Principle 3.1 (Martingale principle of dynamic programming). *The process*

$$\left(\int_0^t f(s, X_s, \alpha_s)\,\mathrm{d}s + u(t, X_t)\right)_{t \in [0,T]}$$

is a super-martingale for any admissible control α, and a martingale for the optimal control α^*.

We can formalise this in the following theorem.

Theorem 3.2 (Davis-Varaiya martingale principle of optimal control). *Suppose the objective is (3.2). Suppose there exists a function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ which is $C^{1,2}$, such that $u(T,x) = F(x)$ for $x \in \mathbb{R}^n$. Suppose also that for any $\alpha \in \mathcal{A}(x)$ the process $R = (R_t)_{t \in [0,T]}$ defined by*

$$R_t := \int_0^t f(s, X_s, \alpha_s)\,\mathrm{d}s + u(t, X_t), \quad 0 \leqslant t \leqslant T, \tag{3.5}$$

is a super-martingale, and that for some $\alpha^* \in \mathcal{A}(x)$ the process R is a martingale. Then α^* is the optimal control, and the value of the problem starting at time zero with initial state $X_0 = x > 0$ is $u(x) \equiv u(0,x)$.

Proof. From the super-martingale property of R we have, for any $\alpha \in \mathcal{A}(x)$,

$$R_0 = u(0,x) \geqslant \mathbb{E}[R_T] = \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t)\,\mathrm{d}t + F(X_T)\right], \tag{3.6}$$

on using the boundary condition that $u(T,x) = F(x)$. Thus, for any admissible control, the value of the objective is less than or equal to $u(0,x)$. If we use the control α^*, then the value of the objective becomes equal to $u(0,x)$, since the super-martingale inequality in (3.6) becomes an equality, and this is the highest achievable expected value of the objective. Hence α^* is optimal.

□

Let us show how the martingale optimality principle leads to the HJB equation. Perform an Itô expansion of the process R in (3.5), assuming that u possesses sufficient regularity:

$$\mathrm{d}R_t = \left(f(t, X_t, \alpha_t) + \left(\frac{\partial u}{\partial t} + \mathcal{L}^\alpha\right)(t, X_t)\right)\mathrm{d}t + \text{ local martingale.}$$

For this to be a super-martingale we must have

$$f(t,x,\alpha) + \frac{\partial u}{\partial t}(t,x) + \mathcal{L}^\alpha(t,x) \leqslant 0$$

for any admissible control α. If we maximise the differential operator on the LHS over $\alpha \in A$ we get a martingale corresponding to the optimal control α^*, so we are once more led to the HJB equation (3.4).

3.1.3. *Verification theorem.* The HJB equation gives a *necessary* condition for optimality: *if* u is sufficiently smooth and *if* α^* is the optimal control, then u satisfies the HJB equation and $\alpha^*(t,x)$ realises the supremum in the HJB equation. The HJB equation also acts as a *sufficient* condition for optimality by virtue of the following *verification theorem.* Its content is that if one finds a candidate (and hence sufficiently smooth) solution of the HJB equation and an optimiser of the relevant differential operator in the HJB equation, then one has found the value function and optimal Markov control $\alpha^*(t,x)$.[1] The optimal control *process* is then $\alpha^*_s \equiv \alpha^*(s, X^*_s)$, for $s \in [t,T]$ and starting state $(t,x) \in [0,T] \times \mathbb{R}^n$, where $X^* \equiv X^{*,t,x}$ is the optimal state process given starting state $(t,x) \in [0,T] \times \mathbb{R}^n$.

Theorem 3.3 (Finite horizon verification theorem)**.** *Let* $w \in C^{1,2}([0,T]\times\mathbb{R}^n) \cap C^0([0,T]\times\mathbb{R}^n)$*. Suppose that* $w(T,x) = F(x)$ *and that there exists a function* $\alpha^* : [0,T] \times \mathbb{R}^n \to A$ *such that*

$$\frac{\partial w}{\partial t}(t,x) + \sup_{\alpha \in A}\left[\mathcal{L}^\alpha w(t,x) + f(t,x,\alpha)\right] = \frac{\partial w}{\partial t}(t,x) + \mathcal{L}^{\alpha^*} w(t,x) + f(t,x,\widehat{\alpha}) = 0.$$

Suppose further that the SDE

$$\mathrm{d}X_s = b(X_s, \alpha^*(s,X_s))\,\mathrm{d}s + \sigma(X_s, \alpha^*(s,X_s))\,\mathrm{d}W_s, \quad X_t = x$$

admits a unique solution $(X^*_s)_{s\in[t,T]} \equiv (\widehat{X}^{*,t,x}_s)_{s\in[t,T]}$*, and that the process* $(\widehat{\alpha}(s, \widehat{X}^{*,t,x}_s))_{s\in[t,T]}$ *lies in* $\mathcal{A}(t,x)$*. Then*

$$w = u \quad \text{on } [0,T] \times \mathbb{R}^n,$$

$\alpha^*(t,x)$ *is the optimal Markov control,* $(\alpha^*(s, \widehat{X}^{*,t,x}_s))_{s\in[t,T]}$ *is the optimal control process over* $[t,T]$*, and* $(\widehat{X}^{*,t,x}_s)_{s\in[t,T]}$ *is the optimal state process.*

Proof. Choose an arbitrary feedback control $\alpha \in \mathcal{A}(t,x)$ and a starting state $(t,x) \in [0,T] \times \mathbb{R}^n$. Using Itô we have

$$\begin{aligned} w(T, X^{t,x}_T) + \int_t^T f(s, X^{t,x}_s, \alpha_s)\,\mathrm{d}s &= w(t,x) \\ + \int_t^T \left(f(s, X^{t,x}_s, \alpha_s) + \left(\frac{\partial w}{\partial t} + \mathcal{L}^\alpha w\right)(s, X^{t,x}_s)\right) \mathrm{d}s &+ \text{local martingale,} \end{aligned}$$

where we write $\alpha_s \equiv \alpha(s, X^{t,x}_s)$ for brevity.

Since w satisfies the HJB equation, $f(t,x,\alpha) + \left(\frac{\partial w}{\partial t} + \mathcal{L}^\alpha w\right)(s, X^{t,x}_s) \leqslant 0$ for any admissible feedback control law. Using this, the boundary condition $w(T,x) = F(x)$ and taking the expectation (and assuming the local martingale is a martingale) we get

$$J(t,x;\alpha) = \mathbb{E}\left[\int_t^T f(s, X^{t,x}_s, \alpha_s)\,\mathrm{d}s + F(X^{t,x}_T)\right] \leqslant w(t,x).$$

Since the chosen control was arbitrary, we have

$$w(t,x) \geqslant \sup_{\alpha \in \mathcal{A}(t,x)} J(t,x;\alpha) = u(t,x).$$

[1]We are abusing notation and using the same symbol α^* for the optimal control process as well as the optimal feedback control function.

To obtain the reverse inequality, repeat the above arguments, this time choosing the control law $\alpha^*(t,x)$ which by assumption achieves the supremum in the HJB equation, to obtain

$$w(t,x) = J(t,x;\alpha^*). \tag{3.7}$$

We also have the trivial inequality

$$u(t,x) \geqslant J(t,x;\alpha^*), \tag{3.8}$$

so combining (3.7) and (3.8) we obtain

$$w(t,x) = u(t,x) = J(t,x;\alpha^*).$$

□

4. Dynamic programming solution to the Merton problem

4.1. **Merton terminal wealth problem.** The wealth process $X = X^\pi$ follows

$$\mathrm{d}X_t = rX_t\,\mathrm{d}t + \sigma\pi_t(\lambda\,\mathrm{d}t + \,\mathrm{d}W_t), \tag{4.1}$$

and the value function is

$$u(t,x) = \sup_{\pi\in\mathcal{A}(t,x)} \mathbb{E}[U(X_T)|X_t = x], \quad 0 \leqslant t \leqslant T,$$

for some utility function $U : \mathbb{R}_+ \to \mathbb{R}$ and admissible strategies $\pi \in \mathcal{A}(t,x)$ such that the wealth process is almost surely non-negative.

The HJB equation is

$$\sup_\pi \left[u_t(t,x) + (rx + \sigma\lambda\pi)u_x(t,x) + \frac{1}{2}\sigma^2\pi^2 u_{xx}(t,x) \right] = 0, \quad u(T,x) = U(x), \tag{4.2}$$

where subscripts denote partial derivatives.

Performing the maximisation over π gives the optimal feedback control function $\pi^*(t,x)$:

$$\pi^*(t,x) = -\frac{\lambda}{\sigma}\frac{u_x(t,x)}{u_{xx}(t,x)}. \tag{4.3}$$

The optimal control *process* $\pi^* = (\pi^*_t)_{0\leqslant t\leqslant T}$ is given by $\pi^*_t \equiv \pi^*(t, \widehat{X}^*_t)$, where $X^* \equiv X^{\pi^*}$ is the wealth process in (4.1) with $\pi_t = \pi^*_t$.[2]

Insert (4.3) into (4.2), converting the HJB equation to

$$u_t(t,x) + rxu_x - \frac{1}{2}\lambda^2\frac{u_x^2(t,x)}{u_{xx}(t,x)} = 0, \quad u(T,x) = U(x). \tag{4.4}$$

Example 4.1 (Logarithmic utility). Take $U(x) = \log x$. Seek a separable solution to (4.4) of the form

$$u(t,x) = \log x + f(t), \tag{4.5}$$

for some function $f(t)$. Using (4.5) in (4.3) and (4.4) gives

$$\pi^*(t,x) = \frac{\lambda}{\sigma}x,$$

and

$$f'(t) + r + \frac{1}{2}\lambda^2 = 0, \quad f(T) = 0,$$

[2]We abuse notation and use the same symbol, π^* for the optimal feedback control *function* $\pi^*(t,x)$ as for the optimal control *process* π^*.

implying

$$f(t) = \left(r + \frac{1}{2}\lambda^2\right)(T - t).$$

Hence, the value function is

$$u(t, x) = \log x + \left(r + \frac{1}{2}\lambda^2\right)(T - t),$$

in agreement with our earlier result (2.16).

The optimal trading strategy is

$$\pi_t^* = \frac{\lambda}{\sigma} X_t^*, \quad t \in [0, T],$$

so that the optimal fraction of wealth in the risky asset is

$$\theta_t^* := \frac{\pi_t^*}{X_t^*} = \frac{\lambda}{\sigma}, \quad 0 \leqslant t \leqslant T, \tag{4.6}$$

reproducing our earlier result (2.15).

Exercise 4.2. Repeat the above calculation with $U(x) = x^p/p, 0 < p < 1$, to reproduce the results in (2.18) and (2.19).

[Hint: seek a solution to the HJB equation of the form $u(t, x) = (x^p/p)f(t)$, for some function $f(t)$.]

Exercise 4.3. Repeat the above calculation with $U(x) = -\exp(-\alpha x), \alpha > 0$, to reproduce the results in (2.20) and (2.21).

[Hint: seek a solution to the HJB equation of the form $u(t, x) = -\exp(-\alpha x)f(t)$, for some function $f(t)$.]

Example 4.4. Two stock prices $S^{(i)} := (S_t^{(i)})_{0 \leqslant t \leqslant T}, i = 1, 2$ follow

$$\mathrm{d}S_t^{(i)} = \mu_i S_t^{(i)}\,\mathrm{d}t + \sigma_i S_t^{(i)}\,\mathrm{d}W_t^{(i)}, \quad i = 1, 2,$$

with $\mu_1, \mu_2, \sigma_1, \sigma_2$ constant, and $W^{(1)}, W^{(2)}$ independent Brownian motions. Stock $S^{(2)}$ is riskier, with $\mu_2 > \mu_1$ and $\sigma_2 > \sigma_1$. An agent decides which fraction $\theta := (\theta_t)_{0 \leqslant t \leqslant T}$ of his wealth X to place in the riskier stock. There is no risk-free asset.

The wealth process is $X \equiv X^{(\theta)}$ given by

$$\mathrm{d}X_t = X_t\left[(\mu_1(1 - \theta_t) + \mu_2\theta_t)\,\mathrm{d}t + \sigma_1(1 - \theta_t)\,\mathrm{d}W_t^{(1)} + \sigma_2\theta_t\,\mathrm{d}W_t^{(2)}\right].$$

Suppose the agent wishes to maximise expected utility of wealth at time T, with power utility. The value function is

$$u(t, x) := \sup_{\theta \in \mathcal{A}(t,x)} \mathbb{E}\left[\left(X_T^{(\theta)}\right)^p \middle| X_t^{(\theta)} = x\right],$$

where $0 < p < 1$, with the set $\mathcal{A}(t, x)$ of admissible strategies such that $X^{(\theta)} \geqslant 0$ almost surely. The HJB equation is then

$$u_t + \sup_\theta \left[(\mu_1(1 - \theta) + \mu_2\theta)\,xu_x + \frac{1}{2}\left(\sigma_1^2(1 - \theta)^2 + \sigma_2^2\theta^2\right)x^2u_{xx}\right] = 0,$$

the maximisation being performed over a scalar variable θ. Performing the maximisation over θ, the optimal control is to choose the optimal weight θ_t^* to be constant for all $t \in [0, T]$, and given by

$$\theta_t^* = \left(\sigma_1^2 + \frac{\mu_2 - \mu_1}{1 - p}\right)\frac{1}{\sigma_1^2 + \sigma_2^2}, \quad 0 \leqslant t \leqslant T.$$

(To obtain this, suppose u is separable: $u(t, x) = (x^p/p)f(t)$.)

4.2. **Merton problem with consumption and terminal wealth.** The wealth dynamics are

$$\mathrm{d}X_t = rX_t\,\mathrm{d}t + \sigma\pi_t(\lambda\,\mathrm{d}t + \mathrm{d}W_t) - c_t\,\mathrm{d}t.$$

The value function is

$$u(t,x) := \sup_{(\pi,c)\in\mathcal{A}(t,x)} \mathbb{E}\left[\left.\int_t^T \mathrm{e}^{-\delta s}U_1(c_s)\,\mathrm{d}s + U_2(X_T)\right| X_t = x\right],$$

where $U_i, i=1,2$ are increasing concave utility functions, and $\delta > 0$. The HJB equation is

$$u_t + \max_{(\pi,c)}\left[\mathrm{e}^{-\delta t}U_1(c) + \mathcal{L}^{(\pi,c)}u\right] = 0, \quad u(T,x) = U_2(x),$$

where $\mathcal{L}^{(\pi,c)}$ denotes the generator of X when using control (π,c).

