# Partial Regularity results for degenerate elliptic systems

Bianca Stroffolini*

**Abstract**

We present a partial Hölder regularity result for solutions of degenerate systems

$$\operatorname{div} A(\,\cdot\,, Du) \,=\, 0 \qquad \text{in } \Omega,$$

on bounded domains in the weak sense. Here certain continuity, monotonicity, growth and structure condition are imposed on the coefficients, including an asymptotic Uhlenbeck behavior close to the origin. Pursuing an approach of Duzaar and Mingione [17], we combine non-degenerate and degenerate harmonic-type approximation lemmas for the proof of the partial regularity result, giving several extensions and simplifications. In particular, we benefit from a direct proof of the approximation lemma [11] that simplifies and unifies the proof in the power growth case. Moreover, we give the dimension reduction for the set of singular points.

**MSC (2010):** 35J45, 35J70

## 1 Introduction

In these notes I will present an approach towards regularity of weak solutions to possibly degenerate elliptic problems, that is mainly contained in the paper [3] for differential forms. We study weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $\Omega$ a bounded domain in $\mathbb{R}^n$, $n, N \geq 2$, to nonlinear systems of the form

$$\operatorname{div} A(\,\cdot\,, Du) \,=\, 0 \qquad \text{in } \Omega, \tag{1.1}$$

where the coefficients are Hölder continuous with respect to the first variable, with some exponent $\beta \in (0,1)$, and of class $C^1$ (possibly apart from the origin) with respect to the second variable with a standard $p$-growth condition. The main focus is set on the ellipticity condition: we allow a monotonicity or ellipticity condition which shows a degenerate (when $p > 2$) or singular (when $p < 2$) behavior in the origin and which is usually expressed by the assumption

$$\langle\, A(x,z) - A(x,\bar{z}), z - \bar{z}\,\rangle \,\geq\, \nu\,(\mu^2 + |z|^2 + |\bar{z}|^2)^{\frac{p-2}{2}}\, |z - \bar{z}|^2$$

for all $x \in \Omega$ and all $z, \bar{z} \in \mathbb{R}^N$ for some $\mu \geq 0$. The nondegenerate situation refers to the case where $\mu > 0$ (and by changing the value $\nu$ these cases can be reduced to the model case $\mu = 1$), whereas we here treat the degenerate case $\mu = 0$, meaning that we are dealing with a lack of ellipticity in the sense that no uniform bound on the ellipticity constant is available for $p \neq 2$. We highlight that the quadratic case does not impose any additional difficulties and is already covered by the standard regularity theory.

Let us first recall some of the well known facts for nondegenerate systems. In the vectorial case $N > 2$ – in contrast to the scalar case $N = 1$ – we cannot in general expect that a weak solution to the nonlinear elliptic system (1.1) is a classical solution (see e.g. the counterexamples in [6, 24]). Instead only a partial regularity result holds true, in the sense that we find an open subset $\Omega_0 \subset \Omega$ with $\mathcal{L}^n(\Omega \setminus \Omega_0) = 0$ such that $Du$ is locally Hölder continuous on $\Omega_0$ with optimal exponent $\beta$ given by the exponent in the Hölder continuity assumption on the $x$-dependency of the coefficients. These results were first obtained by Giusti and Miranda [23] via the indirect blow-up technique, then by Giaquinta,

*B. Stroffolini, Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Via Cintia, 80126 Napoli, Italy. E-mail: bstroffo@unina.it
Version: November 2, 2015

Modica and Ivert [21, 27] via the direct method, and finally Duzaar and Grotowski [12] gave a new proof based on the method of $\mathcal{A}$-harmonic approximation introduced by Duzaar and Steffen [18]. For further references and in particular for related results concerning variational problems we refer to Mingione's survey article [33].

In the degenerate case $\mu = 0$ no (partial) regularity result seems to be known for such general systems. However, supposing some additional assumptions on the structure of the system, Uhlenbeck succeeded in her fundamental paper [37] in showing that Moser-type techniques may be applied and that the classical regularity results of De Giorgi, Nash and Moser can be extended to systems of this special form (often called Uhlenbeck structure). A prototype of these systems is the $p$-Laplace system with $A(z) = |z|^{p-2}z$. More precisely, she stated in the superquadratic case (for systems without explicit dependency on the space variable) that the gradient of the solution is globally Hölder continuous in the interior with an exponent depending only on the space dimension $n$ and the ellipticity ratio $\nu/L$. We emphasize that Uhlenbeck's proof was carried out in the more general setting of $\mathbb{R}^N$-valued closed $\ell$-differential forms $\omega \in L^p(\Omega, \Lambda^\ell \mathbb{R}^n)$ solving the weak formulation to

$$d^* \rho(|\omega|)\omega \, = \, 0 \qquad \text{in } \Omega \, ,$$

where $\rho$ satisfies the Uhlenbeck structure assumptions (see p. 5). Further results concerning the regularity theory under such structure assumptions can for instance be found in [36, 22, 1, 20, 25, 29, 30, 10]. We highlight that Hamburger [25] gave an extension of Uhlenbeck's results in the setting of differential forms on Riemannian manifolds with sufficiently smooth boundary. In particular, he used an elegant duality argument to derive the subquadratic result from the superquadratic one (see also [26]). Restricting ourselves to the special case of 1-forms it is clear that the regularity result also covers weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$.

Minimizers to variational integrals with possibly degenerately quasiconvex integrands were already considered Duzaar and Mingione [14]. They observed that the non-degenerate and the degenerate theory can be combined in the following way: as long as the gradient variable keeps away from the origin, the system is also for $\mu = 0$ not singular/degenerate, and therefore a local partial regularity result holds true without an additional Uhlenbeck structure assumption. In contrast, if the origin is approached, then by requiring this crucial structure assumption even full regularity is locally expected. In fact, this strategy of distinguishing the local type of ellipticity was applied successfully in [14] in case of an asymptotic behavior like the $p$-Laplace system close to the origin, and as a final result minimizers were proved to be locally of class $C^{1,\alpha}$ for some $\alpha > 0$ (specified in the neighborhood of points where $Du$ does not vanish) outside a set of Lebesgue measure zero. In order to obtain an estimate for the decay of a suitable excess quantity, we employ local comparison principles based on harmonic-type approximation lemmas which are inspired by Simon's proof of the regularity theorem of Allard and which extend the method of harmonic approximation (i. e. approximating with functions solving the Laplace equation) in a natural way to bounded elliptic operators with constant coefficients or to even more general monotone operators. Here it is worth to remark that we give a direct proof of the harmonic approximation lemma, motivated from [11], and as a consequence the whole proof of the main partial regularity result is direct and we obtain a good control of the regularity estimates in terms of the structure constants. The important feature of the comparison system resulting from this harmonic-type approximation is the availability of good a priori estimates for its weak solutions (more precisely, solutions to linear systems with constant coefficients are known to be smooth, and solutions to Uhlenbeck systems are known to admit at least Hölder continuous gradients). In case of systems with degeneracy in the origin the above-mentioned distinction of the two different situations is accomplished as follows: if the average of the gradient is not too small compared to the excess quantity, then we deal with the non-degenerate situation and the usual comparison with the solution to the linearized system is performed via the $\mathcal{A}$-harmonic approximation lemma (see Proposition 7.1). If in contrast the average of the gradient is very small (again compared to the excess), then we are in the degenerate situation, meaning that the solution is approximately solving an Uhlenbeck system, and it is therefore compared to the exact solution of this Uhlenbeck system (see Proposition 7.3). These two decay estimates are then matched together in an iteration scheme as in [14], ending up with the desired partial regularity result.

On the one hand, we give a generalization of the existing results concerning possibly degenerate problems. We pursue an approach proposed by Duzaar and Mingione [17] in order to extend the known results dealing with a possible degeneracy at the origin like the $p$-Laplace system to more general ones that may behave at the origin like *any* arbitrary system of Uhlenbeck structure; a similar generalization was also suggested by Schmidt [35] who obtained the corresponding partial regularity result for degenerate variational functionals under $(p,q)$-growth conditions. This first aim is essentially achieved by the use of an extension of the $p$-harmonic approximation lemma from [15] (and similar to the one in [11]), namely the $a$-harmonic approximation Lemma 4.3.

Once the partial regularity result is achieved, it is natural to ask whether the Hausdorff dimension of the singular set can still be improved. We first note that for degenerate Uhlenbeck systems the (interior) singular set is indeed empty – due to the special structure of the coefficients. Turning our attention to the non-degenerate situation without any structure assumptions, much less is known. Indeed, in the course of proving regularity of the gradient $Du$ for classical solutions $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, the set of regular points is characterized, which in turn yields as a first and immediate consequence of a measure density result that the singular set is of Lebesgue measure zero. An estimate of the Hausdorff dimension was firstly investigated in the case of differentiable systems by Campanato in the 80's. The proof relied on the possibility of differentiating the system and obtaining existence of second-order derivatives of the solution. The first one who built a bridge between Hölder continuity of the coefficients and size of the singular set was Mingione [32, 31]: he showed that the singular set $\Omega \setminus \Omega_0$ is not only negligible with respect to the Lebesgue measure, but that its Hausdorff dimension is actually not greater than $n - 2\beta$ (with $\beta$ the degree of Hölder continuity of the coefficients). For related results on dimension reduction of the singular set in the context of convex variational integrals we refer to [28]. By means of the machinery of fractional Sobolev spaces and the differentiability of the system in a fractional sense developed in the previous papers, this upper bound on the Hausdorff dimension of the singular set is shown to be still valid for the solutions under consideration in this paper.

In conclusion, the main regularity result of our paper in the special case of classical weak solution can be stated as follows:

**Theorem 1.1:** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in (1,\infty)$, and consider a weak solution $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ to the system (1.1) under assumptions corresponding to (H1)–(H5) given in Section 2. Then there exists an open subset $\Omega_0 \subset \Omega$ such that*

$$u \in C^{1,\sigma}_{\mathrm{loc}}(\Omega_0, \mathbb{R}^N) \qquad \text{and} \qquad \dim_{\mathcal{H}} \big(\Omega \setminus \Omega_0\big) \le n - 2\beta\,,$$

*where $\sigma$ is an exponent depending only on $n, N, p, L, \nu$ and $\beta$.*

## 2 Structure conditions and main results

We start with $\Omega$ a bounded domain in $\mathbb{R}^n$ and we suppose that $u \in L^p(\Omega, \mathbb{R}^N)$, with $1 < p < \infty$, is a weak solution to the elliptic system

$$\operatorname{div} A(\,\cdot\,, Du) \,=\, 0 \qquad \text{in } \Omega, \tag{2.1}$$

for a vector field $A\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying some structure conditions: the mapping $\mathbf{P} \mapsto \mathbf{A}(\mathbf{x}, \mathbf{P})$ is of class $C^0(\mathbb{R}^N, \mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$, and for fixed numbers $0 < \nu \le L$, all $x, \bar{x} \in \Omega$ and all $\mathbf{P}, \bar{\mathbf{P}} \in \mathbb{R}^N$ the following assumptions concerning growth, ellipticity and continuity hold true:

**(H1)** $A$ is Lipschitz continuous with respect to $\mathbf{P}$ with

$$|A(x,\mathbf{P}) - A(x,\bar{\mathbf{P}})| \,\le\, L\,(|\mathbf{P}|^{\mathbf{2}} + |\bar{\mathbf{P}}|^{\mathbf{2}})^{\frac{\mathbf{p}-\mathbf{2}}{\mathbf{2}}}\, |\mathbf{P} - \bar{\mathbf{P}}|\,,$$

