7302

M.Sc. in Mathematical Modelling and Numerical Analysis

Paper B (Numerical Analysis)

Friday 21 April, 1995, 9.30 a.m. – 12.30 p.m.

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1. By expanding $u(x_j, t_n+\Delta t)$ as a Taylor series, derive a Lax–Wendroff difference scheme for approximating the scalar conservation law $u_t+f_x=0$, where $f \equiv f(u)$ and $\partial f/\partial u = a(u)$. How does your derivation extend to the case of a system of conservation laws for a vector of unknowns $\mathbf{u}$, a vector of flux functions $\mathbf{f}$ and a Jacobian matrix $A(\mathbf{u}) = \partial\mathbf{f}/\partial\mathbf{u}$?

For the scalar law in which $f = au$ and a is a constant, and on a uniform mesh, use Fourier analysis to find the stability condition, the order of accuracy and the leading terms in the amplitude and phase errors. [You may use the fact that if $q(\xi)$ has an expansion $q \sim c_1\xi + c_2\xi^2 + c_3\xi^3 + \ldots$, then $\tan^{-1} q \sim c_1\xi + c_2\xi^2 + (c_3 - \frac{1}{3}c_1^3)\xi^3 + \ldots$.]

2. Write down the Crank–Nicolson difference scheme on a uniform mesh to approximate

$$u_t = (p(x)u_x)_x \quad \text{on} \quad 0 < x < 1,\ t > 0,$$

with $u(0,t) = u(1,t) = 0$ and $u(x,0) = f(x)$ given. Assuming that $p(x) \geq p_0 > 0$, state a maximum principle for the scheme, give conditions on the mesh ratio $\Delta t/\Delta x$ such that it is satisfied, and show that this implies stability. Outline how the maximum principle is used to derive an error bound.

By multiplying the difference equations by $\frac{1}{2}(U_i^{n+1} + U_i^n)$ and summing, or otherwise, show that the scheme is unconditionally stable in the l_2 norm.

3. Suppose the following numerical methods are applied to the solution of the ordinary differential equation $y''(x) + f(x, y) = 0$ at the points $x_n = x_0 + nh, h > 0$:

(i) $h^{-2}(y_{n+1} - 2y_n + y_{n-1}) + f_n = 0$,

(ii) $h^{-2}(y_{n+1} - 2y_n + y_{n-1}) + \frac{1}{12}(f_{n+1} + 10f_n + f_{n-1}) = 0$,

when $f_n = f(x_n, y_n)$. Find the truncation error of method (i) and the order of the truncation error of method (ii).

The methods are applied to $y'' + \lambda y = 0$, with $y(x_0)$, $y'(x_0)$ and $\lambda > 0$ given. Show that for the solution of (i) to remain bounded, h must satisfy $\lambda h^2 < 4$. Find the condition that h must satisfy for method (ii).

The two-point boundary value problem

$$y'' + f(x, y) = 0, \qquad y(0) = y(1) = 0,$$

is solved by shooting. Discuss the numerical difficulties associated with the method, using the case $f(x, y) = \lambda y + g(x)$, $\lambda < 0$, as an illustration. Show that for the case $\lambda > 0$ these difficulties do not arise. Can any other problems occur in the latter case?

4. Give a description of the multigrid method for solving a set of linear algebraic equations arising from the discretization of a linear elliptic boundary value problem. In particular, say what is meant by a "smoother", and explain its role in the multigrid iteration.

Consider the following finite difference discretization of the convection-diffusion equation on a uniform grid in one dimension :

$$-\frac{\epsilon}{h^2}(u_{j+1} - 2u_j + u_{j-1}) + \frac{1}{h}(u_j - u_{j-1}) = f_j, \text{ for } j = 1, \ldots, n,$$

where $\epsilon > 0$ is a constant, h is the mesh spacing, and $u_0 = u_{n+1} = 0$.
By using local mode analysis, give an expression for the smoothing factor of the Gauss-Seidel iteration applied to this discrete problem with the sweep in the direction of increasing j. Comment on the effectiveness of the Gauss-Seidel smoother in the two cases $\epsilon \ll h$ and $\epsilon \gg h$ when the iteration is used in a multigrid solver.

5. Vibrations on a string of unit length and mass per unit length $\rho(x)$, under a constant tension T, with one end fixed and the other sliding freely along a rod, give a transverse displacement $w(x,t)$ satisfying

$$\rho(x)w_{tt} = Tw_{xx}, \quad w(0,t) = 0 = w_x(1,t).$$

Suppose solutions of the form $w(x,t) = e^{i\omega t}u(x)$ are sought using piecewise linear finite elements on a uniform mesh of size $1/N$. State the eigenvalue problem for the vibration frequencies $\omega^{(i)}$ and the continuous eigenmodes $u^{(i)}(x)$, together with the Rayleigh–Ritz approximation scheme giving $\Omega^{(i)}$ and $U^{(i)}(x)$. In the case that $\rho(x)$ is constant derive the finite difference equations for $\Omega^{(i)}$ and $U^{(i)}$.

Define the Rayleigh quotient; explain the minimax principle and what it implies about the relationship between the eigenfrequencies of the continuous and discrete problems. Verify your statements in the case that $\rho(x)$ is constant by means of a Fourier analysis. [You may assume that $\sin^2 x > x^2(1 - \frac{2}{3}\sin^2 x)$ for $0 < x < \frac{1}{2}\pi$.]

6. Consider the problem $\nabla^2 u + f = 0$ on a convex region Ω with $f \in L_2(\Omega)$ and $u = 0$ on $\partial\Omega$. Give the variational (or weak) formulation of this problem and also state an extremal principle for u.

Suppose now that Ω is triangulated using triangles whose diameters are no greater than h; describe what is meant by a regular triangulation and the form taken by a piecewise linear (conforming) approximation U to u, parametrised by its values at the vertices of the triangulation.

Briefly describe the setting up of the discrete equations for U (using a local coordinate system) and how an error bound in the energy norm is obtained; give only enough detail to show how (i) the shape of the region Ω, (ii) the regularity of the triangulation, and (iii) the conforming character of the approximation each plays a rôle in the analysis.