Example 4.5 (Logarithmic utilities). Take $U_1(c) = \log c$, $U_2(x) = \log x$. Performing the maximisation over (π,c) in the HJB equation gives the optimal feedback control functions

$$c^*(t,x) = \frac{\mathrm{e}^{-\delta t}}{u_x(t,x)}, \quad \pi^*(t,x) = -\frac{\lambda}{\sigma}\frac{u_x(t,x)}{u_{xx}(t,x)}.$$

Inserting the optimal feedback controls into the HJB equation converts it to

$$-\mathrm{e}^{-\delta t}(1+\delta t+\log(u_x(t,x))) + u_t(t,x) + rxu_x(t,x) - \frac{1}{2}\lambda^2\frac{u_x^2(t,x)}{u_{xx}(t,x)} = 0, \quad u(T,x) = \log x.$$

This equation can be solved by looking for a solution of the form

$$u(t,x) = f(t)\log\left(\frac{x}{f(t)}\right) + g(t),$$

for functions $f(t)$ and $g(t)$ satisfying $f(T) = 1$ and $g(T) = 0$ respectively. Then one finds that f and g must satisfy

$$f'(t) = -\mathrm{e}^{-\delta t}, \quad g'(t) = \delta t\mathrm{e}^{-\delta t} - \left(r + \frac{1}{2}\lambda^2\right)f(t).$$

Solving these equations gives

$$\begin{aligned} f(t) &= 1 + \frac{1}{\delta}(\mathrm{e}^{-\delta t} - \mathrm{e}^{-\delta T}),\\ g(t) &= \left(T + \frac{1}{\delta}\right)\mathrm{e}^{-\delta T} - \left(t + \frac{1}{\delta}\right)\mathrm{e}^{-\delta t}\\ &+ \left(r + \frac{1}{2}\lambda^2\right)\left[\frac{1}{\delta^2}\left(\mathrm{e}^{-\delta t} - \mathrm{e}^{-\delta T}\right) + \left(1 - \frac{1}{\delta}\mathrm{e}^{-\delta T}\right)(T-t)\right]. \end{aligned}$$

The optimal consumption and investment feedback control functions become

$$c^*(t,x) = \frac{\mathrm{e}^{-\delta t}}{f(t)}x, \quad \pi^*(t,x) = \frac{\lambda}{\sigma}x,$$

so that the agent places a constant proportion of wealth in the stock at all times, and consumes at a rate that, as a fraction of current wealth, is a deterministic function of time.

5. The dual approach to optimal investment

We now describe the dual approach to solving portfolio problems, which exploits the martingale properties of the wealth process. This approach works even in non-Markovian models, so we shall illustrate it in a fairly general continuous-time market.

5.1. **Multi-dimensional complete Itô process market.** Consider a complete multi-dimensional Itô process market. We have a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, supporting a d-dimensional Brownian motion W, for some $d \in \mathbb{Z}_+$.

There is a cash asset associated with non-negative adapted interest rate process $r = (r_t)_{t\geqslant 0}$ and d stocks, with non-negative price processes S^i, $i = 1 \ldots, d$, evolving according to

$$\mathrm{d}S^i_t = S^i_t \left(\mu^i_t \,\mathrm{d}t + \sum_{j=1}^{d} \sigma^{ij}_t \,\mathrm{d}W^j_t \right), \quad i = 1, \ldots, d. \tag{5.1}$$

We can condense these equations into a single vector equation:

$$\mathrm{d}S_t = \mathrm{diag}_d(S_t)(\mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t), \tag{5.2}$$

where $\mathrm{diag}_d(S)$ denotes the $d \times d$ diagonal matrix with $S^1, \ldots, S^d$ along the diagonal.

The appreciation rates μ^i and the entries σ^{ij} $(i, j = 1, \ldots, d)$ of the invertible $d \times d$ volatility matrix σ are $\mathbb{F}$-adapted processes satisfying

$$\int_0^T \|\mu_t\| \,\mathrm{d}t < \infty, \quad \sum_{i=1}^{d}\sum_{j=1}^{d} \int_0^T \left(\sigma^{ij}_t\right)^2 \mathrm{d}t < \infty, \quad \text{a.s.}$$

Here $\|\cdot\|$ denotes the Euclidean norm, so that for instance,

$$\|\mu_t\|^2 = \left(\mu^1_t\right)^2 + \cdots + \left(\mu^d_t\right)^2.$$

The $\mathbb{R}^d$-valued adapted market price of risk (MPR) process λ is defined by

$$\lambda := \sigma^{-1}(\mu - r\mathbf{1}_d).$$

Associated with this MPR process is the deflator Y, defined by

$$Y_t := \exp\left(-\int_0^t r_s \,\mathrm{d}s\right) \mathcal{E}(-\lambda \cdot W)_t, \quad t \geqslant 0. \tag{5.3}$$

Here, $\lambda \cdot W \equiv \int_0^{\cdot} \lambda^{\mathrm{tr}}_s \,\mathrm{d}W_s$ denotes the stochastic integral, with λ^{tr} the transpose of the vector λ, and $\mathcal{E}(\cdot)$ denotes the stochastic exponential, so that

$$\mathcal{E}(-\lambda \cdot W)_t = \exp\left(-\int_0^t \lambda^{\mathrm{tr}}_s \,\mathrm{d}W_s - \frac{1}{2}\int_0^t \|\lambda^2_s\| \,\mathrm{d}s\right), \quad t \geqslant 0.$$

The deflator thus satisfies

$$\mathrm{d}Y_t = -Y_t(r_t \,\mathrm{d}t + \lambda^{\mathrm{tr}}_t \,\mathrm{d}W_t), \quad Y_0 = 1.$$

Note that since $Z := \mathcal{E}(-\lambda \cdot W)$ is a positive local martingale, Y is a positive super-martingale.

A small investor with initial wealth $x > 0$ invests in the stocks and cash, generating self-financing wealth process $X = (X_t)_{t\geqslant 0}$ satisfying

$$\begin{aligned}
\mathrm{d}X_t &= \sum_{i=1}^{d} H^i_t \,\mathrm{d}S^i_t + r_t\left(X_t - \sum_{i=1}^{d} H^i_t S^i_t\right) \mathrm{d}t \\
&= r_t X_t \,\mathrm{d}t + \sum_{i=1}^{d} \pi^i_t \left((\mu^i_t - r_t)\,\mathrm{d}t + \sum_{j=1}^{d} \sigma^{ij}_t \,\mathrm{d}W^j_t \right) \\
&= r_t X_t \,\mathrm{d}t + \pi^{\mathrm{tr}}_t \left[(\mu_t - r_t\mathbf{1}_d)\,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t\right], \quad X_0 = x > 0,
\end{aligned} \tag{5.4}$$

where H^i is the process for the number of shares held of stock $i \in \{1, \ldots, d\}$ and $\pi^i = H^i S^i$ is the wealth held in stock i, with associated vector $\pi := (\pi^1, \ldots, \pi^d)^{\mathrm{tr}}$, which forms the trading strategy, satisfying, for any $T > 0$,

$$\int_0^T \left(\|\pi_t^{\mathrm{tr}} \sigma_t\|^2 + |\pi^{\mathrm{tr}}(\mu_t - r_t \mathbf{1}_d)| \right) \mathrm{d}t < \infty, \quad \text{a.s.}$$

For a given $x > 0$ and π as above, the process $X \equiv X^{x,\pi}$ of (5.4) is called the wealth process corresponding to initial capital x and portfolio π.

We can, when the wealth process is strictly positive, parametrise the portfolio in terms of the stock proportion process $\theta := \pi/X$ and then we will sometimes write $X \equiv X^{x,\theta}$.

Recalling the deflator Y defined in (5.3), if we compute the dynamics of the deflated wealth process XY, we obtain that

$$X_t Y_t = x + \int_0^t Y_s(\pi_s^{\mathrm{tr}} \sigma_s - X_s \lambda_s^{\mathrm{tr}}) \,\mathrm{d}W_s, \quad t \geqslant 0. \tag{5.5}$$

If the wealth process is positive we can recast this in terms of the proportion of wealth process $\theta := \pi/X$, to write

$$X_t Y_t = x + \int_0^t X_s Y_s(\theta_s^{\mathrm{tr}} \sigma_s - \lambda_s^{\mathrm{tr}}) \,\mathrm{d}W_s, \quad t \geqslant 0. \tag{5.6}$$

We see that the deflated wealth is a local martingale.

Remark 5.1 (Including consumption). If the agent also consumes wealth at a non-negative consumption rate $c = (c_t)_{t \geqslant 0}$ then the wealth dynamics (5.4) are altered to

$$\mathrm{d}X_t = (r_t X_t - c_t) \,\mathrm{d}t + \pi_t^{\mathrm{tr}} \left[(\mu_t - r_t \mathbf{1}_d) \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t \right], \quad X_0 = x > 0,$$

and the deflated wealth equation (5.6) is altered to

$$X_t Y_t + \int_0^t c_s Y_s \,\mathrm{d}s = x + \int_0^t X_s Y_s(\theta_s^{\mathrm{tr}} \sigma_s - \lambda_s^{\mathrm{tr}}) \,\mathrm{d}W_s, \quad t \geqslant 0. \tag{5.7}$$

Thus, in this case, the deflated wealth plus cumulative deflated consumption is a local martingale.

For the most part, we shall restrict our attention to non-negative wealth processes, which motivates the following definition.

Definition 5.2 (Admissible strategies)**.** The class of portfolio strategies π (or portfolio-consumption strategies (π, c) if including consumption), starting with initial capital $x > 0$, such that the associated wealth process X satisfies

$$X_t \geqslant 0, \quad t \geqslant 0, \quad \text{a.s.}, \quad X_0 = x,$$

will be called *admissible*, and denoted by $\mathcal{A}(x)$.

If $\pi \in \mathcal{A}(x)$, then the process in (5.6) (or (5.7) if including consumption) is a non-negative $\mathbb{P}$-local martingale and hence a supermartingale, so satisfies

$$\mathbb{E}\left[X_T Y_T\right] \leqslant x, \tag{5.8}$$

or

$$\mathbb{E}\left[X_T Y_T + \int_0^T c_t Y_t \,\mathrm{d}t \right] \leqslant x, \tag{5.9}$$

if including consumption.

These conditions will be used as constraints on allowable wealth processes in what follows.

5.2. **Utility maximisation from terminal wealth.** We consider the following problem. With zero consumption, maximise over admissible $\theta \in \mathcal{A}(x)$ the functional $J(x;\theta)$ representing expected utility of terminal wealth:

$$J(x;\theta) := \mathbb{E}\left[U(X_T)\right].$$

Denote the value function of this problem by

$$u(x) := \sup_{\theta\in\mathcal{A}(x)} J(x;\theta).$$

With zero consumption, the wealth dynamics are characterised according to (5.6):

$$X_tY_t = x + \int_0^t X_sY_s(\theta_s^{\mathrm{tr}}\sigma_s - \lambda_s^{\mathrm{tr}})\,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant T, \tag{5.10}$$

and the budget constraint is (5.8):

$$\mathbb{E}\left[X_TY_T\right] \leqslant x. \tag{5.11}$$

We introduce a Lagrange multiplier $y > 0$ whose role is to enforce this constraint and consider, for any $\theta \in \mathcal{A}(x)$, and any $x, y > 0$, the Lagrangian

$$L(X_T, y) := \mathbb{E}\left[U(X_T)\right] + y\left(x - \mathbb{E}[X_TY_T]\right) = \mathbb{E}[U(X_T) - yX_TY_T] + xy.$$

We can now maximise this pointwise over X_T and y. The first order conditions for an optimum give the optimal terminal wealth X_T^* as satisfying

$$U'(X_T^*) = yY_T \quad\Longleftrightarrow\quad X_T^* = I(yY_T), \tag{5.12}$$

and optimising over the Lagrange multiplier yields that X_T^* also satisfies

$$\mathbb{E}[X_T^*Y_T] = x. \tag{5.13}$$

That is, at the optimum, the process X^*Y is a martingale. This condition can be used in (5.12) to fix the value of the Lagrange multiplier, thus giving a complete characterisation of the optimal terminal wealth. Using (5.12) in (5.13) gives

$$\mathcal{X}(y) := \mathbb{E}\left[Y_TI(yY_T)\right] = x.$$

Denote the inverse of $\mathcal{X}(\cdot)$ by $\mathcal{Y}(\cdot)$, so that

$$\mathcal{X}(y) = x \quad\Longleftrightarrow\quad y = \mathcal{Y}(x).$$

Thus, the optimal terminal wealth can be expressed as

$$X_T^*(x) = I(\mathcal{Y}(x)Y_T). \tag{5.14}$$

Then the *optimal wealth process* is obtained from the fact that X^*Y is a martingale:

$$X_t^* = \frac{1}{Y_t}\mathbb{E}\left[X_T^*Y_T\middle|\,\mathcal{F}_t\right], \quad 0 \leqslant t \leqslant T,$$

and the optimal strategy θ^* is obtained from (5.10) with $X = X^*$:

$$X_t^*Y_t = x + \int_0^t X_s^*Y_s(\theta_s^{*\mathrm{tr}}\sigma_s - \lambda_s^{\mathrm{tr}})\,\mathrm{d}W_s, \quad 0 \leqslant t \leqslant T. \tag{5.15}$$

These ideas have been written down in abstract form. They come to life if we consider an example.

Example 5.3 (Logarithmic utility of terminal wealth). Take $U(x) = \log x$. This is the simplest case. Then we have $U'(x) = 1/x$, $I(y) = 1/y$, and hence

$$X_T^* = \frac{1}{yY_T}.$$

Then y is determined by the constraint $\mathbb{E}[Y_T X^*_T] = x$, giving $y = 1/x$. Hence

$$X^*_T = \frac{x}{Y_T},$$

and the optimal wealth process is given by

$$X^*_t = \frac{1}{Y_t}\mathbb{E}\left[Y_t X^*_T | \mathcal{F}_t\right] = \frac{x}{Y_t}, \quad 0 \leqslant t \leqslant T.$$

Comparing with (5.15), we see that the optimal strategy is given by

$$\theta^*_t = (\sigma^{\mathrm{tr}}_t)^{-1}\lambda_t, \quad 0 \leqslant t \leqslant T.$$

The primal value function can be directly computed using $u(x) = \mathbb{E}[\log X^*_T]$ to give

$$u(x) = \log x - \mathbb{E}\left[\log Y_T\right].$$

6. Full versus partial information models

In classical models of financial mathematics, one usually specifies a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leqslant t\leqslant T}$, and then writes down some stochastic process $S = (S_t)_{0\leqslant t\leqslant T}$ for an asset price, such that S is adapted to the filtration $\mathbb{F}$. A typical example would be the Black-Scholes (BS) model of a stock price, following the geometric Brownian motion (in the case of zero interest rate)

$$\mathrm{d}S_t = \sigma S_t(\lambda\,\mathrm{d}t + \mathrm{d}W_t), \tag{6.1}$$

where W is a $(\mathbb{P}, \mathbb{F})$-Brownian motion and the volatility $\sigma > 0$ and the Sharpe ratio λ are assumed to be known constants. Implicit in this set-up is the strong assumption that a financial agent is able to observe the Brownian motion process W, as well as the stock price process S. We refer to this as a *full information* scenario. In this case, an agent uses $\mathbb{F}$-adapted trading strategies in S, which is an $\mathbb{F}$-adapted process with known drift and diffusion coefficients.

We wish to relax the full information assumption. Suppose we now assume that the agent can only observe the stock price process, and not the Brownian motion W. Hence, the values of the parameters σ, λ are not known with certainty. Moreover, we wish to insist that the agent's trading strategies be adapted to the *observation filtration* $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0\leqslant t\leqslant T}$ generated by S. We refer to this as a *partial information* scenario.

In this case, the parameter λ would be regarded as an unknown constant whose value needs to be determined from price data. In principle, one would also have to apply this philosophy to the volatility σ, but we shall make the approximation that price observations are continuous, so that σ can be computed from the quadratic variation $[S]$ of the stock price, since we have

$$[S]_t = \sigma^2 S^2_t t, \quad t \in [0, T].$$

One way to model the uncertainty in our knowledge of the value of the (supposed constant) parameter λ is to take a so-called Bayesian approach. This means we consider λ to be an $\mathcal{F}_0$-measurable random variable with a given initial distribution (the *prior* distribution) conditional on $\widehat{\mathcal{F}}_0$. The prior distribution initialises the probability law of λ conditional on $\widehat{\mathcal{F}}_0$, and this is updated in the light of new price information, that is, as the observation filtration $\widehat{\mathbb{F}}$ evolves. (In the case that λ is some unknown process $(\lambda_t)_{0\leqslant t\leqslant T}$ (as opposed to an unknown constant), then we would consider it to be some $\mathbb{F}$-adapted process such that its starting value λ_0 has a given prior distribution conditional on $\widehat{\mathcal{F}}_0$.)