**(H2)** $D_{\mathbf{P}} A$ is Hölder continuous with some exponent $\alpha \in (0, |p-2|)$ such that

$$|D_{\mathbf{P}} A(x,\mathbf{P}) - D_{\mathbf{P}} A(x,\bar{\mathbf{P}})| \,\le\, L\,(|\mathbf{P}|^{\mathbf{2}} + |\bar{\mathbf{P}}|^{\mathbf{2}})^{\frac{\mathbf{p}-\mathbf{2}-\alpha}{\mathbf{2}}}\, |\mathbf{P} - \bar{\mathbf{P}}|^{\alpha}$$

holds for $p > 2,$ whereas in the subquadratic case $p \in (1,2)$ there holds for all $\mathbf{P}, \bar{\mathbf{P}} \neq 0$

$$|D_{\mathbf{P}}A(x,\mathbf{P}) - D_{\mathbf{P}}A(x,\bar{\mathbf{P}})| \,\leq\, L\,|\mathbf{P}|^{p-2}|\bar{\mathbf{P}}|^{p-2}(|\mathbf{P}|^2+|\bar{\mathbf{P}}|^2)^{\frac{2-p-\alpha}{2}}\,|\mathbf{P}-\bar{\mathbf{P}}|^{\alpha},$$

**(H3)** $A$ is degenerately monotone:

$$\langle\, A(x,\mathbf{P}) - A(x,\bar{\mathbf{P}}), \mathbf{P}-\bar{\mathbf{P}}\,\rangle \,\geq\, \nu\,(|\mathbf{P}|^2+|\bar{\mathbf{P}}|^2)^{\frac{p-2}{2}}\,|\mathbf{P}-\bar{\mathbf{P}}|^2\,,$$

**(H4)**

$A$ is Hölder continuous with respect to its first argument with exponent $\beta \in (0,1)$:

$$|A(x,\mathbf{P}) - A(\bar{x},\mathbf{P})| \,\leq\, L\,|\mathbf{P}|^{p-1}\,|x-\bar{x}|^{\beta}\,,$$

**(H5)** $A$ is of Uhlenbeck structure at 0, i.e. there exists a non-decreasing function $\widetilde{\mu}\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $\tilde{\mathbf{P}} \in \mathbb{R}^N$ with $|\tilde{\mathbf{P}}| \leq \widetilde{\mu}(t)$ there holds

$$|A(x,\tilde{\mathbf{P}}) - \rho_x(|\tilde{\mathbf{P}}|)\,\tilde{\mathbf{P}}| \,\leq\, t\,|\tilde{\mathbf{P}}|^{p-1}$$

uniformly for all $x \in \Omega$, where $\rho_x$ is a family of functions satisfying (G1)–(G3) introduced on p. 5 further below.

We first note that – due to the growth condition (H1), the monotonicity in (H3) and the Uhlenbeck type behavior at 0 in (H5) – the coefficients $A(x,\mathbf{P})$ exhibit a polynomial growth with respect to the variable $\mathbf{P}$, namely for all $x \in \Omega$, $\mathbf{P} \in \mathbb{R}^N$ there holds

$$\nu\,|\mathbf{P}|^{p-1} \,\leq\, |A(x,\mathbf{P})| \,\leq\, L\,|\mathbf{P}|^{p-1}\,. \tag{2.2}$$

Secondly, in view of the differentiability of $\mathbf{P} \mapsto A(x,z)$, we remark that (H1) and (H3) imply a growth and (degenerate) ellipticity condition for $D_{\mathbf{P}}A(x,\mathbf{P})$, more precisely, we have

$$|D_{\mathbf{P}}A(x,\mathbf{P})| \leq\, L\,|\mathbf{P}|^{p-2}\,, \tag{2.3}$$

$$\langle\, D_{\mathbf{P}}A(x,\mathbf{P})\,\xi,\xi\,\rangle \geq\, \nu\,|\mathbf{P}|^{p-2}\,|\xi|^2 \tag{2.4}$$

for all $\xi \in \mathbb{R}^N$, every $x \in \Omega$ and all $\mathbf{P} \in \mathbb{R}^N \setminus \{0\}$ (for $p > 2$ these inequalities are also valid for $\mathbf{P} = 0$).

**Example:** *A simple example or model case for the systems under consideration in this paper are the following type of $x$-depending versions of the $p$-Laplace system:*

$$A(x,\mathbf{P}) \,:=\, \beta(x)\,|\mathbf{P}|^{p-2}\,\mathbf{P}$$

*for all $\mathbf{P} \in \mathbb{R}^N$ and with $\beta(\cdot)$ a continuous function in $\Omega$ taking values in $[\nu, L]$ with Hölder exponent $\beta$.*

For a field $\mathbf{P} \in L^p(B_r(x_0),\mathbb{R}^N)$ we now introduce the excess

$$\Phi(\mathbf{P};x_0,r,\mathbf{P}_0) \,:=\, \fint_{B_r(x_0)} |V_{|\mathbf{P}_0|}(\mathbf{P}-\mathbf{P}_0)|^2 \qquad \text{for every } \mathbf{P}_0 \in \mathbb{R}^N\,,$$

where $V_\mu(\xi) := (\mu^2 + |\xi|^2)^{(p-2)/4}\xi$. In the sequel this excess shall frequently be used for the choice $\mathbf{P}_0 = (\mathbf{P})_{x_0,\rho}$, where $(\mathbf{P})_{x_0,r} = \fint_{B_r(x_0)} \mathbf{P}$ is an abbreviation for the meanvalue of $\mathbf{P}$ on the ball $B_r(x_0)$. As mentioned in [35] , [11] this excess is equivalent to

$$\fint_{B_r(x_0)} |V_0(\mathbf{P}) - V_0(\mathbf{P}_0)|^2 \tag{2.5}$$

up to a constant depending only on $n, N, p$ , and also to the one used in [14] . With this notation at hand we can now state our main regularity result for weak solutions to (2.1) on a bounded domain in $\mathbb{R}^n$:

**Theorem 2.1:** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in (1,\infty)$ and consider a weak solution $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ to the homogeneous system (2.1) under the assumptions* (H1)–(H5). *Then there exists $\sigma = \sigma(n,N,p,L,\nu,\beta)$ and an open subset $\Omega_0 \subset \Omega$ such that*

$$\nabla u \in C^{0,\sigma}_{\mathrm{loc}}(\Omega_0(\nabla u),\mathbb{R}^N) \qquad \textit{and} \qquad |\Omega \setminus \Omega_0| \,=\, 0$$

*with the following characterization of the set of regular points:*

$$\Omega_0 = \mathcal{R} := \left\{ x_0 \in \Omega \colon : \liminf_{r \searrow 0} \Phi(x_0, r, (\nabla u)_{x_0,r}) = 0 \quad \textit{and} \quad \limsup_{r \searrow 0} \left| (\nabla u)_{x_0,r} \right| < \infty \right\}.$$

*Moreover, if* $x_0 \in \Omega_0(\nabla u)$ *and*

$$\limsup_{r \searrow 0} \frac{\left| (\nabla u)_{x_0,r} \right|^p}{\Phi(x_0, r, (\nabla u)_{x_0,r})} = \infty \,, \tag{2.6}$$

*then* $\nabla u$ *is locally Hölder continuous with exponent* $\min\{\beta, 2\beta/p\}$*. Furthermore, if* $\nabla u(x_0) \neq 0$*, then* $\nabla u \in C^{0,\beta}(B_s(x_0), \mathbb{R}^N)$ *for some* $s > 0$*.*

**Remark:** More precisely, in points $x_0$ where (2.6) is not satisfied, the local Hölder continuity on the regular set $\Omega_0$ is determined by the exponent from the Hölder continuity of the coefficients with respect to the first variable and the asymptotic degenerate system in the origin in a neighborhood of $x_0$, namely the exponent $\sigma$ is given by $\min\{\gamma, \beta\}$ in the subquadratic case and by $\min\{2\gamma/p, 2\beta/p\}$ in the superquadratic case. Here $\gamma \in (0,1)$ is the number from the a priori estimate for weak solutions to systems of Uhlenbeck-type given in Proposition 3.1 below (we note that $\gamma$ does not depend on the point $x_0$ since the parameters $n, N, p, L$ and $\nu$ remain fixed for all functions $\rho_x$).

As a second result we give the dimension reduction for the singular set, which states a relation between the degree of regularity of the coefficients and the size of the Hausdorff dimension of the singular set:

**Theorem 2.2:** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain,* $p \in (1, \infty)$ *and consider a weak solution* $\nabla u \in L^p(\Omega, \mathbb{R}^N)$ *to the system (2.1) under the assumptions* (H1)–(H5)*. Then we have*

$$\dim_{\mathcal{H}} \big( \Omega \setminus \Omega_0(\nabla u) \big) \leq n - 2\beta \,.$$

## 3 Uhlenbeck result

*A regularity result for degenerate Uhlenbeck systems.* I will state a comparison estimate for special nonlinear degenerate systems which exhibit a particular structure that allows to prove everywhere regularity of the solution. More precisely, we consider vector fields of the form

$$a(\bar{\mathbf{P}}) = \rho(|\bar{\mathbf{P}}|) \, \bar{\mathbf{P}}$$

for every $\bar{\mathbf{P}} \in \mathbb{R}^{nN}$. For the function $\rho \colon [0,\infty) \to [0,\infty)$ we shall assume the following continuity, ellipticity and growth conditions:

- **(G1)** The function $t \mapsto \rho(t)$ is of class $C^0([0,\infty]) \cap C^1((0,\infty])$,
- **(G2)** There hold the inequalities
$$\nu \, t^{p-2} \leq \rho(t) \leq L \, t^{p-2}$$
and
$$\nu \, t^{p-2} \leq \rho(t) + \rho'(t) \, t \leq L \, t^{p-2} \,,$$
- **(G3)** There exists a Hölder exponent $\beta_\rho \in (0, \min\{1, |p-2|\})$ such that
$$|\rho'(s) \, s - \rho'(t) \, t| \leq L \, (|s|^2 + |t|^2)^{\frac{p-2-\beta_\rho}{2}} \, |s-t|^{\beta_\rho} \,.$$

for all $s, t \in (0, \infty)$, and some $p \geq 2$, $0 < \nu \leq L$. The model case of a vector field satisfying these conditions is the $p$-Laplace system, i. e. the vector field give by $a(\bar{\mathbf{P}}) = |\bar{\mathbf{P}}|^{p-2} \bar{\mathbf{P}}$ for all $\bar{\mathbf{P}} \in \mathbb{R}^{nN}$. For systems satisfying the above Uhlenbeck structure assumptions the following regularity result can be retrieved from [37, Theorem 3.2], [30]) and [25, Theorem 4.1]:

**Proposition 3.1:** *Let $p \in (1,\infty)$. There exists a constant $c \geq 1$ and an exponent $\gamma \in (0,1)$ depending only on $n, N, p, L$ and $\nu$ such that the following statement holds true: whenever $h \in W^{1,p}(B_R(x_0), \mathbb{R}^N)$ is a weak solution of the system*

$$\operatorname{div}\big(\rho(|\nabla h|)\,\nabla h\big) \,=\, 0 \qquad\qquad \text{in } B_R(x_0)\,,$$

*where $\rho(\cdot)$ fullfills the assumptions (G1)–(G3), then for every $0 < r < R$ there hold*

$$\sup_{B_{R/2}(x_0)} |\nabla h|^p \,\leq\, c \fint_{B_R(x_0)} |\nabla h|^p \qquad \text{and} \qquad \Phi(x_0, r, (\nabla h)_{x_0,r}) \,\leq\, c\left(\frac{r}{R}\right)^{2\gamma} \Phi(x_0, R, (\nabla h)_{x_0,R})\,.$$

## 4 Harmonic approximation lemmas

In this section we shall state two harmonic-type approximation lemmas which are adapted to the degenerate and the non-degenerate situation and which will allow us to compare the solution to the original system to the solution of an easier systems (for which good a priori estimates are available). To this aim we first need a result on Lipschitz-truncation, which from its original formulation can be restated as follows:

**Proposition 4.1 (Lipschitz truncation, cf. [19], Prop. 4.1):** *Let $B \subset \mathbb{R}^n$ be a ball. There exists a constant $c$ depending only on $n, N, p$ and $B$ such that whenever $\chi_k \rightharpoonup 0$ weakly in $W^{1,p}_T(B, \mathbb{R}^N)$, then for every $\lambda > 0$ there exists a sequence $\{\chi_k^\lambda\}_{k\in\mathbb{N}}$ of maps $\chi_k^\lambda \in W^{1,\infty}_T(B, \mathbb{R}^N)$ such that*

$$\|\chi_k^\lambda\|_{W^{1,\infty}} \,\leq\, c\,\lambda\,.$$

*Moreover, up to a set of Lebesgue measure zero we have*

$$\{z \in B\colon \chi_k^\lambda(z) \neq \chi_k(z)\} \,\subset\, \{z \in B\colon M(\nabla\chi_k)(z) > \lambda\}\,,$$

*where $M$ denotes the maximal operator restricted to $B$.*

Due to the direct approach for the proof of Lemma 4.3 we in fact need it only in a simpler version, namely for single functions instead of weakly converging sequences. However, there are much more involved Lipschitz truncation lemmas available in the literature, such as on general domains, versions involving sequences of truncations and variable exponent in [9, Theorem 2.5, Theorem 4.4]), versions truncating at two different levels (one for the function itself, the second one as above for its gradient) etc. In this paper we shall use a consequence of the previous truncation Lemma 4.1 from [11] for a version concerning the existence of a good truncation level in the setting of Sobolev-Orlicz spaces $W^{1,\phi}_T(B, \Lambda^\ell)$. I will state it in the case of power functions.

