JMAT 7301

M.Sc. in Mathematical Modelling and Scientific Computing

Paper A (Mathematical Methods)

Thursday 27 April, 2000, 9.30 a.m. – 12.30 p.m.

Candidates may attempt as many questions as they wish. All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Mathematical Methods I

1. The differential equation

$$y'' + \lambda y = f, \quad 0 \leq x \leq 1$$

is subject to the boundary conditions

$$\begin{aligned} y'(0) &= ay(0) + by(1) \\ y'(1) &= cy(0) + dy(1), \end{aligned}$$

where a, b, c, d and λ are real constants and f is a continuous function. What is the homogeneous adjoint problem? In terms of the solutions of this adjoint problem, state the conditions on f under which the original problem has solutions.

Show that the original problem is self-adjoint if and only if $b + c = 0$, and assume this holds for the rest of the question. Let $y_1, y_2, \ldots$ be a complete set of orthonormal eigenfunctions, with corresponding eigenvalues λ_n such that $y_n'' + \lambda_n y_n = 0$. The Green's function $G_\lambda(x, x')$ is to be the solution of

$$\frac{d^2 G_\lambda}{dx^2} + \lambda G_\lambda = \delta(x - x')$$

on $[0, 1]$ with the same boundary conditions. For what values of λ does such a Green's function exist? When it does exist, express the solution of the original problem as an integral. Show also that

$$G_\lambda(x, x') = \sum_{n=1}^{\infty} c_n y_n(x) y_n(x')$$

and calculate the coefficients c_n in the terms of λ and λ_n. (You may assume that G_λ has a uniformly convergent eigenfunction expansion.)

2. (i) If the Fourier transform of $y(x)$ is defined as

$$\tilde{y}(k) = \int_{-\infty}^{\infty} y(x)e^{ikx}dx,$$

state the inversion integral expressing $y(x)$ in terms of $\tilde{y}(k)$. Assuming that the Fourier transforms of $xy(x), y'(x)$and $y''(x)$ exist, express them in terms of $\tilde{y}(k)$.

If $y'' + xy = 0$ and y has a Fourier transform $\tilde{y}$ show that $\tilde{y}(k) = A\exp(ik^3/3)$ for some constant A. Why does this involve only one arbitrary constant, whereas the differential equation is of second order?

(ii) What is meant by saying that a function $y(x, \epsilon)$ has the asymptotic expansion

$$y(x, \epsilon) \sim \sum_{n=0}^{\infty} y_n(x)\epsilon^n \text{ as } \epsilon \to 0?$$

Consider the boundary value problem

$$\begin{aligned} \epsilon y'' + y' + \exp(cy) &= 0, \quad (0 \leq x \leq 1) \\ y(0) = a, \quad y(1) &= b, \end{aligned}$$

where a, b, c and ϵ are constants with $\epsilon > 0$ and $e^{-bc} > c$. State, with a reason, whether the boundary layer as $\epsilon \to 0$ will be at $x = 0$ or at $x = 1$. Find the leading term in the asymptotic expansion of y, both inside and outside the boundary layer.

3. Use the method of multiple scales to find the leading term in the solution valid up to times of order ϵ^{-1} of the initial value problem

$$\ddot{x} + \epsilon\dot{x} + x + \epsilon a x^3 = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0,$$

where a is constant.

Now a control u is introduced such that

$$\ddot{x} + \epsilon\dot{x} + x + \epsilon a x^3 = u,$$

and the value of u is restricted by $|u(t)| \leq b\epsilon$. The system is to be driven from the same initial values to $x(t_1) = 0, \dot{x}(t_1) = 0$ in minimum time t_1. Using the Pontryagin maximum principle, find the boundary value problem whose solution would achieve this.

Mathematical Methods II

4. The equations of plane plastic strain may be written

$$\begin{aligned}\frac{\partial p}{\partial x} &= \frac{\partial \tau_1}{\partial x} + \frac{\partial \tau_2}{\partial y},\\ \frac{\partial p}{\partial y} &= \frac{\partial \tau_2}{\partial x} - \frac{\partial \tau_1}{\partial y},\end{aligned}$$

with

$$\tau_1^2 + \tau_2^2 = 1,$$

where the stresses are given by $\sigma_{xx} = -p + \tau_1$, $\sigma_{yy} = -p - \tau_1$, $\sigma_{xy} = \tau_2$. By writing $\tau_1 = \sin 2\phi$, $\tau_2 = \cos 2\phi$ show that the resulting 2×2 system is hyperbolic with characteristic slopes

$$\frac{dy}{dx} = \frac{\cos 2\phi + 1}{\sin 2\phi} = \cot\phi \qquad \text{and} \qquad \frac{dy}{dx} = \frac{\cos 2\phi - 1}{\sin 2\phi} = -\tan\phi$$

and that the Riemann invariants are $p + 2\phi$ and $p - 2\phi$ respectively.