This is an example of a *filtering* problem: to compute the best estimate of a random process given observations up to time $t \in [0, T]$, and hence given the σ-algebra $\widehat{\mathcal{F}}_t, t \in$

$[0, T]$. In the case of the BS model (6.1), where we model λ as an $\mathcal{F}_0$-measurable random variable, we are interested in computing the conditional expectation

$$\widehat{\lambda}_t := \mathbb{E}[\lambda|\widehat{\mathcal{F}}_t], \quad t \in [0, T].$$

We shall see (though we shall not give a proof here of the underlying filtering theorem) that the effect of filtering is that the model (6.1) may be replaced by a model specified on the filtered probability space $(\Omega, \widehat{\mathcal{F}}_T, \widehat{\mathbb{F}}, \mathbb{P})$ and written as

$$\mathrm{d}S_t = \sigma S_t(\widehat{\lambda}_t \,\mathrm{d}t + \mathrm{d}\widehat{W}_t),$$

where $\widehat{W}$ is a $(\mathbb{P}, \widehat{\mathbb{F}})$-Brownian motion. This model may now be treated as a full information model, since both $\widehat{W}$ and $\widehat{\lambda}$ are $\widehat{\mathbb{F}}$-adapted processes. The price to be paid for restoring a full information scenario is that the constant parameter λ has been replaced by a random process $\widehat{\lambda}$. The procedure by which a partial information model is replaced with a tractable full information model under the observation filtration is typically only achievable in special circumstances, such as Gaussian prior distributions and certain linearity properties in the relation between the observable and unobservable processes, as we shall see in the next section.

6.1. **Drift parameter uncertainty.** How severe is the aforementioned issue of drift parameter uncertainty (equivalent to uncertainty in the MPR process λ if the volatility is assumed known) in the BS model (6.1)? The short answer is: extremely severe. This point has been well made by Rogers [23, 24] and by Monoyios [18] (the latter from which the following arguments are taken).

Consider an agent in the BS model (6.1) with zero interest rate (so $\lambda = \mu/\sigma$, where μ is the stock price drift) who attempts to infer the value of λ from observations of the share price. Assume (unrealistically, of course) for simplicity that the agent observes the stock returns continuously, and that the volatility σ is known. The agent records the normalised returns

$$\frac{\mathrm{d}S_t}{\sigma S_t} = \lambda \,\mathrm{d}t + \mathrm{d}W_t,$$

and uses these to estimate λ. Using observations over a time interval $[0, t]$, the best estimate of λ is $\bar{\lambda}(t)$ given by

$$\bar{\lambda}(t) = \frac{1}{t}\int_0^t \frac{\mathrm{d}S_s}{\sigma S_s} = \lambda + \frac{W_t}{t}. \tag{6.2}$$

The estimator is normally distributed, $\bar{\lambda}(t) \sim N(\lambda, 1/t)$, so $(\bar{\lambda}(t) - \lambda)/(1/\sqrt{t})$ is a standard normal random variable. Hence, a 95% confidence interval for λ is

$$\left[\bar{\lambda}(t) - \frac{1.96}{\sqrt{t}}, \bar{\lambda}(t) + \frac{1.96}{\sqrt{t}}\right].$$

Suppose that the true parameter values are $\mu = 20\%$ per annum and $\sigma = 20\%$ per annum, so that $\lambda = 1$. We ask, for how long do we have to observe the share price to be 95% certain that we know the value of λ to within 5% of its true value? That is, we require $|\bar{\lambda}(t) - \lambda| \leqslant 0.05$. This implies that

$$\bar{\lambda}(t) + \frac{1.96}{\sqrt{t}} - \left(\bar{\lambda}(t) - \frac{1.96}{\sqrt{t}}\right) = 0.1,$$

which gives $t \approx 1537$ years! This gives a measure of the severity of drift parameter uncertainty in lognormal models, and it is remarkable that (to the best of our knowledge) the above calculation does not appear in any of the standard financial mathematics texts (which the exception of the recent book by Rogers [24]).

7. Filtering theory

Filtering problems concern estimating something about an unobserved stochastic process X given observations of a related process Y. In particular, one seeks the conditional expectation $\mathbb{E}[X_t|\widehat{\mathcal{F}}_t], 0 \leqslant t \leqslant T$, where $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0\leqslant t\leqslant T}$ is the filtration generated by Y. This problem was solved for linear systems in continuous time by Kalman and Bucy [12]. Subsequent work sought generalisations to systems with nonlinear dynamics, see Zakai [26] for instance. Kailath [10] developed the so-called innovations approach to linear filtering, which formulated the problem in the context of martingale theory. This approach to nonlinear filtering was given a definitive treatment by Fujisaki, Kallianpur and Kunita [8]. Textbook treatments can be found in Bain and Crisan [1], Kallianpur [11], Lipster and Shiryaev [15, 16] and Rogers and Williams [25], Chapter VI.8 (which follows the program of Fujisaki, Kallianpur and Kunita [8]).

The setting is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leqslant t\leqslant T}$. All processes are assumed to be $\mathbb{F}$-adapted. Note that $\mathbb{F}$ is not the observation filtration. Let us call $\mathbb{F}$ the *background filtration*. We consider two processes, both taken to be one-dimensional (for simplicity):

- a *signal process* $X = (X_t)_{0\leqslant t\leqslant T}$ which is not directly observable;
- an *observation process* $Y = (Y_t)_{0\leqslant t\leqslant T}$, which is observable and somehow correlated with X, so that by observing Y we can say something about the distribution of X.

Let $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0\leqslant t\leqslant T}$ denote the observation filtration generated by Y. That is,

$$\widehat{\mathcal{F}}_t := \sigma(Y_s; 0 \leqslant s \leqslant t), \quad 0 \leqslant t \leqslant T.$$

The *filtering problem* is to *compute the conditional distribution* of the signal $X_t, t \in [0, T]$, given observations up to that time. Or, equivalently, to compute the conditional expectation

$$E[f(X_t)|\widehat{\mathcal{F}}_t], \quad 0 \leqslant t \leqslant T,$$

where $f : \mathbb{R} \to \mathbb{R}$ is some test function.

To proceed further, we need to specify some particular model for the observation and signal processes. We shall specialise to the case where both signal and observations follow linear SDEs, and where the initial distribution of the signal is Gaussian. This leads to the celebrated Kalman-Bucy filter. See for instance Bain and Crisan [1] for a proof.

7.1. Linear observations and linear signal.

Suppose that the signal process has a Gaussian initial distribution and, for deterministic functions $A(\cdot), C(\cdot), G(\cdot)$, assume that the signal and observation processes follow

$$\begin{aligned} \mathrm{d}X_t &= A(t)X_t\,\mathrm{d}t + C(t)\,\mathrm{d}B_t, \quad X_0 \sim \mathrm{N}(\mu, \mathrm{v}), \\ \mathrm{d}Y_t &= G(t)X_t\,\mathrm{d}t + \mathrm{d}W_t, \quad Y_0 = 0, \end{aligned}$$

with B, W correlated BMs with constant correlation ρ, with X_0 independent of B and of W, and where $\mathrm{N}(\mu, \mathrm{v})$ denotes the normal probability law with mean μ and variance v. The two-dimensional process (X, Y) is then Gaussian, so the conditional distribution of the signal X given the observation filtration $\widehat{\mathbb{F}}$ (generated by the observation process Y) will also be normal (so, in particular, is completely characterised by its mean and variance), with conditional mean

$$\widehat{X}_t := \mathbb{E}[X_t|\widehat{\mathcal{F}}_t], \quad t \geqslant 0,$$

and conditional variance

$$V_t := \mathrm{var}[X_t|\widehat{\mathcal{F}}_t] = \mathbb{E}[(X_t - \widehat{X}_t)^2|\widehat{\mathcal{F}}_t] = \widehat{X_t^2} - (\widehat{X}_t)^2, \quad t \geqslant 0.$$

Observe that the initial values are

$$\widehat{X}_0 = \mathbb{E}[X_0|\widehat{\mathcal{F}}_0] = \mathbb{E}[X_0] = \mu,$$

and

$$V_0 = \mathbb{E}[(X_0 - \widehat{X}_0)^2|\widehat{\mathcal{F}}_0] = \mathbb{E}[(X_0 - \mu)^2] = \mathrm{var}(X_0) = \mathrm{v}.$$

The problem then boils down to finding an algorithm for computing the sufficient statistics $\widehat{X}_t, V_t$ from their initial values $\widehat{X}_0 = \mu$, $V_0 = \mathrm{v}$.

Theorem 7.1 (One-dimensional Kalman-Bucy filter)**.** *On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $\mathbb{F} = (\mathcal{F}_t)_{0\leqslant t\leqslant T}$, let $X = (X_t)_{0\leqslant t\leqslant T}$ be an $\mathbb{F}$-adapted signal process satisfying*

$$\mathrm{d}X_t = A(t)X_t\,\mathrm{d}t + C(t)\,\mathrm{d}B_t,$$

and let $Y = (Y_t)_{0\leqslant t\leqslant T}$ be an $\mathbb{F}$-adapted observation process satisfying

$$\mathrm{d}Y_t = G(t)X_t\,\mathrm{d}t + \mathrm{d}W_t, \quad Y_0 = 0,$$

where B, W are $\mathbb{F}$-Brownian motions with correlation ρ, and the coefficients $A(\cdot)$, $C(\cdot)$, $G(\cdot)$ are deterministic functions satisfying

$$\int_0^T \left(|A(t)| + C^2(t) + G^2(t)\right)\,\mathrm{d}t < \infty.$$

Define the observation filtration $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0\leqslant t\leqslant T}$ by

$$\widehat{\mathcal{F}}_t := \sigma(Y_s; 0 \leqslant s \leqslant t).$$

Suppose X_0 is an $\mathcal{F}_0$-measurable random variable, and that the distribution of X_0 is Gaussian with mean μ and variance v*, independent of B and W. Then the conditional expectation $\widehat{X}_t := \mathbb{E}[X_t|\widehat{\mathcal{F}}_t]$ for $0 \leqslant t \leqslant T$ satisfies*

$$\mathrm{d}\widehat{X}_t = A(t)\widehat{X}_t\,\mathrm{d}t + [G(t)V_t + \rho C(t)]\,\mathrm{d}N_t, \quad \widehat{X}_0 = \mu,$$

where $N = (N_t)_{0\leqslant t\leqslant T}$ is the innovations process, an $\widehat{\mathbb{F}}$-Brownian motion satisfying the defining relation

$$\mathrm{d}N_t := \mathrm{d}Y_t - G(t)\widehat{X}_t\,\mathrm{d}t,$$

and $V_t := \mathrm{var}[X_t|\widehat{\mathcal{F}}_t]$, for $0 \leqslant t \leqslant T$, is the conditional variance, which is independent of $\widehat{\mathcal{F}}_t$ and satisfies the deterministic Riccati equation

$$\frac{\mathrm{d}V_t}{\mathrm{d}t} = (1-\rho^2)C^2(t) + 2\left[A(t) - \rho C(t)G(t)\right]V_t - G^2(t)V_t^2, \quad V_0 = \mathrm{v}.$$

A multi-dimensional version of the Kalman-Bucy filter can be derived along similar lines to the one-dimensional case. See Theorem V9.2 in Fleming and Rishel [7], for instance.

Theorem 7.2 (Multi-dimensional Kalman-Bucy filter)**.** *Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $\mathbb{F} = (\mathcal{F})_{0\leqslant t\leqslant T}$, and two $\mathbb{F}$-adapted processes X, Y as given below.*

Let $X = (X_t)_{0\leqslant t\leqslant T}$ be an n-dimensional signal process satisfying

$$\mathrm{d}X_t = A(t)X_t\,\mathrm{d}t + C(t)\,\mathrm{d}B_t, \quad X_0 \sim \mathrm{N}(\mu, \mathrm{v}), \quad \textit{(linear signal)}, \tag{7.1}$$

where $X_0 \sim \mathrm{N}(\mu, \mathrm{v})$ denotes an n-dimensional $\mathcal{F}_0$-measurable Gaussian vector with mean $\mu \in \mathbb{R}^n$ and covariance matrix $\mathrm{v} \in \mathbb{R}^n \times \mathbb{R}^n$, independent of the d-dimensional Brownian motion B, and where $A(t) \in \mathbb{R}^n \times \mathbb{R}^n$, $C(t) \in \mathbb{R}^n \times \mathbb{R}^d$.

Let $Y = (Y_t)_{0\leqslant t\leqslant T}$ be an m-dimensional observation process satisfying

$$\mathrm{d}Y_t = G(t)X_t\,\mathrm{d}t + D(t)\,\mathrm{d}W_t, \quad Y_0 = 0, \quad \textit{(linear observations)},$$

where $G(t) \in \mathbb{R}^m \times \mathbb{R}^n$, $C(t) \in \mathbb{R}^m \times \mathbb{R}^k$, and B is a k-dimensional Brownian motion independent of B and X_0.

We assume that A, C, G, D are bounded on bounded intervals, that $DD^\top$ is non-singular, and that $(D(t)D(t)^\top)^{-1}$ is bounded on every bounded t-interval.

Let $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{0\leqslant t\leqslant T}$ denote the observation filtration generated by Y, so that $\widehat{\mathcal{F}}_t = \sigma(Y_s; 0 \leqslant s \leqslant t)$.

The conditional expectation vector $\widehat{X}_t := \mathbb{E}[X_t|\widehat{\mathcal{F}}_t], 0 \leqslant t \leqslant T$, satisfies the SDE

$$\begin{aligned} \mathrm{d}\widehat{X}_t &= A(t)\widehat{X}_t\,\mathrm{d}t + V_tG^{\mathrm{T}}(t)\left(D(t)D^{\mathrm{T}}(t)\right)^{-1}\left(\mathrm{d}Y_t - G(t)\widehat{X}_t\,\mathrm{d}t\right), \quad \widehat{X}_0 = \mu, \\ &= A(t)\widehat{X}_t\,\mathrm{d}t + V_tG^\top(t)\left(D(t)D^{\mathrm{T}}(t)\right)^{-1}\mathrm{d}N_t, \quad \widehat{X}_0 = \mu, \end{aligned} \tag{7.2}$$

where N is the innovations process, defined by

$$N_t := Y_t - \int_0^t G(s)\widehat{X}_s\,\mathrm{d}s, \quad 0 \leqslant t \leqslant T,$$

and satisfying

$$N_t = \int_0^t D(s)\,\mathrm{d}\widehat{W}_s, \tag{7.3}$$

where $\widehat{W}$ is a standard k-dimensional $\widehat{\mathbb{F}}$-Brownian motion.