**Corollary 4.2 ([11]):** *For every $\varepsilon > 0$ there exists $c > 0$ depending only on $n, N, p$ such that the following statement holds: If $B \subset \mathbb{R}^n$ is a ball and $\chi \in W^{1,p}_0(B, \Lambda^\ell)$, then for every $m_0 \in \mathbb{N}$ and $\gamma > 0$ there exists $\lambda \in [\gamma, 2^{m_0}\gamma]$ such that the Lipschitz truncation $\chi^\lambda \in W^{1,\infty}_T(B, \mathbb{R}^N)$ of Theorem 4.1 satisfies*

$$\begin{aligned}
&\|\nabla\chi^\lambda\|_\infty \leq\, c\,\lambda\,,\\
&\fint_B |\nabla\chi^\lambda|^p\, \mathbb{1}_{\{\chi^\lambda\neq\chi\}}\, dx \leq\, c \int_B \lambda^p\, \mathbb{1}_{\{\chi^\lambda\neq\chi\}}\, dx \,\leq\, \frac{c}{m_0}\int_B |\nabla\chi|^p\, dx\,.
\end{aligned}$$

Restricting ourselves to the case of power growth in order to keep the setting as simple as possible, we now derive by similar techniques a suitable version which will apply not only to the $p$-Laplace system, but also to more general monotone operators. In what follows we consider vector fields $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ which are measurable with respect to the first variable, continuous in the second, and which satisfy growth and monotonicity conditions of the form

$$\left\{\begin{aligned}
&\big|a(x,\mathbf{P})\big| \,\leq\, L\left(\mu^2 + |\mathbf{P}|^2\right)^{\frac{p-1}{2}}\,,\\
&a(x,\mathbf{P})\cdot\mathbf{P} \,\geq\, \nu\left(\mu^2 + |\mathbf{P}|^2\right)^{\frac{p-2}{2}}|\mathbf{P}|^2\,,\\
&\big(a(x,\mathbf{P}) - a(x,\bar{\mathbf{P}})\big)\cdot(\mathbf{P}-\bar{\mathbf{P}}) \,\geq\, \nu\left(\mu^2 + |\bar{\mathbf{P}}|^2 + |\mathbf{P}|^2\right)^{\frac{p-2}{2}}|\mathbf{P}-\bar{\mathbf{P}}|^2
\end{aligned}\right. \tag{4.1}$$

for all $x \in \Omega$, $\mathbf{P}, \bar{\mathbf{P}} \in \mathbb{R}^N$, $p > 1$, $\mu \in [0,1]$ and $0 < \nu \leq L$. Furthermore, we assume that $a(\cdot,\cdot)$ is uniformly continuous on bounded subsets, i. e. that

$$|a(x,\mathbf{P}) - a(x,\bar{\mathbf{P}})| \,\leq\, K(|\mathbf{P}| + |\bar{\mathbf{P}}|)\,\vartheta(|\mathbf{P} - \bar{\mathbf{P}}|) \tag{4.2}$$

whenever $x \in \Omega$, $\mathbf{P}, \bar{\mathbf{P}} \in \mathbb{R}^N$, where $K\colon [0,\infty) \to [0,\infty)$ is a locally bounded, nondecreasing function and $\vartheta\colon [0,\infty) \to [0,1]$ is a nondecreasing function with $\lim_{t\searrow 0} \vartheta(t) = 0$. We note that these assumptions are in particular satisfied with $\mu = 0$ for vector fields $a(\mathbf{P}) := \rho(|\mathbf{P}|)\,\mathbf{P}$ where $\rho$ fullfills conditions (G1) and (G2) from the previous section. Following the notation of [7], we define for a convex function $\phi \in C^1((0,\infty))$ and $\mu \geq 0$ the *shifted function* $\phi_\mu$ by

$$\phi_\mu(t) \,:=\, \int_0^t \phi'_\mu(s)\,ds \qquad \text{with} \qquad \phi'_\mu(t) \,:=\, \frac{\phi'(\mu+t)}{\mu+t}\,t$$

for $t > 0$. In the case of powers $\phi(t) := t^p$, the excess function $V_\mu(t)$ introduced in Section 3 is equivalent to the shifted function $(\phi_\mu(t))^{1/2}$ (up to a constant depending only on $p$) and relates to the operator $a(\cdot,\cdot)$ satisfying the assumption (4.1) above via the inequalities:

$$\begin{cases} |a(x,\mathbf{P})| \,\leq\, c(p,L)\,\phi'_\mu(|\mathbf{P}|)\,, \\ a(x,\mathbf{P})\cdot\mathbf{P} \,\geq\, \nu\,|V_\mu(\mathbf{P})|^2 \,\geq\, c^{-1}(p)\,\nu\,\phi_\mu(|\mathbf{P}|)\,, \\ \big(a(x,\mathbf{P}) - a(x,\bar{\mathbf{P}})\big)\cdot(\mathbf{P} - \bar{\mathbf{P}}) \,\geq c^{-1}(p)\,\nu\,|V_\mu(\mathbf{P}) - V_\mu(\bar{\mathbf{P}})|^2\,. \end{cases} \tag{4.3}$$

We may now introduce the notion of an $a$-harmonic field: a field $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ is called $a$-harmonic in a domain $\Omega$ if $a(\cdot,\cdot)$ fulfills the growth assumption $(4.1)_1$ and if

$$\int_\Omega \langle\, a(x,\nabla u), \nabla\eta\,\rangle \,=\, 0 \qquad \text{for every } \eta \in C_0^\infty(\Omega,\mathbb{R}^N)\,.$$

**Lemma 4.3 ($a$-harmonic approximation; cf. [11]):** *Let $p \in (1,\infty)$ and $\phi(t) = t^p$ for all $t \geq 0$. For every $\varepsilon > 0$ and every $\theta \in (0,1)$ there exists $\delta > 0$ which depends only on $n, N, p, \nu, L, \theta$ and $\varepsilon$ such that the following statement holds true: Let $B \subset \mathbb{R}^n$ be a ball. Whenever $a(\cdot,\cdot)\colon B \times \mathbb{R}^N \to \mathbb{R}^N$ is a vector field satisfying (4.1) and (4.2) and whenever $\chi \in W^{1,p}(B,\mathbb{R}^N)$ is a vector field that is approximately $a$-harmonic in the sense that*

$$\Big|\,\fint_B \langle\, a(x,\nabla\chi), \nabla\eta\,\rangle\Big| \,\leq\, \delta\,\Big(\fint_B \phi_\mu(|\nabla\chi|) \,+ \phi_\mu(\|\nabla\eta\|_\infty)\Big) \tag{4.4}$$

*holds for all $\eta \in C_0^1(B,\mathbb{R}^N)$, then the unique $a$-harmonic $h \in W^{1,p}(B,\mathbb{R}^N)$, $h = \chi$ on $\partial B$ satisfies*

$$\fint_B \phi_\mu(|\nabla h|) \,\leq\, c(p,\nu,L)\fint_B \phi_\mu(|\nabla\chi| \quad \textit{and} \quad \Big(\fint_B |V_\mu(\nabla\chi) - V_\mu(\nabla h)|^{2\theta}\Big)^{\frac{1}{\theta}} \,\leq\, \varepsilon\fint_B \phi_\mu(|\nabla\chi|)\,.$$

Secondly, we state a suitable version of the $\mathcal{A}$-harmonic approximation lemma for both the super- and the subquadratic case. This version is proved by adjusting the proof of [18, Lemma 3.3], [13, Lemma 6] and [34, Lemma 6.8], respectively, or in a similar way as in the proof of the $a$-harmonic approximation lemma presented above.

**Lemma 4.4 ($\mathcal{A}$-harmonic approximation):** *Let $\nu \leq L$ be positive constants, $p \in (1,\infty)$ . Then for every $\varepsilon > 0$ there exists a positive number $\delta \in (0,1]$ depending only on $n, N, p, \frac{\nu}{L}$ and $\varepsilon$ with the following property: whenever $\mathcal{A}$ is a bilinear form on $\mathbb{R}^{nN}$ which is elliptic in the sense of Legendre-Hadamard with ellipticity constant $\nu$ and upper bound $L$ and whenever $\chi \in W^{1,p}(B_r,\mathbb{R}^N)$ satisfying $\fint_{B_r} |V_1(\nabla\chi)|^2 \,\leq \varsigma^2 \,\leq\, 1$ is approximately $\mathcal{A}$-harmonic in the sense that*

$$\Big|\,\fint_{B_r} \mathcal{A}(\nabla\chi,\nabla\eta)\,dx\Big| \,\leq\, \varsigma\,\delta\,\sup_{B_r} |\nabla\eta| \tag{4.5}$$

*holds for all* $\eta \in C_0^1(B_r, \mathbb{R}^N)$*, then there exists an* $\mathcal{A}$*-harmonic map* $h \in W^{1,p}(B_r, \mathbb{R}^N)$*,* $h = \chi$ *on* $\partial B_r$*, which satisfies*

$$\sup_{B_{r/2}} |\nabla h| + r \sup_{B_{r/2}} |D^2 h| \le c \qquad \textit{and} \qquad \fint_{B_{r/2}} \Big| V_1\Big(\frac{\chi - \varsigma h}{r}\Big)\Big|^2 dx \le \varsigma^2 \varepsilon \,.$$

*for a constant* $c$ *depending only on* $n, N, p, \nu$ *and* $L$*.*

## 5 A Caccioppoli inequality

As usual the first step in proving a regularity theorem for solutions of elliptic systems is to establish a suitable reverse-Poincaré or Caccioppoli-type inequality. This version of the Caccioppoli inequality takes into account the possible degeneracy of the monotonicity condition (H3) (and therefore also of the ellipticity condition).

