Some unsolved problems and some references

June 2, 2015

1. Layers and Spikes

One problem here would be to obtain spikes in higher dimensions (the pde case). Dancer and Yan (see our book for the references) use calculus of variations to obtain single and multiple layers. But according to Dancer (private communication) their technique does not extend to spikes. Something new appears to be needed.

2. local pulse models (pde's)

Existence of a "fast" pulse, with speed $c = O(1)$ as $\varepsilon \to 0$, has been proved for the FitzHugh-Nagumo equations by Hastings (1976); Carpenter (1977); Jones, Kopell, Langer (1991). Existence of a slow pulse ($c = o(1)$ as $\varepsilon \to 0$) was proved by Hastings (1976, 1982), and by Krupa, Sandstede and Szmolyan (1997, for $\left(\frac{1}{2} - a\right)$ small).

The FitzHugh-Nagumo equations are a simplification of the Hodgkin-Huxley equations (1952, part V). Existence of a fast pulse for the Hodgkin-Huxley equations was proved by Hastings (1976) and under slightly different hypotheses, by Carpenter (1977). There has been no proof of the existence of a slow pulse for the Hodgkin-Huxley equations.

In addition to traveling pulses, the FitzHugh-Nagumo equation supports periodic traveling waves. This was proved by Hastings (1974). Presumably it is also true for other models, such as Hodgkin-Huxley, but I am not aware of any proofs of this.

The linearization of the FitzHugh-Nagumo traveling wave equations around the equilibrium point has real eigenvalues if ε is sufficiently small, but for slightly larger ε, these roots become complex. This is also true for the model of Pinto and Ermentrout, but it is not true for the model of Faye, where the roots stay real for all $\varepsilon > 0$.

In a class of neural models, complex roots can lead to interesting additional waves, as shown by Evans, Fenichel and Feroe, 1982. It was shown by Hastings (1982) that FitzHugh-Nagumo is in this class. It appears that Pinto-Ermentrout may be also, but this has not been proved. It seems that the model of Faye is not in this class.

Local stability of the fast pulse has been proved for the FitzHugh-Nagumo model (C.K.R.T. Jones) and for the model of Faye (G. Faye). They use a technique called the Evans functions to show that there are no positive or zero eigenvalues for the relevant eigenvalue problems.

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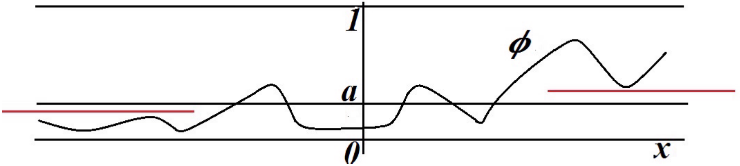
However there is no result about global stability. What sorts of initial conditions can trigger a pulse?

Recall the result of Fife and Mcleod for

$$\begin{aligned} u_t &= u_{xx} + u(1-u)(u-a) \\ u(x,0) &= \phi(x) \end{aligned}$$

Theorem: *Suppose that ϕ is continuous and $0 < \phi(x) < 1$ for all x. Suppose also that*

$$\limsup_{x \to -\infty} \phi(x) < a, \quad \liminf_{x \to \infty} \phi(x) > a$$



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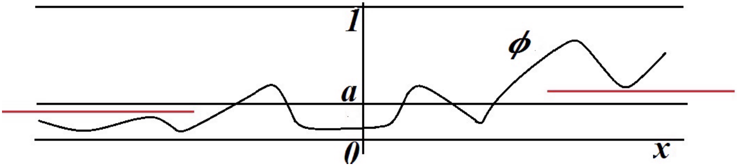

Then for some x_0 and positive constants K and ω,

$$|u(x,t) - U(x + ct - x_0)| < Ke^{-\omega t}$$

for all x and all $t > 0$.

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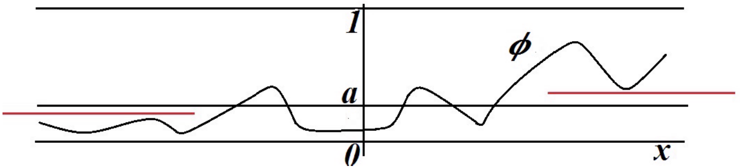

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FitzHugh-Nagumo case:

$$\begin{aligned} u_t &= u_{xx} + u\left(1-u\right)\left(u-a\right) - w \\ w_t &= \varepsilon\left(u - \gamma w\right). \end{aligned}$$

$$q_t : x \to \left(u\left(x,t\right), w\left(x,t\right)\right), \quad -\infty < s < \infty$$

Consider spatially independent solutions:

$$\begin{aligned} U' &= U\left(1-U\right)\left(U-a\right) - W \\ W' &= \varepsilon\left(U - \gamma W\right) \end{aligned}$$

Alternative formulation: Consider on a semi-infinite interval, with a boundary condition

$$u\left(0,t\right) = p\left(t\right).$$

For what p is a pulse generated?

3. Nonlocal pulse models

Here there are still questions about existence. For traveling fronts a major advance was made by Ermentrout and McLeod. This was extended by X. Chen (1997, Advances in Differential Equations) to include many models where the waves are monotonic. This paper also extended the Fife-McLeod global stability result to the Ermentrout-McLeod model and many others. For pulses, in addition to the paper of Faye, Faye and Scheel have new results which are particularly interesting because they do not rely on reduction of the integral equation to a pde. Thus a much wider class of kernels k and firing functions S is allowed. However, they require, in the language of part 4, that the "downjump" of the singular solution occurs before the knee. This seems to exclude most, if not all, cases of the Pinto-Ermentrout model, where the pulse appears to "fall off the knee".