The error $X_t - \widehat{X}_t$ is independent of $\widehat{\mathcal{F}}_t$, and the error covariance

$$V_t := \mathbb{E}[(X_t - \widehat{X}_t)(X_t - \widehat{X}_t)^\top|\widehat{\mathcal{F}}_t] = \mathrm{var}[X_t|\widehat{\mathcal{F}}_t],$$

satisfies the deterministic matrix Riccati equation

$$\frac{\mathrm{d}V_t}{\mathrm{d}t} = A(t)V_t + V_tA^\top(t) - V_tG^\top(t)(D(t)D^\top(t))^{-1}G(t)V_t + C(t)C^\top(t), \quad V_0 = \mathrm{v}.$$

Remark 7.3. Notice that:

- by (7.3) we can rewrite (7.2) as
$$\mathrm{d}\widehat{X}_t = A(t)\widehat{X}_t\,\mathrm{d}t + V_tG^\top(t)\left(D(t)D^\top(t)\right)^{-1}D(t)\,\mathrm{d}\widehat{W}_t, \quad \widehat{X}_0 = \mu,$$
which is a linear SDE of the same type as (7.1);
- since $X, \widehat{X}$ satisfy (7.1), (7.2) and X_0 is Gaussian, then $X_t, \widehat{X}_t$ are Gaussian vectors for each t, and the error $X_t - \widehat{X}_t$ is also Gaussian: $X_t - \widehat{X}_t$ has mean 0 and covariance V_t, and $\mathrm{Law}(X_t|\widehat{\mathcal{F}}_t) = \mathrm{N}(\widehat{X}_t, V_t)$.

7.2. **Merton problem with uncertain drift.** We consider the Merton problem when the agent has uncertainty over the true value of the drift parameter. Optimal investment models under partial information have been considered by many authors. We refer the reader to Rogers [23] and Björk, Davis and Landén [3], for example.

A stock price process $S = (S_t)_{0\leqslant t\leqslant T}$ follows

$$\mathrm{d}S_t = \sigma S_t(\lambda\,\mathrm{d}t + \mathrm{d}W_t), \tag{7.4}$$

on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} := (\mathcal{F}_t)_{0\leqslant t\leqslant T}$, with $W = (W_t)_{0\leqslant t\leqslant T}$ an $\mathbb{F}$-Brownian motion. For simplicity take the interest rate to be zero.

Define the process $\xi = (\xi_t)_{0\leqslant t\leqslant T}$, by

$$\xi_t := \frac{1}{\sigma}\int_0^t \frac{\mathrm{d}S_s}{S_s} = \lambda t + W_t, \quad t \in [0, T].$$

The process ξ will shortly be considered as the observation process in a filtering framework, corresponding to noisy observations of λ, with W representing the noise.

In a partial information model with continuous stock price observations, an agent must use $\widehat{\mathbb{F}}$-adapted trading strategies, where where $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0 \leqslant t \leqslant T}$ is the observation filtration, defined by

$$\widehat{\mathcal{F}}_t := \sigma(\xi_s; 0 \leqslant s \leqslant t) = \sigma(S_s; 0 \leqslant s \leqslant t), \quad t \in [0,T].$$

Then σ is known from the quadratic variation of S, but λ is an unknown constant, and hence modelled as an $\mathcal{F}_0$-measurable random variable. We assume the distribution of λ is Gaussian, $\lambda \sim \mathrm{N}(\lambda_0, \mathrm{v}_0)$, independent of W.

We are faced with a Kalman-Bucy type filtering problem whose unobservable signal process is the market price of risk λ. The signal process SDE is

$$\mathrm{d}\lambda = 0, \tag{7.5}$$

and the observation process SDE is

$$\mathrm{d}\xi_t = \lambda \,\mathrm{d}t + \mathrm{d}W_t. \tag{7.6}$$

We apply Theorem 7.1 to the signal process λ in (7.5) and observation process ξ in (7.6). Then the optimal filter

$$\widehat{\lambda}_t := \mathbb{E}[\lambda|\widehat{\mathcal{F}}_t], \quad 0 \leqslant t \leqslant T,$$

satisfies

$$\mathrm{d}\widehat{\lambda}_t = \mathrm{v}_t \,\mathrm{d}\widehat{W}_t, \quad \widehat{\lambda}_0 = \lambda_0, \tag{7.7}$$

where

$$\mathrm{v}_t := \mathbb{E}[(\lambda - \widehat{\lambda}_t)^2|\widehat{\mathcal{F}}_t], \quad 0 \leqslant t \leqslant T,$$

is the conditional variance. This satisfies the Riccati equation

$$\frac{\mathrm{d}\mathrm{v}_t}{\mathrm{d}t} = -\mathrm{v}_t^2, \tag{7.8}$$

with initial value v_0, so that

$$\mathrm{v}_t = \frac{\mathrm{v}_0}{1 + \mathrm{v}_0 t}, \quad 0 \leqslant t \leqslant T. \tag{7.9}$$

The process $\widehat{W}$ is an $\widehat{\mathbb{F}}$-Brownian motion, the innovations process, satisfying

$$d\widehat{W}_t = \mathrm{d}\xi_t - \widehat{\lambda}_t \,\mathrm{d}t. \tag{7.10}$$

Using this in (7.7), the optimal filter can also be written in terms of the observable ξ as

$$\widehat{\lambda}_t = \frac{\lambda_0 + \mathrm{v}_0 \xi_t}{1 + \mathrm{v}_0 t}, \quad 0 \leqslant t \leqslant T. \tag{7.11}$$

The effect of the filtering is that the agent is now investing in a stock with dynamics given by $\mathrm{d}S_t = \sigma S_t \,\mathrm{d}\xi_t$ which, using (7.10), becomes

$$\mathrm{d}S_t = \sigma S_t(\widehat{\lambda}_t \,\mathrm{d}t + \mathrm{d}\widehat{W}_t). \tag{7.12}$$

Let our agent have power utility function $U(\cdot)$. She may invest a fraction of her wealth in shares and the remaining wealth in the cash account. The ($\widehat{\mathbb{F}}$-adapted) wealth process X follows

$$\mathrm{d}X_t = \sigma \theta_t X_t(\widehat{\lambda}_t \,\mathrm{d}t + \mathrm{d}\widehat{W}_t), \quad X_0 = x, \tag{7.13}$$

where θ_t is the proportion of wealth invested in shares at time $t \in [0,T]$, an $\widehat{\mathbb{F}}$-adapted process satisfying $\int_0^T \theta_t^2 \,\mathrm{d}t < \infty$ almost surely, and such that $X_t \geqslant 0$ almost surely for all $t \in [0,T]$. Denote by $\mathcal{A}(x)$ the set of such admissible strategies.

The objective is to maximise expected utility of terminal wealth over the $\widehat{\mathbb{F}}$-adapted admissible strategies. The value function is

$$u(x) := \mathbb{E}[U(X_T)|\widehat{\mathcal{F}}_0].$$

This may now be treated as a full information problem, with state dynamics given by (7.13).

Example 7.4 (Logarithmic utility). With $U(x) = \log x$, the dual approach to the terminal wealth problem immediately gives that the optimal $\widehat{\mathbb{F}}$-adapted process for the Merton problem with the MPR considered as an unknown Gaussian random variable is $\theta^* = (\theta^*_t)_{0\leqslant t\leqslant T}$, given by

$$\theta^*_t = \frac{\widehat{\lambda}_t}{\sigma}, \quad t \in [0,T],$$

where $\widehat{\lambda} = (\widehat{\lambda}_t)_{0\leqslant t\leqslant T}$ satisfies (7.7) and v_t is given by (7.9).

The classical Merton strategy is thus altered in that the constant λ is replaced by its filtered estimate $\widehat{\lambda}$. This result is only true with logarithmic utility. For other utility functions, there is an additional correction as well as the replacement $\lambda \to \widehat{\lambda}$. See Monoyios [19] for computations involving power utility.

8. Basis risk model

We shall study a simple example of an incomplete market in which the ideas of utility-based pricing can be illustrated with great clarity and explicit solutions. A number of papers [5, 9, 17, 18, 20, 21] have studied such basis risk models.

The setting is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$, where the filtration $\mathbb{F}$ is the $\mathbb{P}$-augmentation of that generated by a two-dimensional Brownian motion $(W, W^{\perp})$. A traded stock price $S := (S_t)_{0\leqslant t\leqslant T}$ follows a log-Brownian process given by

$$\mathrm{d}S_t = \sigma S_t(\lambda\,\mathrm{d}t + \mathrm{d}W_t), \tag{8.1}$$

where $\sigma > 0$ and λ are known constants. For simplicity, the interest rate is taken to be zero.

A non-traded asset price $Y := (Y_t)_{0\leqslant t\leqslant T}$ follows the correlated log-Brownian motion

$$\mathrm{d}Y_t = \eta Y_t(\theta\,\mathrm{d}t + \mathrm{d}B_t), \tag{8.2}$$

with $\eta > 0$ and θ known constants. The Brownian motion B is correlated with W according to

$$\mathrm{d}[B, W]_t = \rho\,\mathrm{d}t, \quad B = \rho W + \sqrt{1-\rho^2}W^{\perp}, \quad \rho \in [-1, 1].$$

The market prices of risk of the stock S (respectively, non-traded asset Y) are λ (respectively, θ).

A European contingent claim pays the non-negative random variable $h(Y_T)$ at time T, where h is a bounded continuous function. If $|\rho| = 1$, the model is complete and a BS-style perfect hedge is possible (as we shall show). But for $|\rho| \neq 1$ the market is incomplete.

Examples of underlying assets that are either not traded (or are difficult to trade) include weather indices or baskets of many stocks. There is no trade-able asset which can be used to perfectly replicate the claim payoff. Traders may resort to using a correlated traded asset to hedge the claim, where the correlation is presumed to be close to 1, in effect taking the traded asset as a perfect proxy for the non-traded one. A typical case is the hedging of a basket option using a futures contract on a stock index, where the composition of the basket and the index are not identical.

The set $\mathbb{M}$ of local martingale measures $\mathbb{Q}$ is defined via the density process $Z = (Z_t)_{0\leqslant t\leqslant T}$ given by

$$Z_t := \mathcal{E}(-\lambda \cdot W - \psi \cdot W^{\perp})_t, \quad 0 \leqslant t \leqslant T,$$

where $\mathcal{E}$ denotes the stochastic exponential, and $\psi = (\psi_t)_{0 \leqslant t \leqslant T}$ is a process satisfying $\int_0^T \psi_t^2 \,\mathrm{d}t < \infty$ a.s. If, in addition, Z is a martingale, then we may define probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ by

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t, \quad t \in [0, T].$$

The set $\mathbb{M}$ of martingale measures is then in one-to-one correspondence with the set of integrands ψ.

By the Girsanov theorem, the two-dimensional process $(W^{\mathbb{Q}}, W^{\mathbb{Q},\perp})$ defined by

$$W_t^{\mathbb{Q}} := B_t + \lambda t, \quad W_t^{\mathbb{Q},\perp} := W_t^{\perp} + \int_0^t \psi_u \,\mathrm{d}u, \quad 0 \leqslant t \leqslant T,$$

is a two-dimensional $\mathbb{Q}$-Brownian motion. Therefore, under $\mathbb{Q} \in \mathbb{M}$ the dynamics of the asset prices are

$$\mathrm{d}S_t = \sigma S_t \,\mathrm{d}W_t^{\mathbb{Q}}, \quad \mathrm{d}Y_t = \eta Y_t[(\theta - \rho\lambda - \sqrt{1-\rho^2}\psi_t)\,\mathrm{d}t + \mathrm{d}B_t^{\mathbb{Q}}],$$

where $W^{\mathbb{Q}}, B^{\mathbb{Q}}$ are correlated $\mathbb{Q}$-Brownian motions:

$$\langle W^{\mathbb{Q}}, B^{\mathbb{Q}} \rangle_t = \rho t, \quad 0 \leqslant t \leqslant T.$$

The traded asset price is a local $\mathbb{Q}$-martingale, but the drift of the non-traded asset is arbitrary and parametrised by the integrand ψ appearing in the density process of any ELMM $\mathbb{Q} \in \mathbb{M}$.

The minimal martingale measure $\mathbb{Q}^M$, which will feature in many of our formulae, corresponds to $\psi = 0$, so has density process with respect to $\mathbb{P}$ given by

$$\left.\frac{\mathrm{d}\mathbb{Q}^M}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}\left(-\lambda \cdot W\right)_t, \quad 0 \leqslant t \leqslant T.$$

Hence, under $\mathbb{Q}^M$, (S, Y) follow

$$\mathrm{d}S_t = \sigma S_t \,\mathrm{d}W_t^{\mathbb{Q}}, \quad \mathrm{d}Y_t = \eta Y_t[(\theta - \rho\lambda)\,\mathrm{d}t + \mathrm{d}B_t^{\mathbb{Q}^M}],$$

where $W^{\mathbb{Q}} \equiv W^{\mathbb{Q}^M}$ and $B^{\mathbb{Q}^M} = \rho W^{\mathbb{Q}} + \sqrt{1-\rho^2} W^{\perp}$ are correlated Brownian motions under $\mathbb{Q}^M$. The stock price S is a local $\mathbb{Q}^M$-martingale but this is not the case for the non-traded asset.

8.1. **Perfect correlation case.** In the perfect correlation case, $\rho = 1$, Y is effectively a traded asset, so no-arbitrage requires the $\mathbb{Q}^M$-drift of Y to be zero. Therefore, given σ, η, in the $\rho = 1$ case the Sharpe ratios of the assets are related by

$$\theta = \lambda. \tag{8.3}$$

In fact, with $\rho = 1$, $B = W$, so we have

$$\frac{Y_t}{Y_0} = \left(\frac{S_t}{S_0}\right)^{\eta/\sigma} \exp(ct), \quad 0 \leqslant t \leqslant T,$$

where c is given by

$$c = \frac{1}{2}\eta\left(\sigma - \eta\right).$$

In this case the market becomes complete, and perfect hedging is possible. Let the claim price process be $v(t, Y_t), 0 \leqslant t \leqslant T$, where $v : [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ is smooth enough to apply the Itô formula, so that

$$\mathrm{d}v(t, Y_t) = \left[v_t(t, Y_t) + \mathcal{L}^Y v(t, Y_t)\right]\mathrm{d}t + \eta Y_t v_y(t, Y_t)\,\mathrm{d}B_t,$$

where $\mathcal{L}^Y$ is the generator of the process Y, given by

$$\mathcal{L}^Y v(t,y) = \eta\theta y v_y(t,y) + \frac{1}{2}\eta^2 y^2 v_{yy}(t,y).$$

Form a portfolio with $H = (H_t)_{0\leqslant t\leqslant T}$ units of S plus cash and denote the corresponding wealth process by X. Then X satisfies

$$\mathrm{d}X_t = H_t\,\mathrm{d}S_t.$$

The replication conditions are

$$X_t = v(t,Y_t), \quad 0\leqslant t\leqslant T, \quad \mathrm{d}X_t = \mathrm{d}v(t,Y_t).$$

Standard arguments then show that to perfectly hedge the claim one must hold H_t shares of S at $t\in[0,T]$, given by

$$H_t = \frac{\eta}{\sigma}\frac{Y_t}{S_t}\frac{\partial v}{\partial y}(t,Y_t), \tag{8.4}$$

and the claim pricing function $v(t,y)$ satisfies

$$v_t(t,y) + \eta(\theta-\lambda)y v_y(t,y) + \frac{1}{2}\eta^2 y^2 v_{yy}(t,y) = 0, \quad v(T,y) = h(y).$$

But when $\rho = 1$, we have $\theta - \lambda = 0$, so we get the BS PDE, and

$$v(t,Y_t) = \mathrm{BS}(t,Y_t;\eta),$$

where $\mathrm{BS}(t,y;\sigma)$ denotes the BS option pricing formula at time t, with underlying asset price y and volatility σ. An important feature of (8.4) is that the perfect hedge does not require knowledge of the values of the parameters λ,θ.