**Lemma 5.1:** *Let* $p \in (1, \infty)$ *and consider a weak solution* $u \in W^{1,p}(B_r(x_0), \mathbb{R}^N)$*,* $r < 1$*, to the system* (2.1) *under the assumptions* (H1)*,* (H3) *and* (H4)*. Then, for* $\xi \in L^p(B_r(x_0), \mathbb{R}^N)$ *and* $\zeta \in \mathbb{R}^{nN}$ *there holds*

$$\fint_{B_{r/2}(x_0)} \big|V_{|\zeta|}(\nabla u - \zeta)\big|^2 \le c \fint_{B_r(x_0)} \Big|V_{|\zeta|}\Big(\frac{u - \xi - \zeta\cdot(x - x_0)}{r}\Big)\Big|^2 + c\,|\zeta|^p\, r^{2\beta} \tag{5.1}$$

*for a constant* $c$ *depending only on* $p, L$ *and* $\nu$*.*

Proof: Without loss of generality we may assume $x_0 = 0$. We consider a cut-off function $\eta \in C_0^\infty(B_r, [0,1])$ such that $\eta \equiv 1$ on $B_{r/2}$ and $|D\eta| \le \frac{c}{r}$. We may take $\eta^p(u - \xi - \zeta \cdot x)$ as a test function in (2.1). Using the assumptions and keeping in mind the properties of the cut-off function $\eta$, we thus arrive at the desired inequality. □

**Remark 5.2:** Applying the Poincaré inquality we get immediately a reverse Hölder' s inequality for $\nabla u - \zeta$ for a $\theta < 1$

$$\fint_{B_{r/2}(x_0)} \big|V_{|\zeta|}(\nabla u - \zeta)\big|^2 \le c\Big(\fint_{B_r(x_0)} \big|V_{|\zeta|}(\nabla u - \zeta)\big|^{2\theta}\Big)^{\frac{1}{\theta}} + c\,|\zeta|^p\, r^{2\beta} \tag{5.2}$$

This would get a higher integrability result for $\nabla u - \zeta$.

## 6 Approximate $\mathcal{A}$- and a-harmonicity

Our next aim is to find a framework in which the $\mathcal{A}$-harmonic and the $a$-harmonic approximation lemma, respectively, can be applied. This means that we have to identify systems for which the smallness conditions in the sense of (4.5) and (4.4) hold true (provided that additional smallness assumptions are satisfied). This shall be accomplished in the non-degenerate case by linearization of the coefficients, whereas in the degenerate case assumption (H5) allows to define a suitable Uhlenbeck-type system.

To start with the non-degenerate case we first recall the definition of the excess: for every ball $B_r(x_0) \subset \mathbb{R}^n$, a fixed function $u \in W^{1,p}(B_r(x_0), \mathbb{R}^N)$, $p \in (1, \infty)$, and every $\mathbf{P}_0 \in \mathbb{R}^{nN}$ the excess of $u$ is defined via

$$\Phi(x_0, r, \mathbf{P}_0) := \fint_{B_r(x_0)} \big|V_{|\mathbf{P}_0|}(\nabla u - \mathbf{P}_0)\big|^2 .$$

**Lemma 6.1 (Approximate $\mathcal{A}$-harmonicity):** *Let* $p \in (1, \infty)$*. There exists a constant* $c_A$ *depending only on* $p$ *and* $L$ *such that for every weak solution* $u \in W^{1,p}(B_r(x_0), \mathbb{R}^N)$*,* $r < 1$*, to system* (2.1) *under the assumptions* (H2) *and* (H4)*, and every* $\mathbf{P}_0 \in \mathbb{R}^{nN}$ *such that* $|\mathbf{P}_0| \neq 0 \neq \Phi(x_0, r, \mathbf{P}_0)$ *we have*

$$\Big|\fint_{B_r(x_0)} \langle D_{\mathbf{P}} A(x_0, \mathbf{P}_0)\, |\mathbf{P}_0|^{1-p}\, (\nabla u - \mathbf{P}_0), \nabla\eta \rangle\Big| \le c_A \Big[\Big(\frac{\Phi(r)}{|\mathbf{P}_0|^p}\Big)^{\frac12 + \frac{|p-2|}{2p}} + \Big(\frac{\Phi(r)}{|\mathbf{P}_0|^p}\Big)^{\frac12 + \frac{\alpha}{2}} + r^\beta\Big] \sup_{B_r(x_0)} |\nabla\eta|$$

*for all* $\eta \in C_0^1(B_r(x_0), \mathbb{R}^N)$*. Here we have abbreviated* $\Phi(x_0, r, \mathbf{P}_0)$ *by* $\Phi(r)$*.*

To treat the degenerate case where the system under consideration is close to a possibly degenerate system of Uhlenbeck structure, we define analogously to [14]

$$\Psi(x_0, r) = \fint_{B_r(x_0)} |\nabla u|^p .$$

Then, the structure assumption (H5) allows us to prove the following

**Lemma 6.2 (Approximate $a$-harmonicity):** *Let $p \in (1, \infty)$. There exists a constant $c_H$ depending only on $L$ such that for every weak solution $u \in W^{1,p}(B_r(x_0), \mathbb{R}^N)$, $r < 1$, to system (2.1) under the assumptions (H1), (H4) and (H5), and for every $t > 0$ we have*

$$\Big| \fint_{B_r(x_0)} \langle\, \rho_{x_0}(|\nabla u|)\, \nabla u, \nabla \eta \,\rangle \Big| \le c_H \Big[ t\, \Psi(r)^{\frac{p-1}{p}} + r^{\beta}\, \Psi(r)^{\frac{p-1}{p}} + \frac{\Psi(r)}{\widetilde{\mu}(t)} \Big] \sup_{B_r(x_0)} |\nabla \eta|$$

*for all $\eta \in C^1_0(B_r(x_0), \mathbb{R}^N)$. Here we have abbreviated $\Psi(x_0, r)$ by $\Psi(r)$.*

Proof: We assume without loss of generality $x_0 = 0$ and that the test function $\eta \in C^1_0(B_r, \mathbb{R}^N)$ satisfies $\sup_{B_r} |\nabla \eta| \le 1$. We fix $t > 0$. Since $u$ is a weak solution to (2.1) we first observe

$$\begin{aligned} \Big| \fint_{B_r} \langle\, \rho_0(|\nabla u|)\, \nabla u, \nabla \eta \,\rangle \Big| &= \Big| \fint_{B_r} \langle\, A(x, \omega) - \rho_0(|\nabla u|)\, \nabla u, \nabla \eta \,\rangle \Big| \\ &\le \fint_{B_r} \big| A(x, \nabla u) - A(0, \nabla u) \big| + \Big| \fint_{B_r} \langle\, A(0, \nabla u) - \mathbf{P}_0(|\nabla u|)\, \nabla u, \nabla \eta \,\rangle \Big| . \qquad (6.1) \end{aligned}$$

Using assumption (H4) on the Hölder continuity of the coefficients $A(\cdot, \cdot)$ with respect to the $x$-variable and Jensen's inequality, we easily find

$$\fint_{B_r} \big| A(x, \nabla u) - A(0, \nabla u) \big| \le L\, r^{\beta}\, \Psi(r)^{\frac{p-1}{p}} . \qquad (6.2)$$

To estimate the second integral on the right-hand side of the previous inequality we now distinguish the cases where $|\nabla u| \le \widetilde{\mu}(t)$ and where $|\nabla u| > \widetilde{\mu}(t)$. In the first case, we may apply (H5) and see

$$|B_r|^{-1} \Big| \int_{B_r \cap \{|\nabla u| \le \widetilde{\mu}(t)\}} \langle\, A(0, \nabla u) - \rho_0(|\nabla u|)\, \nabla u, \nabla \eta \,\rangle \Big| \le t \fint_{B_r} |\nabla u|^{p-1} \le t \Big( \fint_{B_r} |\nabla u|^p \Big)^{\frac{p-1}{p}} .$$

In order to give an estimate for the integral on the remaining set $\{|\nabla u| > \widetilde{\mu}(t)\}$ we first recall the weak $L^p$-type estimate stating

$$\big| B_r \cap \{|\nabla u| > \widetilde{\mu}(t)\} \big| \le \widetilde{\mu}(t)^{-p} \int_{B_r} |\nabla u|^p .$$

Thus, we infer from the upper bound (2.2) on the growth of $A(x, \nabla u)$ and Hölder's inequality that there holds

$$\begin{aligned} &|B_r|^{-1} \Big| \int_{B_r \cap \{|\nabla u| > \widetilde{\mu}(t)\}} \langle\, A(0, \nabla u) - \rho_0(|\nabla u|)\, \nabla u, \nabla \eta \,\rangle \Big| \\ &\le 2L\, |B_r|^{-1} \int_{B_r \cap \{|\nabla u| > \widetilde{\mu}(t)\}} |\nabla u|^{p-1} \\ &\le 2L\, |B_r|^{-1} \big| B_r \cap \{|\nabla u| > \widetilde{\mu}(t)\} \big|^{\frac{1}{p}} \Big( \int_{B_r} |\nabla u|^p \Big)^{\frac{p-1}{p}} \le \frac{2L}{\widetilde{\mu}(t)} \fint_{B_r} |\nabla u|^p . \end{aligned}$$

Merging the previous estimates together, we finally arrive at the inequality

$$\Big| \fint_{B_r} \langle\, A(0, \nabla u) - \rho_0(|\nabla u|)\, \nabla u, \nabla \eta \,\rangle \Big| \le t\, (\Psi(r))^{\frac{p-1}{p}} + \frac{2L}{\widetilde{\mu}(t)}\, \Psi(r) ,$$

where we have used the definition of $\Psi(r)$. In combination with (6.2), the assertion of the lemma follows (after rescaling) immediately from (6.1). $\square$

## 7 Excess decay estimates

In this section we take advantage of the results of the previous sections and deduce decay estimates for the excess of the solution on different balls in terms of the ratio of the radii. To this aim, the crucial ingredients in the non-degenerate and the degenerate situation – identified by a criterion involving the ratio excess to a suitable power of the meanvalue (which of course change with the radius) – are the a priori estimates available for solutions to linear systems and to Uhlenbeck systems, respectively. In a second step these excess decay estimates have to be iterated. Once the non-degeneracy criterion is satisfied, the iteration proceeds in a standard way, but the criterion for degeneracy might fail as the radius decreases, i. e. at a certain radius the situation might become non-degenerate (and as we will see then remains non-degenerate for all smaller ones), and therefore, the two iterations finally have to be combined in a suitable iteration schemes.

**Proposition 7.1:** *Let $p \in (1,\infty)$. For every $\beta' \in (0,1)$ there exist constants $\theta = \theta(n,N,p,\nu,L,\beta') \in (0,\frac{1}{4}]$, $\varepsilon_0 = \varepsilon_0(n,N,p,\nu,L,\alpha,\beta') \in (0,\frac{1}{2})$ and $r_0 = r_0(n,N,p,\nu,L,\alpha,\beta,\beta') \in (0,1)$ such that the following is true: for every weak solution $u \in W^{1,p}(B_r(x_0),\mathbb{R}^N)$, $r \le r_0$, to system (2.1) under the assumptions (H1)–(H4) which satisfies the smallness condition*

$$\Phi(x_0,r,(\nabla u)_{x_0,r}) \, < \, \varepsilon_0 \, |(\nabla u)_{x_0,r}|^p \,, \tag{7.1}$$

*we have the following growth condition:*

$$\Phi(x_0,\theta r,(\nabla u)_{x_0,\theta r}) \, \le \, \tfrac{1}{2}\, \theta^{2\beta'} \, \Phi(x_0,r,(\nabla u)_{x_0,r}) + c_0 \, |(\nabla u)_{x_0,r}|^p \, (\theta r)^{2\beta} \,, \tag{7.2}$$

*and the constant $c_0$ depends on $n,N,p,\nu,L$ and $\beta'$.*

Proof: Without loss of generality we take $x_0 = 0$, and we shall further use the abbreviation $\Phi(r) = \Phi(0,r,(\nabla u)_{0,r})$. Moreover, we assume $\Phi(r) > 0$, otherwise $\Phi(\theta r) = 0$ and the assertion in (7.2) is trivially satisfied. Now let $\varepsilon > 0$ (to be determined later) and choose $\delta \in (0,1]$ according to the $\mathcal{A}$-harmonic approximation Lemma 4.4. From (7.1) follows $|(\nabla u)_{0,r}| > 0$. We define $\widetilde{u}$ via

$$\widetilde{u} \, = \, \frac{u - (\nabla u)_{0,r} \cdot x}{|(\nabla u)_{0,r}|} \qquad \text{on } B_r \,.$$

