JMAT 7301
JACM 7301
JACM 7C61
JACM 7C62

**M.Sc. in Mathematical Modelling and Scientific Computing**

**Paper A (Mathematical Methods)**

**Thursday 18 April, 2002, 9.30 a.m. – 12.30 p.m.**

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*Candidates may attempt as many questions as they wish. All questions will carry equal marks.*

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**Mathematical Methods I**

**1.** Suppose that

$$L[u] = \frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \alpha(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \beta(x)u \equiv u'' + \alpha u' + \beta u\,,$$

where $' \equiv \mathrm{d}/\mathrm{d}x$, and $\alpha$ and $\beta$ are continuously differentiable real-valued functions.

(a) Show that the adjoint operator of $L$ is $L^*[v] = v'' - (\alpha v)' + \beta v$. Find the adjoint boundary conditions when the primary boundary conditions are:

(i) $u(0) = u(1) = 0$;

(ii) $u(0) = u'(0) = 0$;

(iii) $u(0) = u'(1) = 0$;

(iv) $u(0) + u(1) = 0$, $u'(1) + u(0) = u'(0)$.

(b) For each of the following problems, use the Fredholm alternative to state the conditions on $b$ and $\mu$ under which it has a unique solution, the conditions under which it has no solution, and the conditions under which it has multiple solutions.

(i) $u''(x) + \mu u(x) = b(x)$, $u(0) = u(1) = 0$; $\mu$ a constant.

(ii) $u''(x) + \mu u(x) = b(x)$, $u(0) = u'(1) = 0$; $\mu$ a constant.

(c) Suppose that $u_1$ and $u_2$ both satisfy $L[u] = 0$ with $u_1(0) = 0$ and $u_2'(1) = 0$. The Wronskian of $u_1$ and $u_2$ is defined by

$$w = u_1u_2' - u_2u_1'\,.$$

Under what condition on $u_1$ and $u_2$ is $w$ nonzero? Suppose that this condition holds, and define

$$K(x,y) = \begin{cases} \dfrac{u_1(x)u_2(y)}{w(y)}\,, & 0 \le x < y \le 1\,, \\[2ex] \dfrac{u_1(y)u_2(x)}{w(y)}\,, & 0 \le y < x \le 1\,. \end{cases}$$

Show that $u(x) = \int_0^1 K(x,y)b(y)\,\mathrm{d}y$ is a solution to $L[u] = b$, $u(0) = u'(1) = 0$. What goes wrong with this approach when $\alpha = 0$ and $\beta = (n+1/2)^2\pi^2$?

**2.** Consider a self-adjoint problem $L[u] = f(x)$ on $0 \le x \le 1$, where $L$ is a linear second-order differential operator, with linear homogeneous boundary conditions at $x = 0$ and $x = 1$.

(a) Let $\lambda_n$, $n = 1, 2, \ldots$, be the eigenvalues of $L$ and let $u_n$ be corresponding real eigenfunctions (*i.e.*, $L[u_n] = \lambda_n u_n$, $n = 1, 2, \ldots$), forming a basis for each eigenspace, orthonormal with respect to the usual inner product

$$\langle u, v\rangle = \int_0^1 u(x)v(x)\,\mathrm{d}x\,.$$

Suppose that $\lambda = 0$ is *not* an eigenvalue of the problem.
Show that if $u$ solves the inhomogeneous problem $L[u] = f(x)$ where $f$ has an expansion $f(x) = \sum_{n=1}^{\infty} a_n u_n(x)$, then $\langle u_n, u\rangle = a_n/\lambda_n$. Hence show that if the solution $u$ has an eigenfunction expansion, then it is

$$u(x) = \sum_{n=1}^{\infty} \frac{a_n u_n(x)}{\lambda_n}\,. \tag{1}$$

(b) Comparing (1) with the Green's function representation of $u$, deduce that the Green's function is

$$G(x, y) = \sum_{n=1}^{\infty} \frac{u_n(x)u_n(y)}{\lambda_n}\,.$$

[You may assume that the order of integration and summation can be interchanged where necessary.]

(c) Now, suppose $L[u] = u''$ and the boundary conditions are $u(0) = u(1) = 0$. Show that the eigenvalues are $-n^2\pi^2$, with corresponding normalised eigenfunctions $\sqrt{2}\sin n\pi x$, $n = 1, 2, \ldots$. Show that the corresponding Green's function is

$$G(x, y) = \begin{cases} x(y-1)\,, & 0 \le x < y \le 1\,, \\ y(x-1)\,, & 0 \le y < x \le 1\,. \end{cases}$$

Show that the inner product of this with $\sqrt{2}\sin n\pi x$ is $-\dfrac{\sqrt{2}\sin n\pi y}{n^2\pi^2}$, confirming the eigenfunction expansion of $G$ above.