There are no global stability results for pulses of any kind (in these models)

Two unsolved ode problems

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Finite time blowup of the nonlinear Schrödinger equation

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Applies to:

propagation of a laser beam through a medium
electromagnetic waves in a plasma
motion of a vortex filament for the Euler equations of fluid mechanics
some models of Bose-Einstein condensates.

Cubic form:

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Suitable scaling of t, r, and Φ together with an appropriate similarity substitution results in a complex ode

$$Q_{\xi\xi} + \frac{d-1}{\xi}Q_{\xi} - Q + ia\left(\xi Q\right)_{\xi} + Q\left|Q^2\right| = 0, \qquad (2)$$

where a is a real parameter. The function Q is to satisfy boundary conditions

$$\begin{array}{c} \lim_{\xi\to 0} Q_{\xi}\left(\xi\right) = 0,\ \ \lim_{\xi\to 0} \mathrm{Im}\, Q\left(\xi\right) = 0 \\ \lim_{\xi\to\infty}\ \left|Q\left(\xi\right)\right| = 0. \end{array} \qquad (3)$$

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$$H\left(\psi\right)=\int_{R^n}\left|\nabla\psi\right|^2-\frac{1}{4}\left|\psi\right|^4\ dx \text{ is finite,}$$

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Equivalent to:

$$H(Q) = 0. \tag{4}$$

Linearizing the complex equation (2) around $Q = 0$ for $n = 2$ gives

$$\tilde{Q}'' + \frac{2}{\xi}\tilde{Q}' - \tilde{Q} + ia\left(\xi\tilde{Q}\left(\xi\right)\right)' = 0.$$

Section 7.1.1 of a book by Sulem and Sulem shows that this equation has solutions $\tilde{Q}_1$ and $\tilde{Q}_2$, where as $\xi \to \infty$,

$$\tilde{Q}_1\left(\xi\right) \sim \left|\xi\right|^{-1-\frac{i}{a}}, \ \tilde{Q}_2\left(\xi\right) \sim \left|\xi\right|^{-\left(2-\frac{i}{a}\right)} e^{-ia\frac{\xi^2}{2}}.$$

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$\tilde{Q}_1$ is in $L_4\left(R^3\right)$ and $\tilde{Q}_1'$ is in $L_2\left(R^3\right)$. On the other hand, $\tilde{Q}_2'$ is not in $L_2\left(R^3\right)$, because the volume integral introduces an r^2 into the integrand and so $H\left(Q\right)$ is finite only for solutions which are asymptotic to a multiple of $\tilde{Q}_1$ as $\xi \rightarrow 0$.

By setting $Q = x + iy$ we obtain the following system:

$$\begin{aligned} x'' + \frac{2}{r}x' + x\left(x^2 + y^2 - 1\right) - ay - ary' = 0 \\ y'' + \frac{2}{r}y' + y\left(x^2 + y^2 - 1\right) + ax + arx' = 0 \end{aligned} \tag{5}$$

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We wish to know if this system *has a solution on* $(0, \infty)$ *such that*

$$\begin{aligned} \lim_{r\to 0^+} y(r) = \lim_{r\to 0^+} x'(r) = \lim_{r\to 0^+} y'(r) = 0 \\ \lim_{r\to\infty} rx'(r) + x(r) - \tfrac{1}{a}y(r) = 0 \\ \lim_{r\to\infty} ry'(r) + \tfrac{1}{a}x(r) - y(r) = 0. \end{aligned} \tag{6}$$

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It appears from numerical computation that the problem has a solution for each d with $2 < d \leq 4$. However the only existence proofs are those by Kopell and Landman and by Rottshafer and Kaper, each of which covers a range $2 < d < 2 + \delta$ for a small δ.

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$$\Delta\psi - \lambda^2\psi + \psi^3 = 0 \tag{7}$$

Look for radially symmetric solutions $\psi(x) = u(r)$ where $r = |x|$. Setting $\lambda = 1$ and taking the spatial dimension to be 3 gives the equation

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Physical boundary conditions (Weinstein reference in book):

$$u'(0) = 0, u(\infty) = 0,$$

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Solutions which have sign changes, called "bound states", are also important, as localized finite energy solutions. It was shown by G. H. Ryder that for each $k \geq 1$ there is a solution with exactly k zeroes in $(0, \infty)$. The problem is whether these solutions are unique. There are no results for this problem. In our book we give a new proof of the existence of the bound states.

Following Coffman's paper, K. Mcleod and Serrin considered equations of the form

$$u'' + \frac{n-1}{r}u' + |u|^{p-1}u - u = 0 \tag{8}$$

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There are also solutions which are not positive. Once again the condition $1 < p < \frac{n+2}{n-2}$ is imposed, and then, for every $k \geq 1$ there is a solution with exactly k zeros. This result was proved by Jones and Kuiper by a dynamical systems method and by K. McLeod, Troy and Weissler using a classical method. The uniqueness of these solutions is unknown for any $k > 0$. Troy obtained a uniqueness result for $k = 1, 2$ in the case where the nonlinear term is a piecewise linear function mimicking $u - u^3$.