Then, by definition of $\widetilde{\chi}$ and $\Phi(r)$ there holds

$$\fint_{B_r} |V_1(\nabla \widetilde{u})|^2 \, = \, |(\nabla u)_{0,r}|^{-p} \, \Phi(r) \, \le \, 1 \,.$$

The approximate $\mathcal{A}$-harmonicity result from Lemma 6.1 further ensures

$$\begin{aligned}
&\Big| \fint_{B_r} \langle \, \frac{D_{\mathbf{P}} A(x_0,(\nabla u)_{0,r})}{|(\nabla u)_{0,r}|^{p-2}} \, \nabla \widetilde{u}, \nabla \eta \, \rangle \Big| \\
&\quad \le \, c_A \Big( \frac{\Phi(r)}{|(\nabla u)_{0,r}|^p} + 2\,\delta^{-2}\, c_A^2 \, r^{2\beta} \Big)^{\frac{1}{2}} \Big( \Big( \frac{\Phi(r)}{|(\nabla u)_{0,r}|^p} \Big)^{\frac{|p-2|}{p}} + \Big( \frac{\Phi(r)}{|(\nabla u)_{0,r}|^p} \Big)^{\alpha} + \frac{\delta^2}{2c_A^2} \Big)^{\frac{1}{2}} \sup_{B_r} |\nabla \eta| \\
&\quad =: \, c_A \, \varsigma \Big( \Big( \frac{\Phi(r)}{|(\nabla u)_{0,r}|^p} \Big)^{\frac{|p-2|}{p}} + \Big( \frac{\Phi(r)}{|(\nabla u)_{0,r}|^p} \Big)^{\alpha} + \frac{\delta^2}{2c_A^2} \Big)^{\frac{1}{2}} \sup_{B_r} |\nabla \eta|
\end{aligned}$$

for all functions $\eta \in C^1_0(B_r,\mathbb{R}^N)$ with the obvious abbreviation for $\varsigma$. Now assume that

$$\Big( \frac{\Phi(r)}{|(\nabla u)_{0,r}|^p} \Big)^{\frac{|p-2|}{p}} + \Big( \frac{\Phi(r)}{|(\nabla u)_{0,r}|^p} \Big)^{\alpha} \, < \, \frac{\delta^2}{2c_A^2} \,. \tag{SC.1}$$

Then, provided that $r$ is chosen sufficiently small (in dependency of the parameters $c_A$ and $\delta$) and that consequently $\varsigma$ is bounded from above by 1, we find that $\widetilde{u}$ is approximately $\mathcal{A}$-harmonic with respect to $\mathcal{A} = |(\nabla u)_{0,r}|^{2-p} D_{\mathbf{P}} A(x_0, (\nabla u)_{0,r})$, which is elliptic with ellipticity constant $\nu$ and upper bound L, (see (2.4) and (2.3)). Hence, we infer the existence of a $\mathcal{A}$-harmonic map $h \in W^{1,2}(B_r, \mathbb{R}^N)$ such that it satisfies

$$\sup_{B_{r/2}} |\nabla h| + r \sup_{B_{r/2}} |D^2 h| \le c(n, N, p, \nu, L) \qquad \text{and} \qquad \fint_{B_{r/2}} \Big|V_1\Big(\frac{\widetilde{u} - \varsigma h}{r}\Big)\Big|^2 \le \varsigma^2 \varepsilon \,. \tag{7.3}$$

From the first inequality we obtain by Taylor expansion

$$\sup_{x \in B_{2\theta r}} |\nabla h(x) - (\nabla h)_{0,2\theta r}| \le (2\theta r) \sup_{B_{r/2}} |D^2 h| \le c\,\theta$$

for $c$ depending only on $n, N, p, \nu$ and $L$ as above. Hence, for $\theta \in (0, \frac{1}{4}]$ (to be chosen later) we now use the properties of $V_\mu$, Poincaré's inequality and we find:

$$\begin{aligned}
&\fint_{B_{2\theta r}} \Big|V_1\Big(\frac{\widetilde{u} - \varsigma h_0 - \varsigma (dh)_{0,2\theta r} \cdot x}{2\theta r}\Big)\Big|^2 \\
&\quad \le c(p) \fint_{B_{2\theta r}} \Big|V_1\Big(\frac{\widetilde{u} - \varsigma h}{2\theta r}\Big)\Big|^2 + c(p) \fint_{B_{2\theta r}} \Big|V_1\Big(\frac{\varsigma(h - h_0 - (\nabla h)_{0,2\theta r} \cdot x)}{2\theta r}\Big)\Big|^2 \\
&\quad \le c(p)\, \theta^{-n-\max\{2,p\}} \fint_{B_{r/2}} \Big|V_1\Big(\frac{\widetilde{u} - \varsigma h}{r}\Big)\Big|^2 \\
&\qquad + c(n, N, p) \fint_{B_{2\theta r}} \big(|\varsigma(\nabla h - (\nabla h)_{0,2\theta r})|^2 + |\varsigma(\nabla h - (\nabla h)_{0,2\theta r})|^{\max\{2,p\}}\big) \\
&\quad \le c(p)\, \theta^{-n-\max\{2,p\}} \varsigma^2 \varepsilon + c(n, N, p, \nu, L)\, \varsigma^2 \theta^2 \\
&\quad \le c(n, N, p, \nu, L)\, \varsigma^2 \big(\theta^{-n-\max\{2,p\}} \varepsilon + \theta^2\big)\,.
\end{aligned}$$

Setting $\varepsilon = \theta^{n+2+\max\{2,p\}}$ and recalling the definitions of $\widetilde{u}$ and $\varsigma$ we hence find the preliminary decay estimate

$$\fint_{B_{2\theta r}} \Big|V_{|(\nabla u)_{0,r}|}\Big(\frac{u - (\nabla u)_{0,r} \cdot x - |(\nabla u)_{0,r}| \varsigma\big(h_0 + (\nabla h)_{2\theta r} \cdot x\big)}{2\theta r}\Big)\Big|^2 \le c\,\theta^2 \big(\Phi(\rho) + \delta^{-2} |(\nabla u)_{0,r}|^p r^{2\beta}\big)\,, \tag{7.4}$$

and the constant $c$ depends only on $n, N, p, \nu$ and $L$. In order to apply the Caccioppoli inequality from Lemma 5.1 we now have to pass from $V_\mu(\cdot)$ in the previous inequality with index $\mu_1 = |(\nabla u)_{0,r}|$ to a corresponding one with index $\mu_2 = |(\nabla u)_{0,r} + |(\nabla u)_{0,r}| \varsigma (\nabla h)_{2\theta r}|$. This can be done if the indices are equivalent up to a constant. Therefore, since $|\nabla h|$ is bounded in $B_{2\theta r}$ by a constant depending only on $n, N, p, \nu$ and $L$, we now require an additional smallness condition on $\varsigma$ which guarantees $\frac{1}{2}\mu_1 \le \mu_2 \le \frac{3}{2}\mu_1$. To this end we assume

$$c^2 \frac{\Phi(r)}{|(\omega)_{0,r}|^p} \le \min\Big\{\frac{1}{8}, \theta^n\Big\}\,, \tag{SC.2}$$

$$c^2 \delta^{-2} c_A^2 r^{2\beta} \le \frac{1}{16} \tag{SC.3}$$

where $c$ (without loss of generality we assume $c \ge 4$) is the constant appearing in (7.3) (the reason for requiring the smallness assumption with respect to $\theta^{-n}$ will become clear in the iteration immediately after this lemma). We now apply the shifting Lemma, the Caccioppoli inequality and the decay estimate (7.4) to find:

$$\begin{aligned}
\Phi(\theta r) &= \fint_{B_{\theta r}} \big|V_{|(\nabla u)_{0,\theta r}|}\big(\nabla u - (\nabla u)_{0,\theta r}\big)\big|^2 \\
&\le c(p) \fint_{B_{\theta r}} \big|V_{|(\nabla u)_{0,r} + |(\nabla u)_{0,r}| \varsigma (\nabla h)_{2\theta r}|}\big(\nabla u - (\nabla u)_{0,r} - |(\nabla u)_{0,r}| \varsigma (\nabla h)_{2\theta r}\big)\big|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq c(p,L,\nu) \fint_{B_{2\theta r}} \Big| V_{\mu_2}\Big(\frac{u-(\nabla u)_{0,r}\cdot x - |(\nabla u)_{0,r}|\varsigma\big(h_0+(\nabla h)_{2\theta r}\cdot x\big)}{2\theta r}\Big)\Big|^2 + c(p,L,\nu)\,\mu_2^p\,(\theta r)^{2\beta}\\
&\leq c(p,L,\nu) \fint_{B_{2\theta r}} \Big| V_{\mu_1}\Big(\frac{u-(\nabla u)_{0,r}\cdot x - |(\nabla u)_{0,r}|\varsigma\big(h_0+(\nabla h)_{2\theta r}\cdot x\big)}{2\theta r}\Big)\Big|^2 + c(p,L,\nu)\,\mu_1^p\,(\theta r)^{2\beta}\\
&\leq c\,\theta^2\left(\Phi(r)+\delta^{-2}\,|(\nabla u)_{0,r}|^p\,r^{2\beta}\right) + c\,|(\nabla u)_{0,r}|^p\,(\theta r)^{2\beta} \;=:\; c_1\,\theta^2\,\Phi(r) + c_0\,|(\nabla u)_{0,r}|^p\,(\theta r)^{2\beta}\,,
\end{aligned}
$$

and the constants $c_1$ depends only on $n, N, p, \nu$ and $L$, and $c_0$ depends additionally on $\theta$. Given $\beta' \in (0,1)$ we now choose $\theta \in (0,1)$ sufficiently small such that $2c_1\theta^2 \leq \theta^{2\beta'}$. For later purposes we also assume that $2^{\max\{2,p\}}\theta^{2\beta'} < 1$ is fullfilled. Note that this fixes $\theta$ in dependency of $n, N, p, \nu, L$ and $\beta'$ which in turn determines $\varepsilon = \theta^{n+2+\max\{2,p\}}$ and $\delta$ in dependency of the same quantities. Then we infer from the latter inequality the desired excess decay estimate stated in the proposition, provided that the smallness conditions (SC.1), (SC.2) and (SC.3) hold true. Taking into consideration the dependencies in (SC.1), (SC.2) on $\Phi(r)/|(\nabla u)_{0,r}|^p$, we observe that they are satisfied if $\Phi(r) \leq \varepsilon_0 |(\nabla u)_{0,r}|^p$ is required for a number $\varepsilon_0$ chosen sufficiently small in dependency of $n, N, , p, \nu, L, \alpha$ and $\beta'$. For the iteration we will need an additional smallness condition $3^{p+3}c_0\, r^{2\beta'} \leq \varepsilon_0$, thus, in view of the dependencies in the smallness condition (SC.3) on the radius, it suffices to choose $r < r_0$ for a number $r_0 > 0$ depending only on $n, N, p, \nu, L, \alpha, \beta$ and $\beta'$, and the proof of the proposition is complete. □

**Lemma 7.2:** *Let $p \in (1,\infty)$, $\beta' \in (0,\beta]$ and $m \geq 1$. Then, with the numbers $\varepsilon_0$ and $r_0$ defined above, the following is true: for every weak solution $u \in W^{1,p}(B_R(x_0),\mathbb{R}^N)$, $R \leq r_0$, to system (2.1) under the assumptions (H1)–(H4) which satisfies the smallness conditions*

$$
\Phi(x_0,R,(\nabla u)_{x_0,R}) \;<\; \varepsilon_0\,|(\nabla u)_{x_0,R}|^p \quad \textit{and} \quad |(\nabla u)_{x_0,R}| \;<\; 2m\,, \tag{7.5}
$$