**3.** Let $y$ be the solution of

$$\varepsilon y'' + xy' + xy^2 = 0\,, \qquad y(0) = 0\,, \quad y(1) = 1/2\,,$$

where $y' \equiv \mathrm{d}y/\mathrm{d}x$ and $0 < \varepsilon \ll 1$.

(i) Show that there can be no boundary layer at $y = 1$.

(ii) Find the leading-order outer solution valid for $x = \mathcal{O}(1)$.

(iii) Show that the scaling of the boundary layer at $x = 0$ is $x = \varepsilon^{1/2}\bar{x}$.

(iv) By solving the leading-order boundary layer equation and matching with the outer solution, show that the leading-order boundary layer solution is

$$\sqrt{\frac{2}{\pi}} \int_0^{\bar{x}} \mathrm{e}^{-u^2/2}\,\mathrm{d}u\,.$$

(v) Now, suppose that the boundary condition at $x = 1$ is changed to $y(1) = 1$. What happens to the outer solution $y$ as $x \to 0$? Show that the boundary layer scaling is now $x = \varepsilon^{1/2}\bar{x}$, $y = \varepsilon^{-1/2}\bar{y}$, and find the new boundary layer equation.

## Mathematical Methods II

**4.** Show that the system

$$\begin{aligned} \frac{\partial u}{\partial x} + \sin(2v)\frac{\partial v}{\partial x} - \cos(2v)\frac{\partial v}{\partial y} &= 0 \\ \frac{\partial u}{\partial y} - \cos(2v)\frac{\partial v}{\partial x} - \sin(2v)\frac{\partial v}{\partial y} &= 0 \end{aligned}$$

is hyperbolic, with characteristics whose slope is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin(2v) \pm 1}{\cos(2v)}\,.$$

Deduce that the characteristics make an angle $v \pm \pi/4$ with the $x$-axis. Find the Riemann invariants and show that, if $u(0, y) = v(0, y) = \varphi(y)$, then $u$ and $v$ are given implicitly by

$$u = v = \varphi\left(y - x\tan(v - \frac{\pi}{4})\right).$$

**5.** Given that

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = h(x,y) \quad \text{in } D\,,$$

where $h(x,y) \geq 0$ for all $(x,y)$ in $D$ and $D$ is a closed, simply-connected region in the $(x,y)$–plane, show that $\varphi$ attains its maximum value on $\partial D$. Deduce that, if $h(x,y) \leq 0$ for all $(x,y)$ in $D$, then $\varphi$ attains its minimum value on $\partial D$.

Hence show that there is at most one solution to the problem

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= f(x,y)\,, \qquad (x,y) \in D\,, \\ u &= g(x,y)\,, \qquad (x,y) \in \partial D\,. \end{aligned}\right\} \tag{2}$$

Define the *Green's function* $G(x,y;\xi,\eta)$ for the problem (2) and write down the solution for $u$ in terms of $f$, $g$ and $G$. If $D$ is the half-plane $y \geq 0$, find $G$ and thus solve the problem

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0\,, \qquad y \geq 0\,, \\ u &= g(x)\,, \qquad y = 0\,. \end{aligned}\right\}$$

**6.** Consider the first-order nonlinear partial differential equation

$$F(p,q,u,x,y) = 0\,,$$

where

$$p = \frac{\partial u}{\partial x}\,, \quad q = \frac{\partial u}{\partial y}\,.$$

Show that, along the rays $x = x(t)$, $y = y(t)$ defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial F}{\partial p}\,, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial F}{\partial q}\,,$$

$p$, $q$ and $u$ satisfy the ordinary differential equations

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial u}\,, \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{\partial F}{\partial y} - q\frac{\partial F}{\partial u}\,, \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}\,.$$

Find, in parametric form, the solution of

$$\left(\frac{\partial u}{\partial y}\right)^2 = x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\,, \qquad y \geq 0\,,$$

with

$$u(x,0) = \frac{1}{4}x^2(2-x^2)\,, \qquad 0 < x < 1\,.$$

Find the equation of the ray through the point $(s,0)$, $0 < s < 1$, and the envelope of all such rays. Sketch the rays in the $(x,y)$–plane and find the region in which the solution for $u$ is uniquely defined.