Use the hodograph transformation to show that x and y as functions of p and ϕ satisfy

$$\begin{aligned}\frac{\partial y}{\partial \phi} &= -2\cos 2\phi\,\frac{\partial y}{\partial p} - 2\sin 2\phi\,\frac{\partial x}{\partial p},\\ -\frac{\partial x}{\partial \phi} &= 2\sin 2\phi\,\frac{\partial y}{\partial p} - 2\cos 2\phi\,\frac{\partial x}{\partial p}.\end{aligned}$$

Show that this system is hyperbolic with characteristic slopes $dp/d\phi$ given by ± 2 so that the characteristics are given $p - 2\phi =$ constant and $p + 2\phi =$ constant. Show that along the characteristics

$$\sin\phi\,dx + \cos\phi\,dy = 0,$$

and

$$\cos\phi\,dx - \sin\phi\,dy = 0$$

respectively.

5. A function ϕ satisfies Poisson's equation

$$\nabla^2\phi = f(\boldsymbol{x}) \text{ in } \Omega, \tag{1}$$

with the boundary condition

$$\phi = g(\boldsymbol{x}) \text{ on } \partial\Omega, \tag{2}$$

where Ω is a finite region in $\mathbb{R}^2$ bounded by the closed curve $\partial\Omega$. State the equation and conditions satisfied by the Green's function $G(\boldsymbol{x};\boldsymbol{\xi})$ and show that the solution to (1)–(2) is given by

$$\phi(\boldsymbol{\xi}) = \iint_\Omega G(\boldsymbol{x};\boldsymbol{\xi})f(\boldsymbol{x})dS + \int_{\partial\Omega} \frac{\partial G}{\partial n}(\boldsymbol{x};\boldsymbol{\xi})g(\boldsymbol{x})\,ds.$$

Show that if G exists it is unique.

By choosing f and g suitably, or otherwise, show that

$$\int_{\partial\Omega} \frac{\partial G}{\partial n}(\boldsymbol{x};\boldsymbol{\xi})\,ds = 1.$$

Hence show that if Ω is a square with $f = 0$ and g a different constant on each side, then the value of ϕ at the centre of the square is the average of the values on the four sides.

If the boundary condition $\phi = g$ is replaced by

$$\frac{\partial\phi}{\partial n} = h(\boldsymbol{x}) \text{ on } \partial\Omega,$$

show that a solution can exist only if

$$\iint_\Omega f\,dS = \int_{\partial\Omega} h\,ds.$$

If this condition is satisfied will the solution ϕ be unique? By considering the homogeneous problems with ϕ given on $\partial\Omega$ or $\frac{\partial\phi}{\partial n}$ given on $\partial\Omega$, explain why the existence and uniqueness properties are so different.

6. (i) Derive Charpit's equations for

$$F(x, y, u, p, q) = 0$$

in the form

$$\dot{x} = \frac{\partial F}{\partial p}, \quad \dot{y} = \frac{\partial F}{\partial q}, \quad \dot{u} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}, \quad \dot{p} = -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial u}, \quad \dot{q} = -\frac{\partial F}{\partial y} - q\frac{\partial F}{\partial u},$$

where

$$u = u(x, y), \qquad p = \frac{\partial u}{\partial x}, \qquad q = \frac{\partial u}{\partial y}.$$

(ii) By applying the WKB ansatz $\phi = Ae^{-u/\epsilon}$ to the two-dimensional convection-diffusion equation

$$\epsilon\nabla^2\phi = \boldsymbol{U}(x, y) \cdot \nabla\phi$$

in the limit as $\epsilon \to 0$ show that u satisfies

$$|\nabla u|^2 + \boldsymbol{U} \cdot \nabla u = 0.$$

Find the ray equations and show that these are straight lines in the case that $\boldsymbol{U}$ is constant.

When $\boldsymbol{U} = (1, 0)$ suppose that $u = u_0(s)$ on $x = x_0(s) = s$, $y = y_0(s) = 0$. Show that the condition for real rays to exist is $-1 < u_0'(s) < 0$. Show that when $u_0 = -\frac{s^2}{2}$ with $0 < s < 1$ the ray slopes are given by

$$\pm\frac{2(s - s^2)^{1/2}}{1 - 2s}$$

and that there is a caustic at $t = 2s(1 - s)$, where t is the distance along a ray.