*we have $|(\nabla u)_{x_0,r}| < 6m$ and*

$$
\Phi(x_0,r,(\nabla u)_{x_0,r}) \;\leq\; c_{it}\left(\Big(\frac{r}{R}\Big)^{2\beta'}\Phi(x_0,R,(\nabla u)_{x_0,R}) + r^{2\beta'}\right) \tag{7.6}
$$

*for all $r \leq R$, and the constant $c_{it}$ depends only on $n, N, p, \nu, L, \beta'$ and $m$.*

Proof: The assertion follows by a more or less standard iteration procedure. However, for the convenience of the reader we give the main steps and refer to by now classical regularity papers for the details. In the first step one proves that the smallness condition (7.5) implies for every $k \in \mathbb{N}_0$:

(i) $\Phi(x_0,\theta^k R,(\nabla u)_{x_0,\theta^k R}) \;\leq\; 2^{-k}\,\theta^{2\beta' k}\,\Phi(x_0,R,(\nabla u)_{x_0,R}) + 3^{p+2}\,c_0\,(\theta^k R)^{2\beta'}\,|(\nabla u)_{x_0,R}|^p$,

(ii) $\Phi(x_0,\theta^k R,(\nabla u)_{x_0,\theta^k R}) \;<\; \theta^{2\beta' k}\,\varepsilon_0\,|(\nabla u)_{x_0,R}|^p$,

(iii) $|(\nabla u)_{x_0,R}| \;\leq\; 2^k\,|(\nabla u)_{x_0,\theta^k R}|$,

(iv) $\Phi(x_0,\theta^k R,(\nabla u)_{x_0,\theta^k R}) \;<\; \varepsilon_0\,|(\nabla u)_{x_0,\theta^k R}|^p$,

(v) $|(\nabla u)_{x_0,\theta^k R}| \;\leq\; 3\,|(\nabla u)_{x_0,R}|$,

and $\theta, c_0$ are the constants appearing in the previous Proposition 7.1. These estimates are established by induction and essentially rely on Proposition 7.1 .

In the second step we then derive a continuous version and consider $r \in (0,R]$ arbitrary. Then there exists a unique number $k \in \mathbb{N}_0$ such that $r \in (\theta^{k+1}R, \theta^k R]$, and exactly as above in (v), we find

$$
|(\nabla u)_{x_0,r}| \;\leq\; 3\,|(\nabla u)_{x_0,R}| \;<\; 6\,m\,.
$$

Moreover, in view of $(i)$ and shifting Lemma, we get

$$\begin{aligned}\Phi(x_0,r,(\nabla u)_{x_0,r}) &\leq \Big(\frac{\theta^k R}{r}\Big)^n \fint_{B_{\theta^k R}(x_0)} \big|V_{|(\nabla u)_{x_0,r}|}\big(\nabla u-(\nabla u)_{x_0,r}\big)\big|^2 \\ &\leq c(p)\,\theta^{-n} \fint_{B_{\theta^k R}(x_0)} \big|V_{|(\nabla u)_{x_0,\theta^k R}|}\big(\nabla u-(\nabla u)_{x_0,\theta^k R}\big)\big|^2 \\ &\leq c(p)\,\theta^{-n}\left(2^{-k}\,\theta^{2\beta' k}\,\Phi(x_0,R,(\nabla u)_{x_0,R}) + 3^{p+2}\,c_0\,(\theta^k R)^{2\beta'}\,|(\nabla u)_{x_0,R}|^p\right) \\ &\leq c_{it}\left(\Big(\frac{r}{R}\Big)^{2\beta'}\Phi(x_0,R,(\nabla u)_{x_0,R}) + r^{2\beta'}\right),\end{aligned}$$

and due to the dependencies of $\theta$ we have $c_{it}=c_{it}(n,N,p,\nu,L,\beta',m)$. This completes the proof of the excess decay estimate (7.6) and thus of the lemma. □

As already mentioned before we derive an excess decay estimate for the degenerate situation where the mean value of $\nabla u$ on a ball $B_R(x_0)$ is "small" with respect to the excess (in some sense this assumption is equivalent to the system being degenerate). Duzaar and Mingione [14] had considered a degeneracy as the $p$-Laplace system, and they then concluded that approximate $p$-harmonicity allows to find an excess-decay estimate. We here argue similarly, namely we show that if the system exhibits a degeneracy as a system of Uhlenbeck-structure, then approximate $a$-harmonicity implies the desired excess-decay estimate. Nevertheless, our proof is slightly different in order to succeed in showing that also in the superquadratic situation one smallness condition on the mean value of $\nabla u$ (instead of an additional second condition on a smaller ball) is sufficient to prove the decay estimate. In what follows, we denote by $\gamma\in(0,1)$ the exponent from the excess decay estimate in Proposition 3.1 for weak solutions of systems with Uhlenbeck structure (meaning that the weak solution has Hölder exponent $2\gamma/p$ in the superquadratic case and Hölder exponent $\gamma$ in the subquadratic case).

**Proposition 7.3:** *Let $p\in(1,\infty)$. For every exponent $\gamma'\in(0,\min\{\gamma,\beta\})$ and every number $\kappa>0$ there exist constants $\tau\in(0,\frac14]$ and $r_1<1$ depending on $n,N,p,\nu,L,\gamma,\gamma',\beta$ and $\kappa$, and a constant $\varepsilon_1>0$ depending additionally on $\widetilde{\mu}(\cdot)$ such that the following is true: Let $u\in W^{1,p}(B_r(x_0),\mathbb{R}^N)$, $r\leq r_1$, be a weak solution to system (2.1) under the assumptions (H1)–(H5). If*

$$\kappa\,|(\nabla u)_{x_0,r}|^p \;\leq\; \Phi(x_0,r,(\nabla u)_{x_0,r}) \;<\; \varepsilon_1 \tag{7.7}$$

*is fullfilled, then we have*

$$\Phi(x_0,\tau r,(\nabla u)_{x_0,\tau r}) \;\leq\; \tau^{2\gamma'}\,\Phi(x_0,r,(\nabla u)_{x_0,r})\,. \tag{7.8}$$

Proof: Without loss of generality we take $x_0=0$, and we use the abbreviations $\Phi(r)=\Phi(0,r,(\nabla u)_{0,r})$ and $\Psi(r)=\Psi(0,r)$. From $|(\nabla u)_{0,r}|^p\leq\kappa^{-1}\Phi(r)$ we see for the superquadratic case $p\geq 2$

$$\Psi(r)\leq 2^{p-1}\fint_{B_r}|\nabla u-(\nabla u)_{0,r}|^p + 2^{p-1}\,|(\nabla u)_{0,r}|^p \;\leq\; 2^{p-1}(1+\kappa^{-1})\,\Phi(r)\,,$$

whereas in the subquadratic case we distinguish the cases where $|\nabla u-(\nabla u)_{0,r}|\geq|(\nabla u)_{0,r}|$ and where the opposite inequality holds true, and we obtain

$$\begin{aligned}\Psi(r)&\leq 2^{p-1}\fint_{B_r}|\nabla u-(\nabla u)_{0,r}|^p + 2^{p-1}\,|(\nabla u)_{0,r}|^p \\ &\leq 2^{p-1}\,2^{\frac{2-p}{2}}\fint_{B_r}\big|V_{|(\nabla u)_{0,r}|}\big(\nabla u-(\nabla u)_{0,r}\big)\big|^2 + 2^p\,|(\nabla u)_{0,r}|^p \;\leq\; 2^p\,(1+\kappa^{-1})\,\Phi(r)\,.\end{aligned}$$

Hence, in any case we get

$$\Psi(r)\;\leq\; c_\Psi\,\Phi(r)\,, \tag{7.9}$$

where we have set $c_\Psi = 2^p(1+\kappa^{-1})$. In view of Lemma 6.2 on approximate $a$-harmonicity we have for every $t>0$ and every $\eta \in C_0^1(B_r,\mathbb{R}^N)$:

$$\Big|\fint_{B_r} \langle \rho_{x_0}(|\nabla u|)\,\nabla u, \nabla\eta\,\rangle\Big| \le c_H \Big[t\,\Psi(r)^{\frac{p-1}{p}} + r^\beta\,\Psi(r)^{\frac{p-1}{p}} + \frac{\Psi(r)}{\widetilde{\mu}(t)}\Big] \sup_{B_r}|\nabla\eta|$$

Now let $\tau \in (0,\frac14]$ to be specified later and define $\varepsilon = \tau^{p+\max\{1,\frac{p}{2}\}(n+2\gamma)}$. Furthermore, let $\delta = \delta(n,N,p,\nu,L,\varepsilon) \in (0,1]$ be the constant according to the $a$-harmonic approximation with $\theta = n/(n+p)$: For all assumptions of Lemma 4.3 to be fullfilled it still remains to verify assumption (4.4). For this purpose we fix $t = t(L,\delta) > 0$ and a radius $r_1 = r_1(L,\delta,\beta) > 0$ such that $c_H t \le \delta/3$ and $c_H\, r_1^\beta \le \delta/3$, which in turn fixes $\widetilde{\mu}(t)$. If we assume that the smallness condition

$$c_H\,\frac{(c_\Psi\Phi(r))^{1/p}}{\widetilde{\mu}(t)} \le \frac{\delta}{3} \tag{SC.4}$$

holds, then, after application of Young's inequality, $u$ satisfies all assumption of Lemma 4.3, provided that $r \le r_1$. Consequently, there exists a $a$-harmonic map $h \in W^{1,p}(B_r,\mathbb{R}^N)$ for $a(z) = \rho_0(|z|)\,z$ and which satisfies

$$\fint_{B_r}|\nabla h|^p \le c\fint_{B_r}|\nabla u|^p \quad\text{and}\quad \Big(\fint_B |V(\nabla u)-V(\nabla h)|^{2\theta}\Big)^{\frac1\theta} \le \varepsilon\fint_{2B}(|\nabla u|)^p \tag{7.10}$$

for a constant $c$ depending only on $p,\nu$ and $L$. We here have used the fact that for all possible choices of $\rho$ satisfying the assumptions (G1)–(G3) the statement of Lemma 4.3 holds true with $\mu = 0$ (and hence $V_\mu(z) = |z|^{(p-2)/2}z$) as well as a simple property of the $V$-function. In fact Caccioppoli and Poincaré imply a higher integrability result for $\nabla u$ (see 5.2), so we can consider the previous smallness assumption in this way:

$$\Big(\fint_B |V(\nabla u)-V(\nabla h)|^2\Big) \le \varepsilon\Big(\fint_{2B}|\nabla u|^{p+\eta}\Big)^{\frac{p}{p+\eta}} + c\Psi(r)r^{2\beta} \tag{7.11}$$

Suppose that we start from $B_{\tau r}$. For the minimality of $(\nabla u)_{0,\tau r}$ we can consider any $P \in \mathbb{R}^{nN}$ and use the minimality of the shifted function:

$$\begin{aligned}
\Phi(\tau r) &= \fint_{B_{\tau r}} \big|V_{|(\nabla u)_{0,\tau r}|}\big(\nabla u - (\nabla u)_{0,\tau r}\big)\big|^2\\
&\le c(p)\fint_{B_{\tau r}}\big|V_{|P|}\big(\nabla u - P\big)\big|^2 \le c(p)\fint_{B_{\tau r}}\big|V(\nabla u)-V(P)\big)\big|^2\\
&\le c(p)\fint_{B_{\tau r}}\big|V(\nabla u)-V(\nabla h)\big)\big|^2 + \fint_{B_{\tau r}}\big|V(\nabla h)-V(P)\big)\big|^2 \qquad (7.12)\\
&\le c(p,L,\nu)\varepsilon\tau^{-n}\Psi(r) + c(p,L,\nu)\,[\tau^{2\gamma}+,(2\tau r)^{2\beta}]\Psi(r)\\
&\le c_3(n,N,p,L,\nu,\kappa)\,\big(\tau^{2\gamma}+(\tau r)^{2\beta}\big)\,\Phi(r)\,. \qquad (7.13)
\end{aligned}$$

Here we have chosen $P = (\nabla h)_{0,\tau r}$ and applied the excess decay estimate stated in proposition 3.1.

