M.Sc. in Mathematical Modelling and Numerical Analysis

M.Phil. in Mathematics for Industry

Paper B (Numerical Analysis)

Friday 22 April, 1994, 9.30 a.m. – 12.30 p.m.

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1. Describe the θ-method for the solution of the heat equation $u_t = u_{xx}$ on the region $0 < x < 1$, $t > 0$, given the boundary values $u(0,t)$ and $u(1,t)$ for $t > 0$ and the initial values $u(x,0)$ for $0 < x < 1$. Give details of the numerical algorithm used to calculate the numerical solution.

Use the von Neumann analysis to show that the method is stable if $\frac{1}{2} \leq \theta \leq 1$, or if $0 < \theta < \frac{1}{2}$ and $(1-2\theta)\Delta t \leq \frac{1}{2}(\Delta x)^2$. Explain what is meant by the *truncation error* $T_j^{n+1/2}$ of the method. Assuming the existence of T such that $\left|T_j^{n+1/2}\right| \leq T$ for all j and for $0 \leq n \leq N-1$, show that the error in the numerical solution for $0 \leq n \leq N$ is bounded by $t_N T$, provided that Δt and Δx satisfy a condition which should be clearly stated.

2. Explain what is meant by the *practical stability* of a numerical method for the solution of $u_t + au_x = bu_{xx}$, where a and b are constants with $b > 0$.

A numerical method is defined by using central differences in space and a forward difference in time. Show that the von Neumann stability condition is satisfied if $2b\Delta t \leq (\Delta x)^2$, and that the practical stability condition is satisfied if $a^2(\Delta t)^2 \leq 2b\Delta t \leq (\Delta x)^2$.

How does the stability of the leapfrog scheme

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} + a\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = b\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

for the same problem depend on b?

Turn Over

3. The linear multistep method

$$\sum_{r=0}^{k} \alpha_r y_{n+1-r} = h \sum_{r=0}^{k} \beta_r f(x_{n+1-r}, y_{n+1-r})$$

is used to construct an approximate solution of the first-order ordinary differential equation $y' = f(x, y)$. Define the terms *zero-stability*, *consistency*, *convergence*, *truncation error* and *order of convergence*. How are the first three of these five terms related to each other, and how are the last two terms related for zero-stable methods?

Defining

$$\rho(\theta) := \sum_{r=0}^{k} \alpha_r \theta^{k-r}, \quad \sigma(\theta) := \sum_{r=0}^{k} \beta_r \theta^{k-r},$$

give simple tests for zero-stability and consistency. What conditions do these tests impose on the coefficients in the case $k = 2$? What further conditions are needed if the method is to be of third order? Are these conditions compatible with the conditions for zero-stability?

Show that Euler's method ($k = 1$, $\beta_0 = 0$) and the midpoint method ($k = 2$, $\alpha_1 = 0$, $\beta_0 = \beta_2 = 0$, $\beta_1 = 2$) are zero-stable. What further stability restrictions, if any, must be satisfied if the methods are to be used to obtain acceptable solutions to $y' = \lambda y$ (for real λ) with a fixed Δx?

4. The matrix $A := I - L - U$ is said to be k-*cyclic* if the corresponding Jacobi iteration matrix $B := L + U$ is *weakly cyclic of index* k. Explain the meanings of these terms, and why they are important. Explain further the meaning of *consistent ordering* for such a matrix A.

The Laplace equation $\nabla^2 u = 0$, with u given on the boundary, is solved on the unit square using the five-point finite-difference formula. The mesh has just four interior points: P at $(\frac{1}{3}, \frac{1}{3})$, Q at $(\frac{1}{3}, \frac{2}{3})$, R at $(\frac{2}{3}, \frac{1}{3})$ and S at $(\frac{2}{3}, \frac{2}{3})$. Show that the matrix A for this case is 2-cyclic. Show further that the ordering $PQRS$ is consistent but that $PQSR$ is not.

SOR with parameter ω is applied to a system of equations with the 2-cyclic consistently-ordered symmetric and positive definite matrix A; if μ is an eigenvalue of B then λ is an eigenvalue of the iteration matrix G_ω if $(\lambda + \omega - 1)/\omega = \mu\lambda^{1/2}$. Show how the optimum ω value for SOR can be deduced from this statement.

5. The problem

$$(p(x)u'')'' + q(x)u = f(x) \text{ on } (0,1)$$
$$u(0) = u''(0) = 0, \quad u(1) = u'(1) = 0,$$

with $p > 0$ and $q \geq 0$, corresponds to a simple model for the transverse displacement of a stressed beam. State the variational form of the problem in preparation for its approximation by a finite element method, paying particular attention to the boundary conditions.

Explain why Hermite cubic elements would be appropriate for the approximation and show that

$$N_1(\xi) := \tfrac{1}{4}(1-\xi)^2(2+\xi), \qquad N_2(\xi) := \tfrac{1}{4}(1-\xi)^2(1+\xi),$$
$$N_3(\xi) := \tfrac{1}{4}(1+\xi)^2(2-\xi), \qquad N_4(\xi) := \tfrac{1}{4}(1+\xi)^2(\xi-1)$$

provide a suitable set of element basis functions. Thence give a global expansion for the finite element approximation $U(x)$ in terms of the nodal parameters $\{U(x_i), U'(x_i); i = 0, 1, \ldots, n\}$ on a mesh $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$.

Show briefly how an error bound of the form

$$a(u-U, u-U) \leq \left[\left(\frac{h}{\pi}\right)^4 p_{max} + \left(\frac{h}{\pi}\right)^8 q_{max} \right] \left\| u^{(iv)} \right\|^2_{L_2(0,1)}$$

is derived. [N.B. You may assume the interpolation error bound on each interval

$$||\Delta||_{L_2(x_{i-1},x_i)} \leq \left(\frac{h_i}{\pi}\right)^2 \|\Delta''\|_{L_2(x_{i-1},x_i)}$$

for $\Delta := u - U$.]

6. Two-dimensional potential flow down a channel is modelled by

$$\nabla^2\psi = 0 \text{ in } \Omega$$

for the stream function ψ, with ψ given on the top and bottom of the channel as well as at the inflow on the left, and with $\dfrac{\partial \psi}{\partial n} = 0$ given at the outflow on the right. The domain Ω is approximated by a (straight-sided) quadrilateral mesh and ψ by a finite-element approximation Ψ using isoparametric bilinear elements.

Derive the element basis functions and the local–global coordinate mapping; show why and how the resulting approximation may be made conforming. Show that the mapping for each element is nonsingular if the quadrilateral is convex.

Formulate the complete system of discrete equations and derive integral expressions for the entries in the stiffness matrix and load vector; explain why numerical quadrature is required for their evaluation.