8.2. **Utility-indifference valuation and hedging.** Now suppose the correlation is not perfect, so that the market is incomplete. We embed the problem in a utility maximisation framework. Let the agent have risk preferences expressed via the exponential utility function

$$U(x) = -\exp(-\alpha x), \quad x\in\mathbb{R}, \quad \alpha > 0.$$

The agent maximises expected utility of terminal wealth at time T, with a random endowment of n units of claim payoff. Define $\pi = (\pi_t)_{0\leqslant t\leqslant T}$ as the process of wealth in the stock, so that $\pi_t := H_t S_t$. The wealth process $X = (X_t)_{0\leqslant t\leqslant T}$ of a portfolio containing $H = (H_t)_{0\leqslant t\leqslant T}$ shares of stock S satisfies

$$\mathrm{d}X_t = \sigma\pi_t(\lambda\,\mathrm{d}t + \mathrm{d}W_t).$$

Given a starting time $t\in[0,T]$ the objective to be maximised is

$$J^{(n)}(t,x,y;\pi) = \mathbb{E}[U(X_T + nh(Y_T))|X_t = x, Y_t = y].$$

The value function is $u^{(n)}(t,x,y)$, defined by

$$\begin{aligned} u^{(n)}(t,x,y) &:= \sup_{\pi\in\mathcal{A}} J^{(n)}(t,x,y;\pi), \\ u^{(n)}(T,x,y) &= U(x+nh(y)). \end{aligned} \tag{8.5}$$

Denote the optimal trading strategy that achieves the supremum in (8.5) by $\widehat{\pi}^{(n)}$, and denote the optimal wealth process by $\widehat{X}^{(n)}$. We assume the set $\mathcal{A}$ of admissible strategies are those for which XZ is a martingale, where Z is the density process of any ELMM $\mathbb{Q}$.

Given $X_t = x, Y_t = y$, the *utility indifference price* per claim is $p^{(n)}(t,x,y)$, defined by

$$u^{(n)}(t, x - np^{(n)}(t,x,y), y) = u^{(0)}(t,x). \tag{8.6}$$

We allow for possible dependence on t, x, y of $p^{(n)}$ in the above definition, but with exponential preferences it turns out that there is no dependence on x, as we shall see.

The *optimal hedging strategy* is defined as the adjustment one makes to one's optimal portfolio strategy relative to the problem when $n = 0$. In terms of the variable $\pi := HS$, the optimal hedging strategy for n units of the claim is $\widehat{\pi} := (\widehat{\pi}_t)_{0\leqslant t\leqslant T}$ given by

$$\widehat{\pi}_t := \widehat{\pi}^{(n)}_t - \widehat{\pi}^{(0)}_t, \quad 0 \leqslant t \leqslant T. \tag{8.7}$$

The solution to the optimisation problem (8.5) is well-known, using a so-called distortion transformation (see Zariphopoulou [27]) to linearise the Hamilton-Jacobi-Bellman (HJB) equation for $u^{(n)}$. See [17] for more details of the computation in this model.

The HJB equation for the value function $u^{(n)}$ is

$$u^{(n)}_t + \max_{\pi}[\mathcal{L}^{X,Y} u^{(n)}] = 0,$$

where $\mathcal{L}^{X,Y}$ is the generator of the two-dimensional process (X, Y):

$$\mathcal{L}^{X,Y} f(t,x,y) = \sigma\lambda\pi f_x + \frac{1}{2}\sigma^2\pi^2 f_{xx} + \eta\theta y f_y + \frac{1}{2}\eta^2 y^2 f_{yy} + \rho\sigma\eta\pi y f_{xy}.$$

Performing the maximisation over π yields that the optimal trading strategy $\widehat{\pi}^{(n)}$ is given by $\widehat{\pi}^{(n)}_t = \widehat{\pi}^{(n)}(t, \widehat{X}^{(n)}_t, Y_t)$, where the function $\widehat{\pi}^{(n)} : [0,T] \times \mathbb{R} \times \mathbb{R}^+$ is given by

$$\widehat{\pi}^{(n)}(t,x,y) := -\left(\frac{\lambda u^{(n)}_x + \rho\eta y u^{(n)}_{xy}}{\sigma u^{(n)}_{xx}}\right). \tag{8.8}$$

Plugging this back into the Bellman equation gives the HJB PDE as

$$u^{(n)}_t + \mathcal{L}^Y u^{(n)} - \frac{\left(\lambda u^{(n)}_x + \rho\eta y u^{(n)}_{xy}\right)^2}{2u^{(n)}_{xx}} = 0, \quad u^{(n)}(T,x,y) = U(x + nh(y)), \tag{8.9}$$

where $\mathcal{L}^Y$ is the generator of Y.

We have the following representation for the value function and indifference price.

Proposition 8.1. [17]

The value function $u \equiv u^{(n)}$ and indifference price $p \equiv p^{(n)}$, given $X_t = x, Y_t = y$ for $t \in [0,T]$, are given by

$$u^{(n)}(t,x,y) = -\mathrm{e}^{-\alpha x - \frac{1}{2}\lambda^2(T-t)}\left[F(t,y)\right]^{1/(1-\rho^2)}, \tag{8.10}$$

$$F(t,y) = \mathbb{E}^{\mathbb{Q}^M}\left[\exp\left(-\alpha n(1-\rho^2)h(Y_T)\right)\middle|\, Y_t = y\right], \tag{8.11}$$

$$p^{(n)}(t,y) = -\frac{1}{\alpha n(1-\rho^2)}\log F(t,y). \tag{8.12}$$

Proof. The function $F(t,y)$ satisfies a linear PDE by virtue of the stochastic representation (8.11) and the Feynman-Kac theorem. It is easy to verify that the value function as given in the proposition then satisfies the HJB equation (8.9). The indifference price formula then follows from its definition. □

The indifference pricing function $p^{(n)}(t,y)$ satisfies

$$p^{(n)}_t + \eta(\theta - \rho\lambda)y p^{(n)}_y + \frac{1}{2}\eta^2 y^2 p^{(n)}_{yy} - \frac{1}{2}\eta^2 y^2 n\alpha(1-\rho^2)(p^{(n)}_y)^2 = 0, \tag{8.13}$$

with $p^{(n)}(T,y) = h(y)$. This is a semi-linear PDE, and in this sense the indifference pricing methodology constitutes a non-linear pricing rule.

Notice that for $\rho = 1$ we recover the BS PDE, as the market becomes complete.

For $n = 0$, the indifference pricing PDE becomes linear, and by the Feynman-Kac Theorem we obtain the following representation for the marginal utility-based price $\hat{p}(t,y) := \lim_{n\to 0} p^{(n)}(t,y)$:

$$\hat{p}(t,y) = \mathbb{E}^{\mathbb{Q}^M}[h(Y_T)|Y_t = y].$$

This is a special case of a general representation of the marginal price as the expectation of the payoff under the optimal dual measure for the problem with $n = 0$, as we shall explain in more detail in Section 10.1. For exponential utility the dual optimal measure is the minimal entropy measure $\mathbb{Q}^E$, which minimises the relative entropy $\mathcal{H}(\mathbb{Q}|\mathbb{P}) := \mathbb{E}^{\mathbb{Q}}[\log Z_T]$. In the basis risk model the minimal entropy measure $\mathbb{Q}^E$ coincides with the minimal martingale measure $\mathbb{Q}^M$. This is because the relative entropy between a martingale measure $\mathbb{Q} \in \mathcal{M}$ and $\mathbb{P}$ is given by

$$\mathcal{H}(\mathbb{Q},\mathbb{P}) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\left(\lambda^2 T + \int_0^T \psi_t^2 \,\mathrm{d}t\right)\right],$$

and this is clearly minimised by $\psi = 0$.

Given the form of the value function, it is easy to show that the expression (8.8) for the optimal control loses its dependence on x. Then, applying (8.7) gives the optimal hedging strategy for a position in n claims.

Proposition 8.2. *The optimal hedging strategy for a position in n claims is to hold $\widehat{H}_t$ shares at $t \in [0,T]$, given by*

$$\widehat{H}_t = -n\rho\frac{\eta}{\sigma}\frac{Y_t}{S_t}\frac{\partial p^{(n)}}{\partial y}(t,Y_t). \tag{8.14}$$

Note that for $\rho = 1$ we recover the perfect delta hedge (8.4), and (as already noted) the claim price then satisfies the BS PDE.

Proof of Proposition 8.2. The optimal trading strategy is given by applying (8.7), using (8.8) to compute the optimal feedback control (from which the optimal trading strategy is computed by evaluating the feedback control, at the current value of the state (X,Y)):

$$\widehat{\pi}^{(n)}(t,x,y) = -\left(\frac{\lambda u_x^{(n)} + \rho\eta y u_{xy}^{(n)}}{\sigma u_{xx}^{(n)}}\right). \tag{8.15}$$

Setting $n = 0$ we also have

$$\widehat{\pi}^{(0)}(t,x) = -\left(\frac{\lambda}{\sigma}\frac{u_x^{(0)}}{u_{xx}^{(0)}}\right). \tag{8.16}$$

We also have, from (8.10)

$$u^{(n)}(t,x,y) = u^{(0)}(t,x)\left[F(t,Y)\right]^{1/(1-\rho^2)},$$

and using this in conjunction with (8.12), we have

$$u^{(n)}(t,x,y) = u^{(0)}(t,x)\exp(-\alpha n p^{(n)}(t,y)).$$

We compute partial derivatives to insert into (8.15) and (8.16) and deduce that

$$\widehat{\pi}^{(n)}(t,x,y) - \widehat{\pi}^{(0)}(t,x) = -n\rho\frac{\eta y}{\sigma}p_y^{(n)}(t,y).$$

□

Note that the optimal strategy $\widehat{\pi}$ is the classical Merton trading strategy $\widehat{\pi}^{(0)}$ for the problem without the claim plus a correction which corresponds precisely to the utility-based hedging strategy. Note also that if $\rho = 1$, then we recover the perfect hedge (8.4).

8.3. **Residual risk process.** Suppose the agent trades n claims at time zero for price $p^{(n)}(0, Y_0)$ per claim. The agent hedges this position over $[0, T]$ using the strategy $(\widehat{H}_t)_{0\leqslant t\leqslant T}$. Her overall position has value process $R := (R_t)_{0\leqslant t\leqslant T}$ given by $R_t = \widehat{X}_t + np^{(n)}(t, Y_t)$, so that

$$\mathrm{d}R_t = \mathrm{d}\widehat{X}_t + n\,\mathrm{d}p^{(n)}(t, Y_t), \tag{8.17}$$

where $\widehat{X} = (\widehat{X}_t)_{0\leqslant t\leqslant T}$ is the value of the hedging portfolio in S, satisfying

$$\mathrm{d}\widehat{X}_t = \widehat{H}_t\,\mathrm{d}S_t, \quad \widehat{X}_0 = -np^{(n)}(0, Y_0).$$

Using this in (8.17) along with the Itô formula and the PDE satisfied by $p^{(n)}(t, y)$, we obtain

$$\mathrm{d}R_t = \frac{1}{2}\eta^2 n^2 \alpha(1-\rho^2) Y_t^2 \left(p_y^{(n)}\right)^2 (t, Y_t)\,\mathrm{d}t + \eta n\sqrt{1-\rho^2} Y_t p_y^{(n)}(t, Y_t)\,\mathrm{d}W_t^{\perp}, \tag{8.18}$$

with $R_0 = 0$. We call R the *residual risk* (or hedging error) process. The term in $\mathrm{d}W_t^{\perp}$, orthogonal to the Brownian increments driving the stock price, is interpreted as the unhedgeable component of risk. If $\rho = 1$ we see that the process R becomes riskless (recall that the interest rate is zero), reflecting the fact that the market incompleteness disappears in this case.

8.4. **Power series expansions for the indifference price and hedge.** We are interested in analysing the distribution of the terminal hedging error R_T. This is not possible in closed form, so our approach is to use the SDE (8.18) to simulate R over many asset price histories, and compute the distribution of terminal hedging error R_T. This programme was carried out in [17] and [18].

To simulate R via (8.18) efficiently, one may use analytic approximations for $p^{(n)}(t, y)$ and $p_y^{(n)}(t, y)$, in the form of power series expansions in powers of $a := -\alpha(1-\rho^2)n$. These arise from a Taylor expansion of the indifference pricing function

$$p^{(n)}(t, y) = \frac{1}{a}\log \mathbb{E}^{\mathbb{Q}^M}\left[\exp\left(ah(Y_T)\right)\middle|\, Y_t = y\right]. \tag{8.19}$$

For a random variable X, recall that its cumulant generating function (CGF) is $C_X(a) := \log \mathbb{E}\exp(aX)$. Using linearity of the expectation operator, it is not hard to see that the CGF has a Taylor expansion of the form

$$C_X(a) = \sum_{j=1}^{\infty} \frac{1}{j!} k_j(X) a^j,$$

where $k_j(X) \equiv k_j$ is the j^{th} *cumulant* of X. The cumulants are related to the central moments of X. For instance, writing

$$m_j(X) := \mathbb{E}(X^j), \quad \mu_j(X) := \mathbb{E}[(X - m_1)^j], \quad j \in \mathbb{N},$$

for the j^{th} raw and central moments, it is not hard to show that the first three cumulants are the mean, variance and skewness:

$$\begin{aligned} k_1(X) &= m_1(X), \\ k_2(X) &= \mu_2(X), \\ k_3(X) &= \mu_3(X). \end{aligned}$$

Since the pricing function (8.19) is proportional to the cumulant generating function of the payoff under the minimal measure, it is easy to generate an analytic formula for the indifference pricing function. In [18], Monoyios gives the following representation.

Proposition 8.3. *The indifference pricing function $p^{(n)}(t,y)$ has the power series expansion*

$$p^{(n)}(t,y) = \sum_{j=1}^{5} \frac{1}{j!} k_j\left(h(Y_T)\right) a^{j-1} + O(a^5), \tag{8.20}$$

where $a = -\alpha(1-\rho^2)n$ and k_j is the j^{th} cumulant of the payoff under Q^M, conditional on $Y_t = y$. The expansion is valid for model parameters satisfying

$$\mathbb{E}^{\mathbb{Q}^M}\left[\exp(ah(Y_T))|\, Y_t = y\right] \leqslant 2. \tag{8.21}$$

This means one can produce an accurate perturbation series for $p^{(n)}(t,y)$, as a series of BS-type formulae, which can be differentiated term by term to give an analytic approximation for $p_y^{(n)}(t,y)$. In particular, the leading order term in the price expansion is Davis' [4] marginal price. Once again, this shows how the non-linear pricing rule given by the indifference price reduces to the linear pricing rule of the marginal price as the number of claims tends to zero.