For a given exponent $\gamma' \in (0,\min\{\gamma,\beta\})$ we now fix $\tau \in (0,\frac14]$ such that

$$c_3\,\tau^{\min\{2\gamma,2\beta\}} \le \tau^{2\gamma'}\,. \tag{SC.5}$$

Hence, $\tau$ is fixed in dependency of $n,N,p,L,\nu,\gamma,\gamma',\beta$ and $\kappa$. The choice $\varepsilon = \tau^{p+\max\{1,\frac{p}{2}\}(n+2\gamma)}$ further fixes $\delta$ – and therefore also $t$ and the radius $r_1$ – with exactly the same dependencies as those appearing in $\tau$. We now remark that (SC.4) may be rewritten by

$$\Phi(r) \le c_\Psi^{-1}\Big(\frac{\delta\,\widetilde{\mu}(t)}{3c_H}\Big)^p = 2^{-p}\,\frac{\kappa}{1+\kappa}\Big(\frac{\delta\,\widetilde{\mu}(t)}{3c_H}\Big)^p\,.$$

For later purposes, we additionally assume that

$$\Phi(r)^{\frac{1}{p}} \, \frac{\tau^{-n/2}}{1-\tau^{\gamma'}} \, \kappa^{\frac{p-2}{2p}} + \Phi(r)^{\frac{1}{p}} \, \frac{\tau^{-n/p}}{1-\tau^{2\gamma'/p}} \, \leq \, 1 \, . \tag{SC.6}$$

Hence, we observe that these smallness conditions are fullfilled if we choose $\varepsilon_1$ sufficiently small in dependency of the parameters stated in the proposition. This completes the proof. □

**Remark:** We mention that the radius $r$ appears in inequality (7.12) as a factor. Thus, we may replace (SC.5) by the following smallness condition concerning $r$ and $\tau$:

$$c_3 \, r^{\beta} \, \tau^{2\beta} \, \leq \, \frac{1}{2} \, \tau^{2\beta} \qquad \text{and} \qquad c_3 \, \tau^{\frac{2\gamma}{p}} \, \leq \, \frac{1}{2} \, \tau^{2\beta} \, ,$$

where $c_3 = c_3(n, N, p, L, \nu, \kappa)$. This enables us to state the excess decay estimate also with exponent $\gamma' = \beta$ when $\beta < \gamma$.

**Lemma 7.4 (Excess decay):** *Let $p \in (1, \infty)$. For every exponent $\gamma' \in (0, \min\{\gamma, \beta\})$ and $m \geq 1$ there exist $\varepsilon_1 = \varepsilon_1(n, N, p, \nu, L, \gamma, \gamma', \alpha, \beta, \widetilde{\mu}(\cdot)) > 0$ and a radius $r_2 = r_2(n, N, p, \nu, L, \gamma, \gamma', \alpha, \beta) > 0$ such that the following is true: Let $u \in W^{1,p}(B_R(x_0), \mathbb{R}^N)$, $R \leq r_2$, be a weak solution to system (2.1) under the assumptions (H1)–(H5). If the smallness conditions*

$$\Phi(x_0, R, (\nabla u)_{x_0,R}) \, < \, \varepsilon_1 \quad \textit{and} \quad |(\nabla u)_{x_0,R}| \, < \, m \tag{7.14}$$

*are fullfilled, then we have*

$$\Phi(x_0, r, (\nabla u)_{x_0,r}) \, \leq \, c \left( \left( \frac{r}{R} \right)^{2\gamma'} \Phi(x_0, R, (\nabla u)_{x_0,R}) + r^{2\gamma'} \right) \qquad \textit{for all } r \leq R \, , \tag{7.15}$$

*and $c$ depends on $n, N, p, \nu, L, \gamma, \gamma', \beta, \alpha$ and $m$.*

Proof: We again take $x_0 = 0$ and use the abbreviation $\Phi(R) = \Phi(0, R, (\nabla u)_{0,R})$. Let $\gamma' \in (0, \min\{\gamma, \beta\})$, where $\gamma$ is the exponent from Proposition 3.1, and choose $\beta' = \gamma'$ in Lemma 7.2. This fixes two positive constants

$$\begin{aligned} \varepsilon_0 &= \varepsilon_0(n, N, p, \nu, L, \alpha, \gamma') \, , \\ r_0 &= r_0(n, N, p, \nu, L, \alpha, \beta, \gamma') \, . \end{aligned}$$

Furthermore, we set $\kappa = \varepsilon_0$ and we find from Proposition 7.3 positive constants

$$\begin{aligned} \tau &= \tau(n, N, p, \nu, L, \gamma, \gamma', \beta, \alpha) \, , \\ r_1 &= r_1(n, N, p, \nu, L, \gamma, \gamma', \beta, \alpha) \, , \\ \varepsilon_1 &= \varepsilon_1(n, N, p, \nu, L, \gamma, \gamma', \beta, \alpha, \widetilde{\mu}(\cdot)) \, . \end{aligned}$$

We define $r_2 := \min\{r_0, r_1\}$. We next observe that (7.14) ensures that the second inequality in the smallness assumption (7.7) required for the application of Proposition 7.3 is satisfied. We introduce the set of natural numbers

$$\mathbb{S} \, := \, \left\{ n \in \mathbb{N}_0 \colon \Phi(\tau^n R) \, \geq \, \varepsilon_0 \, |(\nabla u)_{0,\tau^n R}|^p \right\}$$

(we note that due to the different conditions in the excess-decay estimate [14, Proposition 4] we need - in contrast to [14, Lemma 13] – only one condition). In order to prove the desired excess decay estimate we have to distinguish the cases where the mean values of $\nabla u$ is always small (i. e. where the system is purely degenerate) and where the mean value for a certain radius (and then for every smaller radius) dominates the excess of $\nabla u$:

*Case $\mathbb{S} = \mathbb{N}$:* By induction we prove for every $k \in \mathbb{N}_0$

$$\Phi(\tau^k R) \, < \, \varepsilon_1 \qquad \text{and} \qquad \Phi(\tau^k R) \, \leq \, \tau^{2k\gamma'} \, \Phi(R) \, . \tag{7.16}$$

For $k = 0$ these inequalities are trivially satisfied due to (7.14). Now, for a given $k \in \mathbb{N}_0$, we suppose $(7.16)_j$ for $j \in \{0, \dots, k\}$. In view of $k \in \mathbb{S}$ we may apply Proposition 7.3 on the ball $B_{\tau^k R}$ and we find $\Phi(\tau^{k+1}R) \le \tau^{2\gamma'}\Phi(\tau^k R) \le \tau^{2(k+1)\gamma'}\Phi(R)$. Moreover, $\Phi(\tau^{k+1}R) < \varepsilon_1$ follows from $(7.16)_0$ and $\tau < 1$. This shows that (7.16) is valid for $k+1$ and therefore, for every $k \in \mathbb{N}_0$. For proving the excess decay estimate (7.15) we first infer from Lemma **??**

$$\begin{aligned} \Phi(r) &= \fint_{B_r} \big|V_{|(\nabla u)_{0,r}|}\big(\nabla u - (\nabla u)_{0,r}\big)\big|^2 \\ &\le c(p)\left(\frac{r}{R}\right)^{-n} \fint_{B_R} \big|V_{|(\nabla u)_{0,R}|}\big(\nabla u - (\nabla u)_{0,R}\big)\big|^2 = c(p)\left(\frac{r}{R}\right)^{-n}\Phi(R) \end{aligned} \tag{7.17}$$

for all $0 < r \le R$. For a continuous analogue of the decay estimate in (7.16) we now consider $r \in (0, R]$ arbitrary. Then there exists a unique $k \in \mathbb{N}$ such that $r \in (\tau^{k+1}R, \tau^k R]$, and using (7.16) and (7.17) we conclude

$$\Phi(r) \le c(p)\left(\frac{r}{\tau^k R}\right)^{-n}\Phi(\tau^k R) \le c(p)\,\tau^{-n}\,\tau^{2k\gamma'}\,\Phi(R) \le c(p)\,\tau^{-n-2\gamma'}\left(\frac{r}{R}\right)^{2\gamma'}\Phi(R)\,, \tag{7.18}$$

and the statement of the lemma follows taking into account the dependencies of $\tau$ given above.

*Case* $\mathbb{S} \neq \mathbb{N}$*:* We define $k_0 := \min \mathbb{N} \setminus \mathbb{S}$. We obtain $\Phi(\tau^{k_0}R) < \varepsilon_0\,|(\nabla u)_{\tau^{k_0}R}|^p$ by definition of $k_0$, and the calculations leading to (7.16) reveal

$$\Phi(\tau^k R) \le \tau^{2k\gamma'}\,\Phi(R) \qquad \text{for every } k \le k_0\,. \tag{7.19}$$

Furthermore, we observe that (7.14) and (7.19) combined with the smallness condition (SC.6) ensure that the mean values of $\nabla u$ remain uniformly bounded in the sense that we have $|(\nabla u)_{0,\tau^k R}| < 2m$ for every $k \le k_0$: In the subquadratic case this can be seen as follows:

$$\begin{aligned} |(\nabla u)_{0,\tau^k R}| &\le |(\nabla u)_{0,R}| + \sum_{j=0}^{k-1} |(\nabla u)_{0,\tau^j R} - (\nabla u)_{0,\tau^{j+1}R}| \\ &< m + \tau^{-\frac{n}{p}} \sum_{j=0}^{k-1} \Phi(\tau^j R)^{\frac{1}{p}} + \tau^{-\frac{n}{2}} \sum_{j=0}^{k-1} \Phi(\tau^j R)^{\frac{1}{2}}\,|(\nabla u)_{0,\tau^j R}|^{\frac{2-p}{2}} \\ &\le m + \tau^{-\frac{n}{p}}\,(1-\tau^{2\gamma'/p})^{-1}\,\Phi(R)^{\frac{1}{p}} + \tau^{-\frac{n}{2}}\,(1-\tau^{\gamma'})^{-1}\,\Phi(R)^{\frac{1}{p}}\,\varepsilon_0^{\frac{p-2}{2p}} \le 2\,m\,. \end{aligned}$$

In the superquadratic case instead, we find proceed analogously (but the third term in the sum does not appear) and get the same result. Hence, the assumptions of Lemma 7.2 are satisfied on the ball $B_{\tau^{k_0}R}$. In view of (7.19) we thus infer for every $r \in (0, \tau^{k_0}R]$

$$\Phi(r) \le c_{it}\left(\left(\frac{r}{\tau^{k_0}R}\right)^{2\gamma'}\Phi(\tau^{k_0}R) + r^{2\gamma'}\right) \le c_{it}\left(\left(\frac{r}{R}\right)^{2\gamma'}\Phi(R) + r^{2\gamma'}\right), \tag{7.20}$$

where $c_{it}$ is the constant from Lemma 7.2 and depends only on $n, N, p, \nu, L, \gamma'$ and $m$. To finish the proof of the excess decay estimate (7.15) it still remains to consider radii $r \in (\tau^{k_0}R, R]$, but the assertion is then deduced easily from (7.19) following the line of arguments for the case $\mathbb{S} = \mathbb{N}$. Exactly as in the proof of the excess decay result stated in [14], the integer $k_0$ (which cannot be controlled and which depends on the point $x_0$ under consideration) is not reflected in the dependencies of the constant $c$ appearing in (7.15). $\square$

**Remark:** As mentioned in the introduction, the proof presented here simplifies slightly the one of [14, Lemma 13]. The key point here is the definition of the set $\mathbb{S}$ which was previously defined in a way such that the condition was required to hold on two subsequent balls (see the different smallness assumptions (7.7) and [14, (5.25)] in the excess decay estimate).

