M.Sc. in Mathematical Modelling and Numerical Analysis

Paper B (Numerical Analysis)

Friday 24 April, 1998, 9.30 a.m. – 12.30 p.m.

Candidate's may attempt as many questions as they wish. All questions will carry equal marks. Please start the answer to each question on a separate sheet.

Do not turn this page until you are told that you may do so

You may use without proof the result that the roots of the polynomial $az^2 + 2bz + c = 0$ with complex coefficients a, b, c satisfy the condition $|z| \le 1$ if and only if

$$\begin{array}{ll} \text{either} & |c| < |a| \text{ and } 2|\bar{a}b - \bar{b}c| \le |a|^2 - |c|^2, \\ \text{or} & |c| = |a|, \quad \bar{a}b = \bar{b}c \text{ and } |b| \le |a|. \end{array}$$

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1. (a) If $A \in \Re^{m\times n}$, what is the Singular Value Decomposition (SVD) of A? In terms of this SVD, what are the eigenvalues and eigenvectors of $B = A^T A$? If it is desired to calculate the eigenvalues of B using the algorithm

$B = B_1$, For $k = 1, 2, \ldots$

Factor $B_k = Q_k R_k$ where Q_k is orthogonal, R_k is upper triangular

Set $B_{k+1} = R_k Q_k$

end

show that the matrices $B_k, k = 1, 2, \ldots$, are similar and hence have the same eigenvalues.

Further, if B is tridiagonal describe how the QR factorisation of $B(= B_1)$ may be achieved by using a sequence of $n-1$ Givens (plane) rotations.

(b) State and prove the Gershgorin Circle Theorem.

If the Jacobi iteration method is applied to the linear system $Ax = b$ where

$$A = \begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 1 & 4 & 1 \\ 0 & \cdots & 0 & 1 & 4 \end{bmatrix} \in \Re^{n\times n},$$

estimate the number of iterations required to reduce the vector 2-norm of the error to less that $10^{-6} \parallel x - x^{(0)} \parallel_2$ where $x^{(0)}$ is the starting guess. Prove any theorems you use.

2. Define *absolute stability* for a linear multistep method applied to the solution of $y' = f(x, y)$ and determine the *interval of absolute stability* for the following methods:

(a) Euler's method,

(b) the backward Euler method,

and the methods given by

(c) $y_{n+1} - y_n = h(\frac{3}{2}f_n - \frac{1}{2}f_{n-1})$

(d) $\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = hf_{n+1}$

It is required to integrate

$$10^{-4}y' + y = \cos x \qquad y(0) = 0$$

on the interval (0,100). What should determine the choice of h in each of the above methods (i) on the interval (0, 0.1) and (ii) on the interval (10,100)?

Which of the above methods are suitable for integrating this differential equation? Give brief reasons for your answers.

3. The pair of first-order wave equations

$$u_t + av_x = 0$$

$$v_t + au_x = 0$$

is approximated on a staggered mesh using the scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{V_{j+\frac{1}{2}}^{n+\frac{1}{2}} - V_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x} = 0$$

$$\frac{V_{j+\frac{1}{2}}^{n+\frac{3}{2}} - V_{j+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta t} + a\frac{U_{j+1}^{n+1} - U_j^{n+1}}{\Delta x} = 0$$

Show, using Fourier analysis, that the method is stable for $| \nu | \leq 1$, where $\nu = a\Delta t/\Delta x$.

The heat equation $u_t = bu_{xx}$ is approximated using the explicit scheme

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} = b\frac{U_{j-1}^n + U_{j+1}^n - U_j^{n+1} - U_j^{n-1}}{\Delta x^2}.$$

Show that the truncation error for this scheme is

$$b\left|\frac{\Delta t}{\Delta x}\right|^2 u_{tt} + O(\Delta t^2, \Delta x^2).$$

It can be shown that this scheme is unconditionally stable. Comment on this situation.

4. (a) Write down the 5-point approximation $L_h U_P = f_P$ to Poisson's equation $u_{xx} + u_{yy} = f(x, y)$ on the unit square with boundary conditions $u = 0$ on $x = 0, 1$ and $y = 0, 1$. Find its truncation error T_P.

(b) Show that, for any mesh function U_P, satisfying $L_h U_P \geq 0$ for all interior points Ω_h of the mesh,

$$\max_{P \in \Omega_h} U_P \leq \max_{P \in \Gamma_h} U_P,$$

where Γ_h are the points of the mesh on the boundary.

(c) Given any non-negative mesh function Φ_P defined in $\Omega_h \cup \Gamma_h$ satisfying $L_h \Phi_P \geq 1$, show that the solution error $e_P = U_P - u_P$ satisfies

$$| e_P | \leq \max_{P \in \Gamma_h} \Phi_P \cdot \max_{P \in \Omega_h} | T_P | .$$

(d) Taking $\Phi = cy^2$ for suitable c, find a bound for the solution for the 5-point approximation.

5. Consider the two-point boundary value problem

$$-u'' + c(x)u = f(x), \;\; 0 < x < 1; \quad u(0) = 0, \; u'(1) = 0, \qquad (P)$$

where c is a non-negative function defined and continuous on the closed interval $[0, 1]$ and $f \in L_2(0, 1)$. What does it mean to say that u *is a weak solution to problem* (P)? Formulate the finite element approximation (P_h) to (P) on a uniform subdivision of size $h = 1/N$, $N \geq 2$, using continuous piecewise linear basis functions.

You may assume that (P_h) has a unique solution u_h.

In a suitably defined *energy norm* $\| \cdot \|_a$, show that

$$\|u - u_h\|_a = \min_{v_h \in S} \|u - v_h\|_a,$$

where S is a set that you should carefully define. Hence deduce that

$$\|u - u_h\|_a \leq Kh\|u''\|_{L_2(0,1)},$$

where K is a positive constant, independent of h and u.

6. Let $u(x,t)$ denote the solution to the initial-boundary-value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad 0 < t \leq T, \\ u(0,t) &= 0, \qquad u(1,t) = 0, \quad 0 \leq t \leq T, \\ u(x,0) &= u_0(x), \qquad\qquad 0 \leq x \leq 1, \end{aligned}$$

where $T > 0$ and $u_0 \in L_2(0,1)$. Construct a finite element method for the numerical solution of this problem, based on the backward Euler scheme with time step $\Delta t = T/M$, $M \geq 2$, and piecewise linear basis functions $\phi_i(x), i = 1, \ldots, N-1$ in x on a uniform subdivision of the interval $[0,1]$ of spacing $h = 1/N$. Writing the finite element approximation u_h^m to $u(\cdot, t^m)$ where $t^m = m\Delta t$, $0 \leq m \leq M$, in the form

$$u_h^m(x) = \sum_{j=1}^{N-1} U_j^m \phi_j(x),$$

show that u_h^{m+1} can be obtained by solving a linear system of equations of the form

$$(L + \Delta t K)\mathbf{U}^{m+1} = L\mathbf{U}^m$$

where L and K are square matrices and $\mathbf{U}^\ell = (U_1^\ell, U_2^\ell, \ldots, U_{N-1}^\ell)^T$. Calculate the entries of the matrices L and K. Why is the matrix $L + \Delta t K$ non-singular for any $\Delta t > 0$?