The terms in the expansion depend ultimately on the moments $m_j := \mathbb{E}^{\mathbb{Q}^M}[h^j(Y_T)|Y_t = y]$, $j \in \mathbb{N}$ and (in the case of $p_y^{(n)}(t,y)$) on their partial derivatives $\partial m_j := \partial m_j/\partial y$, $j \in \mathbb{N}$. These are easy to compute (in the case of a put option we give some results shortly) since, under $\mathbb{Q}^M$, and conditional upon $Y_t = y$, $\log Y_T$ is normally distributed: with $\mathrm{N}(m, \Sigma^2)$ denoting the normal probability law with mean m and variance Σ^2, we have

$$\begin{array}{rcl} \log Y_T & \sim & \mathrm{N}\left(\log y + b - \frac{1}{2}\Sigma^2, \Sigma^2\right), \\ b & = & \eta(\theta - \rho\lambda)(T-t), \\ \Sigma^2 & = & \eta^2(T-t). \end{array} \tag{8.22}$$

For the optimal hedging strategy, the explicit results are obtained by differentiating (8.20) with respect to y. If we denote by $\partial\kappa_j$ the partial derivative of κ_j with respect to y:

$$\partial\kappa_j \equiv \frac{\partial\kappa_j}{\partial y},$$

where κ_j denotes any of m_j, μ_j, k_j, then we have:

Corollary 8.4. *The partial derivative of the indifference price $p^{(n)}(t,y)$ with respect to y has the power series expansion*

$$\frac{\partial p^{(n)}}{\partial y}(t,y) = \sum_{j=1}^{5} \frac{1}{j!}(\partial k_j) a^{j-1} + O(a^5).$$

The partial derivatives of the cumulants are related to $\mu_j, \partial\mu_j$. For instance, up to $j = 3$ we have

$$\begin{array}{rcl} \partial k_1 & = & \partial m_1, \\ \partial k_2 & = & \partial \mu_2, \\ \partial k_3 & = & \partial \mu_3. \end{array}$$

(See [18] for full details and more formulae.)

The significance of the expansions is that they give easily computed closed form approximations for the indifference price and optimal hedge. In the specific case of a put option, we have the following formulae for the raw moments of the payoff under the minimal measure $\mathbb{Q}^M$.

Lemma 8.5. *For a put option, $h(y) = (K-y)^+$, where $K > 0$ is the strike price, the j^{th} moment $m_j := \mathbb{E}^{\mathbb{Q}^M}[h^j(Y_T)|Y_t = y]$, $j \in \mathbb{N}$, is given by*

$$m_j = \sum_{\ell=0}^{j} \left(\begin{array}{c} j \\ \ell \end{array}\right) (-y)^\ell K^{(j-\ell)} \exp\left[\ell\left(b + \frac{1}{2}(\ell-1)\Sigma^2\right)\right] \Phi(-d_1 - (\ell-1)\Sigma),$$

where $\Phi(\cdot)$ denotes the standard cumulative normal distribution function, and where

$$\begin{aligned} d_1 &= \frac{1}{\Sigma}\left[\log\left(\frac{y}{K}\right)+b+\frac{1}{2}\Sigma^2\right] \\ b &= \eta(\theta-\rho\lambda)(T-t), \\ \Sigma^2 &= \eta^2(T-t). \end{aligned}$$

Proof. For the put payoff, we have, for $j \in \mathbb{N}$,

$$\begin{aligned} (h(Y_T))^j &= \left((K-Y_T)^+\right)^j \\ &= (K-Y_T)^j \mathbb{1}_{\{Y_T \leqslant K\}} \\ &= \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^\ell K^{(j-\ell)} Y_T^\ell \mathbb{1}_{\{Y_T \leqslant K\}}. \end{aligned}$$

Given the lognormal distribution (8.22) of Y_T, it is easy to show that

$$\mathbb{E}^{\mathbb{Q}^M}\left[Y_T^\ell I_{\{Y_T \leqslant K\}}\middle| Y_t = y\right] = y^\ell \exp\left(\ell\left(b+\frac{1}{2}(\ell-1)\Sigma^2\right)\right)\Phi(-d_1-(\ell-1)\Sigma),$$

from which the result follows. □

Lemma 8.6. *Let $j \in \mathbb{N}$. For a put option payoff, $h(y) = (K-y)^+$, ∂m_j is given by*

$$\partial m_j = -\sum_{\ell=1}^{j} \binom{j}{\ell} (-y)^{(\ell-1)} K^{(j-\ell)} \exp\left(\ell\left(b+\frac{1}{2}(\ell-1)\Sigma^2\right)\right) \ell N(-d_1-(\ell-1)\Sigma).$$

Proof. This is a straightforward exercise in differentiation. □

This power series expansions for $p^{(n)}(t,y)$ and $p_y^{(n)}(t,y)$ give a closed form and extremely accurate (see [17]) computation of the optimal price and hedging strategy, with the leading order term in the expansion for the price being the marginal price, $\hat{p}(t,y) = \mathbb{E}^{\mathbb{Q}^M}[h(Y_T)|Y_t = y]$, of the claim.

8.5. **Optimal versus naive hedging.** In [17, 18], a comparison was made between hedging a claim with the optimal strategy versus with the BS-style "naive" strategy (8.4) which takes S as a good proxy for Y.

In the BS-style hedge, let us repeat the calculation leading to the residual risk SDE (8.18), but with the claim traded at the BS price $v(0,Y_0) = \mathrm{BS}(0,Y_0)$ per claim and hedged H^N, using the $\rho \to 1$ limit of the optimal hedging formula (8.14) (even though true value of ρ is *not* equal to 1). We then obtain the "naive" hedging error process R^N, following

$$\mathrm{d}R_t^N = n\eta Y_t(\theta-\lambda)v_y(t,Y_t)\,\mathrm{d}t + n\eta Y_t v_y(t,Y_t)[(\rho-1)\,\mathrm{d}W_t + \sqrt{1-\rho^2}\,\mathrm{d}W_t^\perp].$$

Once again, we note that this is not riskless, but becomes so if the true value of ρ is indeed 1. The "naive" trader hopes this proves a good approximation.

For the case when the agent sells a put option ($n = -1$) on the non-traded asset, in [18] Monoyios generated optimal and naive hedging error distributions (by simulating the processes R and R^N) using 10,000 asset price histories. These showed that the optimal hedge error distribution has a higher mean, lower standard deviation, and a higher median, than the naive hedge error distribution. The increased median, in particular, showed how the relative frequency of profits over losses is increased when hedging optimally. Figure 1 shows a typical path trajectory for one simulated asset price history.

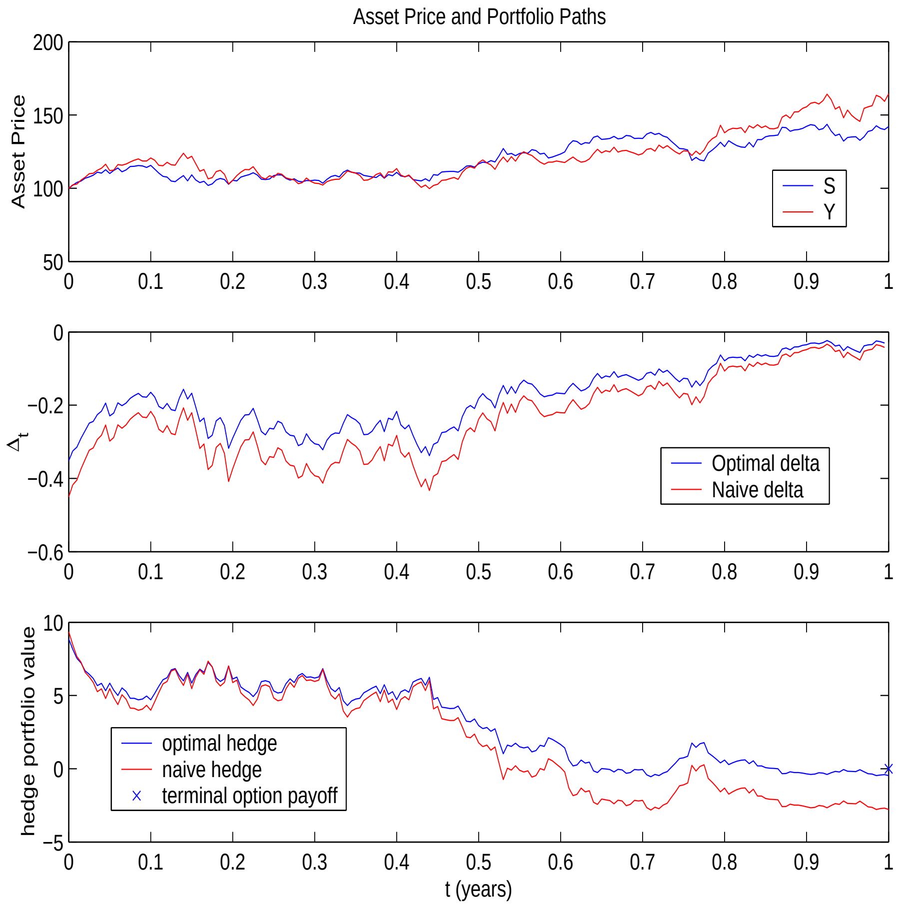

FIGURE 1. A typical trajectory when hedging a short put position with the optimal strategy or the naive strategy. The put is sold optimally for price $p^{(-1)}(0, Y_0)$, or naively for price $\mathrm{BS}(0, Y_0)$.

TABLE 1. Model parameters for histograms in Figure 2.

S_0	Y_0	K	r	μ	σ	ν	η	T
100	100	110	5%	13%	30%	10%	25%	1year

TABLE 2. Summary statistics for histograms in Figure 2.

$\mathbb{E}V_T$	$\mathrm{sd}(V_T)$	$\mathrm{med}(V_T)$	$\max(V_T)$	$\min(V_T)$
0.4346	9.5902	2.1621	33.40	-41.60
$\mathbb{E}V_T^N$	$\mathrm{sd}(V_T^N)$	$\mathrm{med}(V_T^N)$	$\max(V_T^N)$	$\min(V_T^N)$
-0.8574	10.3724	0.1216	39.11	-47.99

Suppose the agent sells a put option (so $n = -1$) on the non-traded asset. Figure 2 shows the optimal and naive hedging error distributions generated from 10,000 asset price histories, for $\rho = 0.75$. $\alpha = 0.01$, with the other parameters as in Table 1. Summary statistics for the hedge error distributions, in Table 2, show that the optimal hedge error distribution has a higher mean, lower standard deviation, and a higher median, than the naive hedge error distribution. The increased median, in particular, shows how the relative frequency of profits over losses is increased when hedging optimally.

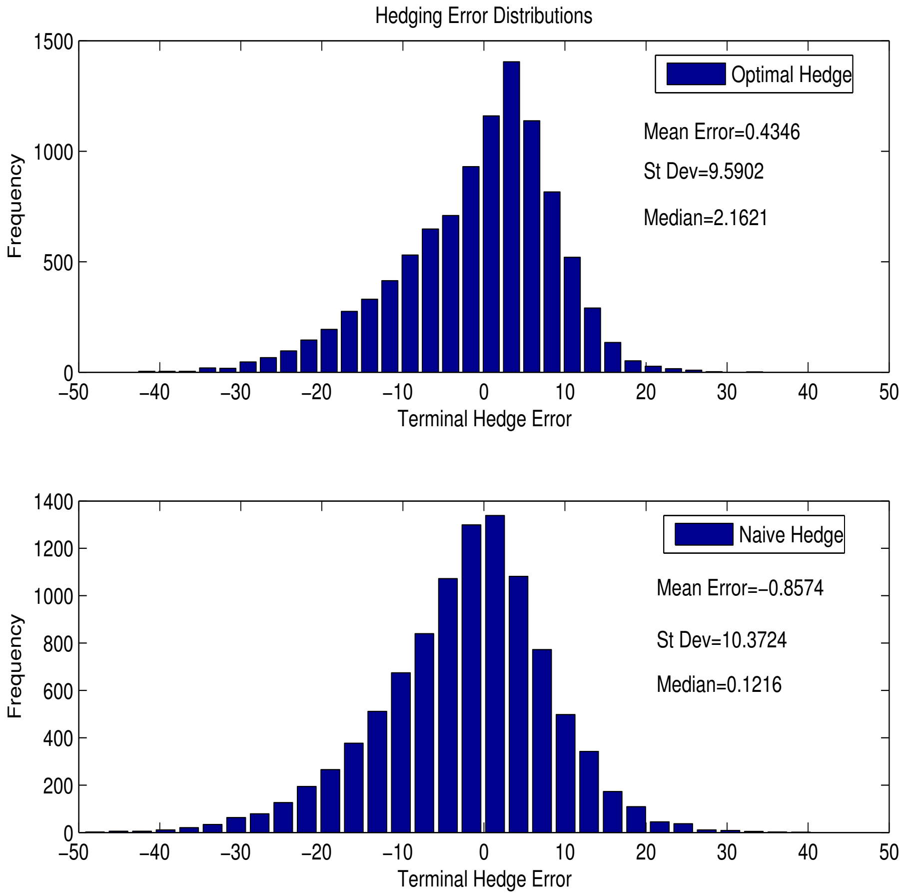

FIGURE 2. Hedging error distributions over 10,000 simulated asset price histories. A short put position is hedged, with $\rho = 0.75$, $\alpha = 0.01$. The remaining parameters are as in Table 1. The put is sold optimally for price $p^{(-1)}(0, Y_0) = 13.14$, and naively for price $\mathrm{BS}(0, Y_0) = 12.66$.

Thus, the hedging strategy in (8.14) is, at first sight, superior to the BS-style hedge (8.4). But the exponential hedge requires knowledge of λ, θ, which are impossible to estimate with any degree of accuracy (Monoyios [18]). This can ruin the effectiveness of indifference hedging, as shown in [18]. A way round this problem is presented by Monoyios [18, 20] using filtering theory.

9. The dual approach to optimal investment in incomplete markets

Here we briefly revisit the dual approach to solving utility maximisation problems. We studied this topic in Section 5 in the complete market case. Here, we give the main ideas in the incomplete market case. The major difference is that there are many equivalent local martingale measures (ELMMs), and the unique density process of the single ELMM of the complete market is replaced by the dual optimiser that achieves the infimum in a suitably defined dual problem. We shall not give a complete proof (which is difficult) of the duality, but shall proceed with the assumption that the primal and dual optimisers exist (this is the hard thing to establish in general, requiring a delicate demonstration of closure of the optimisation sets in a suitable topology). In subsequent sections, these ideas are applied to the case of utility maximisation with the random endowment of a claim payoff, to derive dual representations of utility-based prices, and we revisit the basis risk model via a dual approach.

9.1. **Abstract incomplete market.** We shall work in a general incomplete semi-martingale model, with zero interest rate (for simplicity), and with some (say) n-dimensional stock price process S, a semi-martingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{P})$.

Example 9.1 (Itô process model). An example would be a multi-dimensional Itô process model, with stock prices S^i, $i = 1 \ldots, n$, evolving according to

$$\mathrm{d}S_t^i = S_t^i \left(\mu_t^i \,\mathrm{d}t + \sum_{j=1}^{d} \sigma_t^{ij} \,\mathrm{d}W_t^j \right), \quad i = 1, \ldots, n, \tag{9.1}$$

or, as a single vector equation:

$$\mathrm{d}S_t = \mathrm{diag}_n(S_t)[\mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t], \tag{9.2}$$

where $\mathrm{diag}_n(S)$ denotes the $n \times n$ diagonal matrix with $S^1, \ldots, S^n$ along the diagonal, and W is a d-dimensional Brownian motion.