## 8 Proofs of the main results

We finally come to the proof of the partial regularity results and the dimension reduction stated in Theorem 2.1 and Theorem 2.2.

Proof (of Theorem 2.1): We consider an arbitrary point $x_0$. Then, denoting by $r_2$ the radius from Lemma 7.4, we find $m \geq 1$ and $R \in (0, r_2)$ such that $B_R(x_0) \subset \mathcal{R}$, $\Phi(x_0, R, (\nabla u)_{x_0,R}) < \varepsilon_1$ and $|(\nabla u)_{x_0,R}| < m$, i.e. such that the assumptions (7.14) of Lemma 7.4 are fullfilled. Since (7.14) is an open condition and since the functions $x \mapsto (\nabla u)_{x,R}$, $x \mapsto \Phi(x, R, (\nabla u)_{x,R})$ are continuous, we observe that (7.14) is satisfied in a small neighborhood $B_s(x_0)$ of $x_0$. Hence, due to the equivalence of the excess $\Phi(x, R, (\nabla u)_{x,R})$ and the one given in (2.5), the excess decay estimate (7.15) and Campanato's characterization of Hölder continuous functions imply the local Hölder continuity of $V_0(\nabla u)$, from which in turn the local Hölder continuity of $\nabla u$ is obtained via [15, Lemma 3]. Finally, $|\mathcal{R} \setminus \Omega_0| = 0$ follows from Lebesgue's theorem.

We now consider $x_0 \in \Omega_0$ such that additionally the assumption (2.6) is satisfied. Then we choose $\varepsilon_0$ and $r_0$ according to Lemma 7.2 (with $\beta' = \beta$ and an appropriate number $m \geq 1$). We observe that (2.6) guarantees that the assumptions in (7.5) are fullfilled for $x_0$. Since this is also an open condition we find a small neighborhood $B_s(x_0)$ of $x_0$ such that it is satisfied for all $y \in B_s(x_0)$, and therefore, we end up with the decay estimate (7.6) for all $y \in B_s(x_0)$. Consequently, Campanato's characterization of Hölder continuous functions yields that $V_0(\nabla u)$ is locally Hölder continuous with exponent $\beta$, which implies that $\nabla u$ is Hölder continuous with exponent $\min\{\beta, 2\beta/p\}$. Moreover, if $\nabla u(x_0) \neq 0$, this result may still be improved in the superquadratic case: since $\nabla u$ is already continuous, we may assume $|(\nabla u)_{y,R}| \neq 0$ in $B_s(x_0)$ (after possibly choosing $s$ smaller if necessary), and we conclude that the excess $\Phi(y, R, (\nabla u)_{y,R})$ is dominated by the quadratic term for every $y \in B_s(x_0)$. This immediately yields the improved local Hölder regularity result with exponent $\beta$ and finishes the proof. □

We shall now address the estimate on the Hausdorff dimension for the singular set stated in Theorem 2.2. To this aim we proceed as Mingione in [32, 31] and differentiate the system in a fractional sense, using fractional Sobolev spaces (for the relevant definitions we refer to [2, Chapter 7]). With this reasoning we first come up with the following fractional differentiability result:

**Lemma 8.1:** *Let $p \in (1, \infty)$ and consider a weak solution $u \in L^p(B_R(x_0), \mathbb{R}^N)$ to the system (2.1) under the assumptions (H1), (H3) and (H4). Then we have $V(\nabla u) \in W^{\beta',2}_{\mathrm{loc}}(B_R(x_0), \mathbb{R}^{nN})$ for all $\beta' < \beta$.*

Proof: We start by proving an estimate for finite differences of $V(\nabla u)$. Furthermore, consider $B_r(y) \subset B_R(x_0)$ and let $\eta \in C_0^\infty(B_{3r/4}(y), [0,1])$ be a cut-off function with $\eta \equiv 1$ on $B_{r/2}(y)$ and $|D\eta| \leq c/r$. We then introduce the finite difference operator $\tau_{s,h}$ via $\tau_{s,h}\tau(x) := \tau(x + he_s) - \tau(x)$ for an arbitrary vector $\tau$, every real number $h \in \mathbb{R}$ and $s \in \{1, \ldots, n\}$. Analogously as in [32, proof of Proposition 3.1] we then choose $\tau_{s,-h}(\eta^2 \tau_{s,h} u)$ with $s \in \{1, \ldots, n\}$ and $h$ sufficiently small as a test function in the weak formulation of (2.1). Then taking into account the assumptions (H1), (H3) and (H4) it follows

$$\fint_{B_{r/2}(y)} \big|\tau_{s,h} V(\nabla u)\big|^2 \leq c\,|h|^{2\beta} \fint_{B_r(y)} \big|V(\nabla u)\big|^2$$

for all $s \in \{1, \ldots, n\}$ and a constant $c$ depending only on $p, L, \nu$ and $r$ (but independently of $h$). Due to the uniform estimate in $h$ and the fact that $B_r(y) \subset B_R(x_0)$ was chosen arbitrarily, the assertion then follows from [2, 7.73]. □

Proof (of Theorem 2.2): As a consequence of a measure density result going back to Giusti, see [32, Section 4], the previous lemma implies that the singular set of every weak solution $u$ to (2.1) is actually not only of Lebesgue measure zero, but that its Hausdorff dimension is not greater than $n - 2\beta$. This finishes the the dimension reduction. □

## References


[1] E. Acerbi and N. Fusco, *Regularity for minimizers of non-quadratic functionals: the case* $1 < p < 2$, J. Math. Anal. Appl. **140** (1989), 115–135.

[2] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[3] L. Beck, B. Stroffolini, *Regularity results for differential forms solving degenerate elliptic systems*, Calc. Var. Partial Differential Equations **46** (2013), no. 3-4, 769-808.

[4] F. Browder, *Nonlinear elliptic boundary value problems*, Bull. Am. Math. Soc. **69** (1963), 862–874.

[5] M. Carozza, N. Fusco, and G. Mingione, *Partial Regularity of Minimizers of Quasiconvex Integrals with Subquadratic Growth*, Ann. Mat. Pura Appl. Ser. 4 **175** (1998), 141–164.

[6] E. De Giorgi, *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico*, Boll. Unione Mat. Ital., IV. **1** (1968), 135–137.

[7] L. Diening and F. Ettwein, *Fractional estimates for non-differentiable elliptic systems with general growth*, Forum Math. **20** (2008), no. 3, 523–556.

[8] L. Diening and C. Kreuzer, *Linear convergence of an adaptive finite element method for the p-laplacian equation*, SIAM J. Numer. Anal. **46** (2008), no. 2, 614–638.

[9] L. Diening, J. Málek, and M. Steinhauer, *On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications*, ESAIM, Control Optim. Calc. Var. **14** (2008), no. 2, 211–232.

[10] L. Diening, B. Stroffolini, and A. Verde, *Everywhere regularity of functionals with* $\phi$*-growth*, Manuscr. Math. **129** (2009), no. 4, 449–481.

[11] L. Diening, B. Stroffolini, and A. Verde, *The* $\varphi$*-harmonic approximation and the regularity of* $\varphi$*-harmonic maps*, J. Differential Equations **253** (2012), no. 7, 1943-1958.

[12] F. Duzaar and J. F. Grotowski, *Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximation*, Manuscr. Math. **103** (2000), 267–298.

[13] F. Duzaar, J. F. Grotowski, and M. Kronz, *Regularity of almost minimizers of quasi-convex variational integrals with subquadratic growth*, Ann. Mat. Pura Appl. **11** (2005), no. 4, 421–448.

[14] F. Duzaar and G. Mingione, *Regularity for degenerate elliptic problems via p-harmonic approximation*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **21** (2004), no. 5, 735–766.

[15] F. Duzaar and G. Mingione, *The p-harmonic approximation and the regularity of p-harmonic maps*, Calc. Var. Partial Differ. Equ. **20** (2004), 235–256.

[16] F. Duzaar and G. Mingione, *Second order parabolic systems, optimal regularity, and singular sets of solutions*, Ann. Inst. Henri Poincaré Anal. Non Linéaire **22** (2005), no. 6, 705–751.

[17] F. Duzaar and G. Mingione, *Harmonic type approximation lemmas*, J. Math. Anal. Appl. **352** (2009), no. 1, 301–335.

[18] F. Duzaar and K. Steffen, *Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals*, J. Reine Angew. Math. **546** (2002), 73–138.

[19] J. Frehse, J. Málek, and M. Steinhauer, *On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method*, SIAM J. Math. Anal. **34** (2003), no. 5, 1064–1083.

[20] N. Fusco and J. E. Hutchinson, *Partial regularity for minimisers of certain functionals having nonquadratic grwoth*, Ann. Mat. Pura Appl., IV. Ser. **155** (1989), 1–24.

[21] M. Giaquinta and G. Modica, *Almost-everywhere regularity results for solutions of non linear elliptic systems*, Manuscr. Math. **28** (1979), 109–158.

[22] M. Giaquinta and G. Modica, *Remarks on the regularity of the minimizers of certain degenerate functionals*, Manuscr. Math. **57** (1986), 55–99.

[23] E. Giusti and M. Miranda, *Sulla Regolarità delle Soluzioni Deboli di una Classe di Sistemi Ellitici Quasi-lineari*, Arch. Rational Mech. Anal. **31** (1968), 173–184.

[24] E. Giusti and M. Miranda, *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni*, Boll. Unione Mat. Ital., IV. Ser. **1** (1968), 219–226.

[25] C. Hamburger, *Regularity of differential forms minimizing degenerate elliptic functionals*, J. Reine Angew. Math. **431** (1992), 7–64.

[26] C. Hamburger, *Quasimonotonicity, Regularity and Duality for Nonlinear Systems of Partial Differential Equations*, Ann. Mat. Pura Appl. **169** (1995), 321–354.

[27] P.-A. Ivert, *Regularitätsuntersuchungen von Lösungen elliptischer Systeme von quasilinearen Differentialgleichungen zweiter Ordnung*, Manuscr. Math. **30** (1979), 53–88.

[28] J. Kristensen and G. Mingione, *The Singular Set of Minima of Integral Functionals*, Arch. Rational Mech. Anal. **180** (2006), no. 3, 331–398.

[29] P. Marcellini, *Everywhere regularity for a class of elliptic systems without growth conditions*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. **23** (1996), no. 1, 1–25.

[30] P. Marcellini and G. Papi, *Nonlinear elliptic systems with general growth*, J. Differ. Equations **221** (2006), no. 2, 412–443.

[31] G. Mingione, *Bounds for the singular set of solutions to non linear elliptic systems*, Calc. Var. Partial Differ. Equ. **18** (2003), no. 4, 373–400.

[32] G. Mingione, *The Singular Set of Solutions to Non-Differentiable Elliptic Systems*, Arch. Rational Mech. Anal. **166** (2003), 287–301.

[33] G. Mingione, *Regularity of minima: an invitation to the Dark Side of the Calculus of Variations*, Appl. Math. **51** (2006), no. 4, 355–425.

[34] T. Schmidt, *Regularity of minimizers of $w^{1,p}$-quasiconvex variational integrals with $(p,q)$-growth*, Calc. Var. Partial Differ. Equ. **32** (2008), no. 1, 1–24.

[35] T. Schmidt, *Regularity theorems for degenerate quasiconvex energies with $(p,q)$-growth*, Adv. Calc. Var. **1** (2008), no. 3, 241–270.

[36] P. Tolksdorf, *Everywhere-regularity for some quasilinear systems with a lack of ellipticity*, Ann. Mat. Pura Appl., IV. Ser. **134** (1983), 241–266.

[37] K. Uhlenbeck, *Regularity for a class of nonlinear elliptic systems*, Acta Math. **138** (1977), 219–240.