The appreciation rates μ^i and the entries σ^{ij} $(i = 1, \ldots, n,\ j = 1, \ldots, d)$ of the $n \times d$ volatility matrix σ are $\mathbb{F}$-adapted processes satisfying

$$\int_0^T \|\mu_t\| \,\mathrm{d}t < \infty, \quad \sum_{i=1}^{n} \sum_{j=1}^{d} \int_0^T \left(\sigma_t^{ij}\right)^2 \mathrm{d}t < \infty, \quad \text{a.s.}$$

Here $\| \cdot \|$ denotes the Euclidean norm, so that for instance,

$$\|\mu_t\|^2 = \left(\mu_t^1\right)^2 + \cdots + \left(\mu_t^n\right)^2 .$$

We assume that there exists (at least one) $\mathbb{R}^d$-valued progressively measurable process q such that the equations

$$\sigma_t q_t = \mu_t, \quad 0 \leqslant t \leqslant T, \quad \text{a.s.} \tag{9.3}$$

admit at least one solution. We call (9.3) the *market price of risk* (MPR) equations, and the processes q are called MPRs. The existence of a MPR satisfying (9.3) is equivalent to a no-arbitrage (NA) condition on the market model, and that this equivalence is a form of the first Fundamental Theorem of Asset Pricing (FTAP I) (see Chapter 0 of Karatzas [13] for more details).

Observe that (9.3) constitutes n equations for $d \geqslant n$ unknowns (the components of $q = (q^1, \cdots, q^d)^*$, where * denotes matrix transposition). Hence, in general, there will not be a unique MPR (and this is in one-to-one correspondence with the multiplicity of ELMMs), unless $n = d$ and the volatility matrix is invertible (and this is the special case of a complete market). This ends the Itô process example. You can read more about this model in, say, Karatzas [13], Chapter 0.

Returning to the general semi-martingale model, the wealth process X from trading the stocks is given by

$$X_t = X_0 + \int_0^t \theta_s \,\mathrm{d}S_s, \quad t \in [0, T], \tag{9.4}$$

where θ is the n-dimensional predictable and S-integrable process for the number of shares of each stock.

If the market is incomplete there are many ELMMs $\mathbb{Q} \sim \mathbb{P}$ such that S (and hence also X) is a local $\mathbb{Q}$-martingale. Denote the class of ELMMs by $\mathbb{M}$. Denote the density process of any ELMM $\mathbb{Q} \in \mathbb{M}$ by Z, a positive martingale:

$$Z_t = \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}, \quad t \in [0, T].$$

We shall identify any ELMM $\mathbb{Q}$ with its density process Z.

Example 9.2 (Itô process model, continued). In the Itô process model of Example 9.1, for any q satisfying (9.3), we define the positive local martingale

$$Z_t := \mathcal{E}(-q \cdot W)_t, \quad 0 \leqslant t \leqslant T, \tag{9.5}$$

satisfying

$$\mathrm{d}Z_t = -Z_t q_t^* \, \mathrm{d}W_t.$$

Here, $\mathcal{E}(\cdot)$ denotes the stochastic exponential:

$$\mathcal{E}(M) := \exp\left(-M - \frac{1}{2}\langle M\rangle\right),$$

for any continuous process M, as well as

$$(q \cdot W)_t := \int_0^t q_s^* \, \mathrm{d}W_s, \quad 0 \leqslant t \leqslant T,$$

for the stochastic integral, so that

$$\mathcal{E}(-q \cdot W)_t = \exp\left(-\int_0^t q_s^* \, \mathrm{d}W_s - \frac{1}{2}\int_0^t \|q_s\|^2 \, \mathrm{d}s\right), \quad 0 \leqslant t \leqslant T.$$

If $\mathbb{E}[Z_T] = 1$, then Z is a martingale and we define equivalent local martingale measures (ELMMs) $\mathbb{Q}$ via

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = Z_T.$$

Denote the set of ELMMs by $\mathbb{M}$. Under any $\mathbb{Q} \in \mathbb{M}$, the stock prices are local martingales. To see this, recall that the Girsanov theorem (Theorem 3.5.1 in Karatzas and Shreve [14]) implies that under any $\mathbb{Q} \in \mathbb{M}$, the process

$$W_t^{\mathbb{Q}} := W_t + \int_0^t q_s \, \mathrm{d}s, \quad 0 \leqslant t \leqslant T,$$

is Brownian motion, and hence we have

$$\mathrm{d}S_t = \mathrm{diag}_n(S_t)\sigma_t \, \mathrm{d}W_t^{\mathbb{Q}}.$$

Hence the stock prices are local $\mathbb{Q}$-martingales.

Returning to the general semi-martingale model, the process ZX is a local $\mathbb{P}$-martingale. In this section, we shall take admissible strategies θ to be those that yield non-negative wealth process:

Definition 9.3 (Admissible strategies)**.** The class of portfolio strategies θ, starting with initial capital $x > 0$, such that the associated wealth process X satisfies

$$X_t \equiv X_t(x) \geqslant 0, \quad 0 \leqslant t \leqslant T, \quad \text{a.s.}$$

will be called *admissible*, and denoted by $\mathcal{A}(x)$.

If $\theta \in \mathcal{A}(x)$, then the process ZX is a non-negative $\mathbb{P}$-local martingale and hence a super-martingale, so satisfies

$$\mathbb{E}\left[Z_T X_T\right] \leqslant x, \quad \text{for any } \mathbb{Q} \in \mathbb{M}. \tag{9.6}$$

This condition will be used as a constraint on allowable wealth processes in what follows

For now, we consider the case where the utility function U is defined on $\mathbb{R}^+$ (so keep logarithmic and power utility in mind). We shall restrict attention (for the present) to admissible strategies with non-negative wealth process, so we have the super-martingale constraint (9.6). We consider the following problem.

Problem 9.4 (Utility from terminal wealth). Denote by $\theta \in \mathcal{A}(x)$ the portfolio process. Maximise over admissible $\theta \in \mathcal{A}(x)$ the functional $J(x;\theta)$ representing expected utility of terminal wealth:

$$J(x;\theta) := \mathbb{E}\left[U(X_T)\right].$$

Denote the value function of this problem by

$$u(x) := \sup_{\theta \in \mathcal{A}(x)} J(x;\theta).$$

The wealth dynamics are characterised according to (9.4) and the budget constraint is (9.6). Introduce a Lagrange multiplier $y > 0$ whose role is to enforce this constraint and consider, for any $\theta \in \mathcal{A}(x)$, and any $x, y > 0$,

$$\begin{aligned} \mathbb{E}[U(X_T)] &\leqslant \mathbb{E}\left[U(X_T)\right] + y\left(x - \mathbb{E}[X_T Z_T]\right) \\ &= \mathbb{E}\left[U(X_T) - yZ_T X_T\right] + xy \\ &\leqslant \mathbb{E}\left[V(yZ_T)\right] + xy, \end{aligned} \tag{9.7}$$

where we have used (2.7). The inequality (9.7) motivates us to define the *dual problem* to the primal utility maximisation Problem 9.4, with dual value function

$$v(y) := \inf_{\mathbb{Q} \in \mathbb{M}} \mathbb{E}\left[V(yZ_T)\right], \quad y > 0. \tag{9.8}$$

Now, inequality (9.7) holds for all $\theta \in \mathcal{A}(x)$ and $\mathbb{Q} \in \mathbb{M}$, and for all $x, y > 0$. Then, maximising over trading strategies on the LHS of (9.7) and minimising over ELMMs on the RHS, leads to

$$v(y) \geqslant u(x) - xy, \quad \forall x > 0, y > 0,$$

and thus to

$$v(y) \geqslant \sup_{x>0}[u(x) - xy], \quad y > 0,$$

This suggests that $u(\cdot), v(\cdot)$ are conjugate (inheriting this property from $U(\cdot), V(\cdot)$). This can be made rigorous, and we get equality in (9.7) if and only if the primal and dual optimisers $\widehat{X}_T \equiv \widehat{X}_T(x)$ (we identify the optimal trading strategy with the resultant terminal wealth) and $\widehat{Z}_T \equiv \widehat{Z}_T(y)$ satisfy the martingale constraint

$$\mathbb{E}[\widehat{X}_T \widehat{Z}_T] = x \tag{9.9}$$

holds (so the first inequality becomes an equality) and also if and only if $\widehat{X}_T$ satisfies

$$U'(\widehat{X}_T) = y\widehat{Z}_T \Longleftrightarrow \widehat{X}_T = I(y\widehat{Z}_T), \tag{9.10}$$

so the second inequality becomes an equality. Thus (9.9) and (9.10) identify the *optimal terminal wealth* $\widehat{X}_T$, once we fix $y > 0$ by inserting (9.10) into (9.9):

$$\mathbb{E}\left[\widehat{Z}_T I(y\widehat{Z}_T)\right] = x. \tag{9.11}$$

To establish the conjugacy between $u(\cdot), v(\cdot)$, first write $v(\cdot)$ as

$$\begin{aligned} v(y) &= \mathbb{E}\left[V(y\widehat{Z}_T)\right], \quad y>0, \\ &= \mathbb{E}[U(I(y\widehat{Z}_T))] - y\mathbb{E}[\widehat{Z}_T I(y\widehat{Z}_T)] \\ &=: F(y) - y\mathcal{X}(y), \end{aligned} \tag{9.12}$$

where we have used (2.6) and where we have defined

$$F(y) := \mathbb{E}\left[U(I(y\widehat{Z}_T))\right], \quad \mathcal{X}(y) := \mathbb{E}\left[\widehat{Z}_T I(y\widehat{Z}_T)\right], \quad y>0.$$

Thus, the constraint (9.11) reads as

$$\mathcal{X}(y) = x.$$

Denote the inverse of $\mathcal{X}(\cdot)$ by $\mathcal{Y}(\cdot)$, so that

$$\mathcal{X}(y) = x \Longleftrightarrow y = \mathcal{Y}(x). \tag{9.13}$$

Thus, the optimal terminal wealth can be expressed as

$$\widehat{X}_T(x) = I(\mathcal{Y}(x)\widehat{Z}_T). \tag{9.14}$$

Then, in principle, the optimal wealth process is obtained from the from the fact that $\widehat{X}\widehat{Z}$ is a martingale:

$$\widehat{X}_t = \frac{1}{\widehat{Z}_t}\mathbb{E}\left[\left.\widehat{X}_T\widehat{Z}_T\right|\mathcal{F}_t\right], \quad 0 \leqslant t \leqslant T. \tag{9.15}$$

The primal value function may be expressed in the form

$$u(x) = \mathbb{E}[U(\widehat{X}_T(x))] = \mathbb{E}[U(I(\mathcal{Y}(x)\widehat{Z}_T))] = F(\mathcal{Y}(x)) = v(\mathcal{Y}(x)) + x\mathcal{Y}(x), \tag{9.16}$$

where we have used (9.12) to get the last equality.

Observe from (9.7) that we have

$$u(x) \leqslant v(y) + xy, \quad \text{for all } x>0, y>0.$$

Hence,

$$\sup_{x>0}[u(x) - xy] \leqslant v(y), \quad \text{for all } y>0. \tag{9.17}$$

Evaluating the primal value function u at $\mathcal{X}(y)$, for any $y>0$, and using (9.16) and (9.12), we have

$$u(\mathcal{X}(y)) = F(\mathcal{Y}(\mathcal{X}(y))) = F(y) = v(y) + y\mathcal{X}(y),$$

or

$$v(y) = u(\mathcal{X}(y)) - y\mathcal{X}(y),$$

which implies that

$$v(y) \leqslant \sup_{x>0}[u(x) - xy], \quad \text{for all } y>0. \tag{9.18}$$

From (9.17) and (9.18) we conclude that

$$v(y) = \sup_{x>0}[u(x) - xy], \quad y>0.$$

In other words, u, v are conjugate (inheriting this property from U, V).

Finally, the functions $\mathcal{X}, \mathcal{Y}$ are in fact related to the derivatives of the value functions v, u as we now show.

Differentiating $v(y) = \mathbb{E}[V(y\widehat{Z}_T)]$ with respect to y we have

$$v'(y) = \mathbb{E}[V'(y\widehat{Z}_T)\widehat{Z}_T] = -\mathbb{E}[\widehat{Z}_T I(y\widehat{Z}_T)] = -\mathcal{X}(y).$$

While for u we have, on differentiating the last equality in (9.16) and using $v' = -\mathcal{X}$,

$$u'(x) = v'(\mathcal{Y}(x))\mathcal{Y}'(x) + \mathcal{Y}(x) + x\mathcal{Y}'(x) = \mathcal{Y}(x).$$

Thus, the relations (9.13) between the initial wealth $x > 0$ and the Lagrange multiplier $y > 0$ may also be written as

$$y = u'(x), \quad x = -v'(y).$$

Moreover, the expression (9.14) for the optimal terminal wealth translates to the striking expression for the dual optimiser:

$$\widehat{Z}_T = \frac{U'(\widehat{X}_T(x))}{u'(x)}.$$

10. Optimal investment with random endowment in an incomplete market

We now specialise to the exponential utility function $U(x) = -\exp(-\alpha x)$ for $x \in \mathbb{R}$ and $\alpha > 0$ and consider the problem of optimal investment in an incomplete market when one has also to pay out at time T the payoff of a European claim, some $\mathcal{F}_T$-measurable random variable C. Let π denote the trading strategy (the wealth held in stocks). The set of admissible strategies $\mathcal{A}(x)$ are assumed to be such that XZ is a martingale, for any deflator $Z \in \mathcal{Z}$. (The set of deflators $\mathcal{Z}$ is in one-to-one correspondence with the set $\mathbb{M}$ of ELMMs.) The primal problem is defined by

$$u(x) := \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}[U(X_T - C)], \quad x \in \mathbb{R}, \tag{10.1}$$

and we assume that $\mathbb{E}[X_T Z_T] = x$, for any $Z \in \mathcal{Z}$.

Consider, for any admissible π, any deflator Z, any $x \in \mathbb{R}$ and any $y > 0$,

$$\begin{aligned}
\mathbb{E}[U(X_T - C)] &= \mathbb{E}\left[U(X_T - C)\right] + y\left(x - \mathbb{E}[X_T Z_T]\right) \\
&= \mathbb{E}\left[U(X_T - C) - yZ_T(X_T - C) - yZ_T C\right] + xy \\
&\leqslant \mathbb{E}\left[V(yZ_T) - yZ_T C\right] + xy.
\end{aligned}$$

We define the dual problem to the primal utility maximisation problem by

$$v(y) := \inf_{Z \in \mathcal{Z}} \mathbb{E}[V(yZ_T) - yZ_T C], \quad y > 0, \tag{10.2}$$

Now, (10.2) holds for any $\pi \in \mathcal{A}(x)$, any deflator $Z \in \mathcal{Z}$, any $x \in \mathbb{R}$ and any $y > 0$, so if we maximise the LHS over $\pi \in \mathcal{A}(x)$ and minimise the RHS over $Z \in \mathcal{Z}$, we have the usual inequality

$$u(x) \leqslant v(y) + xy, \quad \forall x \in \mathbb{R}, \quad y > 0.$$

Denote the dual minimiser in (10.2) by $\widehat{Z}^{(C)}$ and the optimal terminal wealth in (10.1) by $\widehat{X}^{(C)}_T$. We suppose (and this can be made rigorous, see for example, Delbaen et al [6]) that we get equality in (10.2) when we choose $X_T = \widehat{X}^{(C)}_T$ and $Z_T = \widehat{Z}^{(C)}_T$ such that $U'(\widehat{X}^{(C)}_T - C) = y\widehat{Z}^{(C)}_T$, with $y > 0$ fixed via $\mathbb{E}[\widehat{X}^{(C)}_T \widehat{Z}^{(C)}_T] = x$. Once again, the theory goes through as in the case without random endowment and we get the following theorem.

Theorem 10.1 (Optimal investment with random endowment in incomplete market)**.** *Define the primal value function by*

$$u(x) := \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}\left[U(X_T - C)\right], \quad x \in \mathbb{R}. \tag{10.3}$$

and suppose that $u(x) < \infty$ for all $x \in \mathbb{R}$.

Define the dual value function by

$$v(y) := \inf_{Z \in \mathcal{Z}} \mathbb{E}\left[V(yZ_T) - yZ_T C\right], \quad y > 0,$$

Then we have

(1) *The value functions u and v are conjugate:*

$$v(y) = \sup_{x\in\mathbb{R}}[u(x) - xy], \quad u(x) = \inf_{y>0}[v(y) + xy], \quad x \in \mathbb{R}, \quad y > 0.$$

(2) *The optimal terminal wealth $\widehat{X}_T^{(C)}$ and the dual minimiser $\widehat{Z}_T^{(C)}$ are related by*

$$U'(\widehat{X}_T^{(C)} - C) = y\widehat{Z}_T^{(C)} \iff \widehat{X}_T^{(C)} - C = I(y\widehat{Z}_T^{(C)}),$$

with $y > 0$ fixed via $\mathbb{E}[\widehat{X}_T^{(C)}\widehat{Z}_T^{(C)}] = x$, or

$$\mathcal{X}(y) = x - \Pi(C) \iff y = \mathcal{Y}(x - \Pi(C)), \quad \mathcal{Y} := \mathcal{X}^{-1}, \tag{10.4}$$

where the function $\mathcal{X}(\cdot)$ and the constant $\Pi(C)$ are defined by

$$\mathcal{X}(y) := \mathbb{E}\left[\widehat{Z}_T^{(C)} I(y\widehat{Z}_T^{(C)})\right], \quad y > 0, \quad \Pi(C) := \mathbb{E}^{\widehat{\mathbb{Q}}^{(C)}}[C].$$

(3) *The primal value function is given as*

$$u(x) = \mathbb{E}\left[U\left(I(\mathcal{Y}(x - \Pi(C))\widehat{Z}_T^{(C)})\right)\right] = F(\mathcal{Y}(x - \Pi(C))),$$

where F is defined by

$$F(y) := \mathbb{E}\left[U(I(y\widehat{Z}_T^{(C)}))\right], \quad y > 0.$$

(4) *The derivatives of the primal and dual value functions are given by*

$$u'(x) = \mathcal{Y}(x - \Pi(C)), \quad v'(y) = -(\mathcal{X}(y) + \Pi(C)),$$

so that the relations $y = \mathcal{Y}(x - \Pi(C))$ and $\mathcal{X}(y) = x - \Pi(C)$ between the initial wealth $x > 0$ and the Lagrange multiplier $y > 0$ in (10.4) may be written as

$$y = u'(x), \quad x = -v'(y).$$

10.1. **Exponential utility-based indifference valuation.** We now apply the theory of the previous subsection to the valuation and hedging of the claim C. Recall that in an incomplete market there is no replication strategy for C, so any valuation method must take into account potential unhedged risk, so should incorporate the agent's preferences towards risk. These are captured by her utility function.

We are still using the exponential utility function $U(x) = -\exp(-\alpha x)$ for $x \in \mathbb{R}$ and $\alpha > 0$. Take the interest rate process to be zero. Thus the deflator Z is a positive local martingale (given by $Z = \mathcal{E}(-\lambda \cdot W)$), and when Z is a martingale then it is also the density process of any ELMM $\mathbb{Q} \in \mathbb{M}$:

$$Z_t = \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}, \quad t \in [0, T].$$

We shall assume that this is the case from now on. Then, with exponential utility, the dual problem may be written as

$$v(y) = V(y) + \frac{y}{\alpha}\inf_{\mathbb{Q}\in\mathbb{M}} \mathbb{E}^{\mathbb{Q}}[\log Z_T - \alpha C].$$

The quantity

$$\mathbb{E}^{\mathbb{Q}}[\log Z_T] =: \mathcal{H}(\mathbb{Q}, \mathbb{P})$$

is called the *relative entropy* between $\mathbb{Q}$ and $\mathbb{P}$. We may therefore write

$$v(y) = V(y) + \frac{y}{\alpha}\inf_{\mathbb{Q}\in\mathbb{M}} \left(\mathcal{H}(\mathbb{Q}, \mathbb{P}) - \mathbb{E}^{\mathbb{Q}}[\alpha C]\right). \tag{10.5}$$

Thus, the dual problem amounts to the minimisation:

$$\inf_{\mathbb{Q}\in\mathbb{M}} \left(\mathcal{H}(\mathbb{Q}, \mathbb{P}) - \mathbb{E}^{\mathbb{Q}}[\alpha C]\right).$$

Observe that for the case $C = 0$, the dual problem without random endowment is thus equivalent to minimising the relative entropy over ELMMs. Denoting the dual value function for the case $C = 0$ by v_0, we have

$$v_0(y) = V(y) + \frac{y}{\alpha} \inf_{\mathbb{Q}\in\mathbb{M}} (\mathcal{H}(\mathbb{Q}, \mathbb{P})) =: V(y) + \frac{y}{\alpha}\mathcal{H}(\mathbb{Q}^E, \mathbb{P}),$$

where $\mathbb{Q}^E$ denotes the *minimal entropy martingale measure* (MEMM).

Now, using that u, v are conjugate, we have

$$u(x) = \inf_{y>0}[v(y) + xy].$$

Using the form of v in (10.5), we obtain

$$u(x) = -\exp\left(-\alpha x - \inf_{\mathbb{Q}\in\mathbb{M}}\left(\mathcal{H}(\mathbb{Q}, \mathbb{P}) - \mathbb{E}^{\mathbb{Q}}[\alpha C]\right)\right). \tag{10.6}$$

Definition 10.2 (Utility indifference price)**.** Let u_0 denote the value function for the problem (10.3) with $C = 0$. The time zero utility indifference price of the claim C is p, defined by

$$u(x + p) = u_0(x).$$

Applying this definition to the dual representation (10.6) of the primal value function we obtain the *dual representation of the indifference price*:

$$p = \sup_{\mathbb{Q}\in\mathbb{M}}\left[\mathbb{E}^{\mathbb{Q}}[C] - \frac{1}{\alpha}\left(\mathcal{H}(\mathbb{Q}, \mathbb{P}) - \mathcal{H}(\mathbb{Q}^E, \mathbb{P})\right)\right].$$

Observe that we have the limits

$$\lim_{\alpha\to\infty} p = \sup_{\mathbb{Q}\in\mathbb{M}} \mathbb{E}^{\mathbb{Q}}[C], \quad \widehat{p} := \lim_{\alpha\to 0} p = \mathbb{E}^{\mathbb{Q}^E}[C].$$

The first of these is called the super-replication price, and the second is called the *marginal utility-based price* (MUBP). For this latter quantity, we have the following representation.

Lemma 10.3. *The marginal utility-based price of the claim C at time zero is given by*

$$\widehat{p} = \frac{\mathbb{E}[U'(\widehat{X}^{(0)}_T)C]}{u_0'(x)},$$

where u_0 and $\widehat{X}^{(0)}_T$ denote the primal value function and optimal terminal wealth respectively, for the problem (10.3) with $C = 0$.

Proof. The dual minimiser for the case $C = 0$ is $\mathbb{Q}^E$, and we have

$$U'(\widehat{X}^{(0)}_T) = y\frac{\mathrm{d}\mathbb{Q}^E}{\mathrm{d}\mathbb{P}},$$

with $y = u_0'(x)$.

□

10.2. **The basis risk model revisited via duality.** We apply the utility indifference valuation results to the basis risk model of Section 8, with a traded asset S and a correlated non-traded asset Y following the geometric Brownian motions

$$\mathrm{d}S_t = \sigma S_t(\lambda\,\mathrm{d}t + \mathrm{d}W_t), \quad \mathrm{d}Y_t = \eta Y_t(\theta\,\mathrm{d}t + \mathrm{d}B_t), \quad \lambda, \theta \in \mathbb{R}, \quad \sigma, \theta \in \mathbb{R}^+$$

where W, B are correlated Brownian motions, satisfying

$$\langle W, B\rangle_t = \rho t, \quad \rho \in [-1, 1], \quad 0 \leqslant t \leqslant T.$$

A European contingent claim pays the non-negative random variable $h(Y_T)$ at time T, where $h(\cdot)$ is a bounded continuous function. The interest rate is zero. This market is incomplete unless the correlation is perfect. The set $\mathbb{M}$ of ELMMs $\mathbb{Q}$ is defined as

$$\mathbb{M} := \{\mathbb{Q} \sim \mathbb{P} | S \text{ is a local } (\mathbb{Q}, \mathbb{F})\text{-martingale}\},$$

where $\mathbb{F}$ is the augmented filtration generated by $(W, W^{\perp})$, with $W^{\perp}$ a Brownian motion independent of W, so that

$$B = \rho W + \sqrt{1-\rho^2} W^{\perp}.$$

The density process of any $\mathbb{Q} \in \mathbb{M}$ is given by

$$Z_t = \mathcal{E}(-\lambda W - \psi \cdot W^{\perp})_t, \quad 0 \leqslant t \leqslant T,$$

where ψ is an adapted process satisfying

$$\int_0^T \psi_t^2 \, \mathrm{d}t < \infty, \quad \text{a.s.}$$

For $\mathbb{Q}$ to be a probability measure we need Z to be a martingale, and we assume this is the case here (a sufficient condition for this is the Novikov condition on ψ (since λ is constant we need no further conditions on it)): $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \psi_t^2 \, \mathrm{d}t\right)\right] < \infty$. By the Girsanov theorem, the two-dimensional process $(W^{\mathbb{Q}}, W^{\mathbb{Q},\perp})$ defined by

$$W_t^{\mathbb{Q}} := B_t + \lambda t, \quad W_t^{\mathbb{Q},\perp} := W_t^{\perp} + \int_0^t \psi_u \, \mathrm{d}u, \quad 0 \leqslant t \leqslant T,$$

is a two-dimensional $\mathbb{Q}$-Brownian motion. Therefore, under $\mathbb{Q} \in \mathbb{M}$ the dynamics of the asset prices are

$$\mathrm{d}S_t = \sigma S_t \, \mathrm{d}W_t^{\mathbb{Q}}, \quad \mathrm{d}Y_t = \eta Y_t[(\theta - \rho\lambda - \sqrt{1-\rho^2}\psi_t) \, \mathrm{d}t + \mathrm{d}B_t^{\mathbb{Q}}],$$

where $W^{\mathbb{Q}}, B^{\mathbb{Q}}$ are correlated $\mathbb{Q}$-Brownian motions:

$$\langle W^{\mathbb{Q}}, B^{\mathbb{Q}} \rangle_t = \rho t, \quad 0 \leqslant t \leqslant T.$$

The traded asset price is a local $\mathbb{Q}$-martingale, but the drift of the non-traded asset is arbitrary and parametrised by the integrand ψ appearing in the density process of any ELMM $\mathbb{Q} \in \mathbb{M}$. The set $\mathbb{M}$ of martingale measures is then in one-to-one correspondence with the set Ψ of integrands ψ.

The minimal martingale measure $\mathbb{Q}^M$ corresponds to $\psi = 0$, so has density process with respect to $\mathbb{P}$ given by

$$\left.\frac{\mathrm{d}\mathbb{Q}^M}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}\left(-\lambda W\right)_t, \quad 0 \leqslant t \leqslant T.$$

Hence, under $\mathbb{Q}^M$, (S, Y) follow

$$\mathrm{d}S_t = \sigma S_t \, \mathrm{d}W_t^{\mathbb{Q}}, \quad \mathrm{d}Y_t = \eta Y_t[(\theta - \rho\lambda) \, \mathrm{d}t + \mathrm{d}B_t^{\mathbb{Q}^M}],$$

where $W^{\mathbb{Q}} \equiv W^{\mathbb{Q}^M}$ and $B^{\mathbb{Q}^M} = \rho W^{\mathbb{Q}} + \sqrt{1-\rho^2} W^{\perp}$ are correlated Brownian motions under $\mathbb{Q}^M$.

As shown in Section 8.1, if $|\rho| = 1$, the model is complete and a BS-style perfect hedge for the claim with payoff $h(Y_T)$ is possible. For $|\rho| \neq 1$ the market is incomplete.

10.2.1. *Utility-indifference valuation and hedging.* Suppose the correlation is not perfect, so that the market is incomplete. The relative entropy between $\mathbb{Q} \in \mathbb{M}$ and $\mathbb{P}$ in this model is given by

$$\mathcal{H}(\mathbb{Q},\mathbb{P}) = \frac{1}{2}\left(\lambda^2 T + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \psi_t^2\,\mathrm{d}t\right]\right).$$

We see immediately that

$$\mathcal{H}(\mathbb{Q}^E,\mathbb{P}) = \frac{1}{2}\lambda^2 T, \quad \frac{\mathrm{d}\mathbb{Q}^E}{\mathrm{d}\mathbb{P}} = \mathcal{E}(-\lambda W)_T,$$

so that $\mathbb{Q}^E = \mathbb{Q}^M$ in this model. Then the MUBP at time zero is given by

$$\widehat{p} = \mathbb{E}^{\mathbb{Q}^M}[h(Y_T)].$$

The time-zero indifference price has the stochastic control representation

$$p = \sup_{\psi\in\Psi}\mathbb{E}^{\mathbb{Q}}\left[h(Y_T) - \frac{1}{2\alpha}\int_0^T \psi_t^2\,\mathrm{d}t\right],$$

Consider the intermediate value function

$$p(t,y) := \sup_{\psi\in\Psi}\mathbb{E}^{\mathbb{Q}}\left[h(Y_T) - \frac{1}{2\alpha}\int_t^T \psi_u^2\,\mathrm{d}u \,\middle|\, Y_t = y\right].$$

This is the indifference pricing function at $t \leqslant T$ given $Y_t = y$. The HJB equation for $p(t,y)$ is

$$p_t + \max_{\psi}\left[\mathcal{L}^{Y,\mathbb{Q}}p - \frac{1}{2\alpha}\psi^2\right] = 0, \quad p(T,y) = h(y),$$

where $\mathcal{L}^{Y,\mathbb{Q}}$ is the generator of Y under $\mathbb{Q} \in \mathbb{M}$:

$$\mathcal{L}^{Y,\mathbb{Q}}p = \eta y(\theta - \rho\lambda - \sqrt{1-\rho^2}\psi)p_y + \frac{1}{2}\eta^2 y^2 p_{yy}.$$

Maximising over ψ in the HJB equation gives the optimal Markov control as $\widehat{\psi}(t,y) = -\alpha\sqrt{1-\rho^2}\eta y p_y(t,y)$. Substituting this into the HJB equation gives the PDE for $p(t,y)$ as

$$p_t + \eta(\theta - \rho\lambda)yp_y + \frac{1}{2}\eta^2 y^2 p_{yy} + \frac{1}{2}\eta^2 y^2 \alpha(1-\rho^2)p_y^2 = 0, \quad p(T,y) = h(y).$$

Observe that this PDE is consistent with (8.13) derived via the primal approach in Section 8.

One can check that the indifference price PDE is solved by

$$p(t,y) = \frac{1}{\alpha(1-\rho^2)}\log\mathbb{E}^{\mathbb{Q}^M}\left[\exp\left(\alpha(1-\rho^2)h(Y_T)\right)\middle| Y_t = y\right].$$

For $\alpha \to 0$, the indifference pricing PDE becomes linear, and by the Feynman-Kac Theorem we obtain the marginal utility-based price (MUBP) $\widehat{p}(t,y) := \lim_{\alpha\to 0} p(t,y)$:

$$\widehat{p}(t,y) = \mathbb{E}^{\mathbb{Q}^M}[h(Y_T)|Y_t = y].$$

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