JMAT 7301
JACM 7301
JACM 7C61
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JMAT 7301

Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods

Thursday, 24th April 2003, 9:30 a.m. – 12:30 p.m.

Candidates may attempt as many questions as they wish.

JACM 7301

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods I & II

Thursday, 24th April 2003, 9:30 a.m. – 12:30 p.m.

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JACM 7C61

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods I

Thursday, 24th April 2003, 9:30 a.m. – 11:30 a.m.

Candidates may attempt questions 1,2,3 only.

JACM 7C62

Degree Master of Science in Applied & Computational Mathematics

Mathematical Methods II

Thursday, 24th April 2003, 9:30 a.m. – 11:30 a.m.

Candidates may attempt questions 4,5,6 only.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Mathematical Methods I

Question 1

The differential equation

$$u'' + u' = f, \quad 0 \le x \le 1,$$

is subject to the boundary conditions

$$\begin{aligned} u'(0) &= au(0) + bu(1), \\ u'(1) &= 0, \end{aligned}$$

where a and b are real constants, f is a real continuous function, and $' \equiv \mathrm{d}/\mathrm{d}x$. What is the homogeneous adjoint problem? Under what condition on a and b does this adjoint problem have only the trivial solution?

The Green's function $G(x, y)$ for the original problem satisfies

$$\frac{\mathrm{d}^2 G}{\mathrm{d}x^2} + \frac{\mathrm{d}G}{\mathrm{d}x} = \delta(x - y)$$

on $[0, 1]$ with the same boundary conditions, where δ is the Dirac δ-function. Under what condition on a and b does such a Green's function exist? Assuming this condition holds, find the Green's function, and express the solution of the original problem in terms of an integral.

When the Green's function does not exist, what condition must f satisfy for a solution of the original problem to exist? If f satisfies this condition, and u_P is a particular solution of the original problem, what is the general solution?

Question 2

Consider the nonlinear oscillator equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + k(x^2 - 1)\frac{\mathrm{d}x}{\mathrm{d}t} + x = \lambda,$$

where λ and k are real constants. Show that in the limit $k \ll 1$ there is a closed trajectory with period $2\pi + O(k^2)$ providing $|\lambda| < 1$, and that its leading-order approximation is $x_0 = \lambda + 2(1 - \lambda^2)^{1/2}\cos t$. [You may use the fact that $\cos^2 y \sin y = \frac{1}{4}(\sin y + \sin 3y)$.]

Find $F(x)$ such that the system may be written in Liénard form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= k(y - F(x)), \\ k\frac{\mathrm{d}y}{\mathrm{d}t} &= \lambda - x. \end{aligned}$$

In the limit $k \gg 1$ sketch the trajectories in the (x, y)-plane for the three cases $\lambda < -1$, $|\lambda| < 1$ and $\lambda > 1$. Show that when $\lambda = 0$ the period of the closed trajectory is approximately $k(3 - 2\log 2)$.

Question 3

(a) i) Calculate the Fourier transform of $f_\epsilon(x) = e^{-\epsilon|x|}$, for $\epsilon > 0$.

ii) Explain briefly why

$$\int_{-\infty}^{\infty} \frac{2\epsilon}{\epsilon^2 + k^2}\phi(k)\,dk \to 2\pi\phi(0) \quad \text{as } \epsilon \to 0+$$

for all smooth test functions ϕ. Deduce that

$$\frac{2\epsilon}{\epsilon^2 + k^2} \to 2\pi\delta(k)$$

in the sense of distributions as $\epsilon \to 0+$, where $\delta(k)$ is the Dirac δ-function.

iii) Combining (i) and (ii) show that the natural definition of the Fourier transform of 1 is $2\pi\delta(k)$. Check that the inversion formula holds formally in this case.

(b) Find the leading-order WKB approximations as $\epsilon \to 0$ of the solutions of the equation

$$\epsilon^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{y}{x} = 0$$

in the form $y \sim Ae^{i\phi/\epsilon}$. Hence show that the large eigenvalues of the generalised eigenvalue problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda\frac{y}{x} = 0, \qquad y(1) = 0, \quad y(4) = 0,$$

are given approximately by

$$\lambda \sim \frac{n^2\pi^2}{4},$$

for integer n.

Mathematical Methods II

Question 4

(a) Find the solution $u(x,t)$ of the problem

$$\frac{\partial u}{\partial t} + (1-u)^2 \frac{\partial u}{\partial x} = 0 \qquad t > 0,$$
$$u = 1 - |x| \qquad t = 0, \ -1 \le x \le 1.$$

Show that the solution is uniquely defined in a region bounded by segments of the curves $t = 0$, $t - x = 1$, $x - t = 1$ and $4xt = -1$, and sketch the characteristic projections in the (x,t) plane.

(b) The function $u(x,y)$ satisfies the Eikonal equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1,$$

and is equal to zero on a plane curve Γ given parametrically by $x = x_0(s)$, $y = y_0(s)$, where s is arc-length (that is, $(\mathrm{d}x_0/\mathrm{d}s)^2 + (\mathrm{d}y_0/\mathrm{d}s)^2 \equiv 1$). State what is meant by a ray, and show that the rays are perpendicular to Γ. Find two possible solutions $u(x,y)$ in parametric form.

[Charpit's equations for the equation $F(p,q,u,x,y) = 0$ are

$$\dot{x} = F_p, \quad \dot{y} = F_q, \quad \dot{p} = -F_x - pF_u, \quad \dot{q} = -F_y - qF_u, \quad \dot{u} = pF_p + qF_q,$$

where $p = \partial u/\partial x$ and $q = \partial u/\partial y$.]

Question 5

Define the *characteristics* of the system of partial differential equations

$$\boldsymbol{A}\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{B}\frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{c},$$

where $\boldsymbol{u}$ is an n-dimensional vector, and the $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and the vector $\boldsymbol{c}$ are all functions of x, t and $\boldsymbol{u}$. Show that the slope of a characteristic satisfies

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda \quad \text{where} \quad \det(\boldsymbol{B} - \lambda\boldsymbol{A}) = 0.$$

What does it mean for such a system to be *hyperbolic*? Suppose the solution $\boldsymbol{u}(x,t)$ of an n-dimensional hyperbolic system is discontinuous across a curve C in the (x,t)-plane. Explain, using a causality argument, how many characteristics in general must enter and leave C at each point.

Show that the system

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uv) &= 0 \\ \frac{\partial}{\partial t}(uv) + \frac{\partial}{\partial x}\left(uv^2 + \frac{u^3}{3}\right) &= 0 \end{aligned}\right\} \qquad (\star)$$

is hyperbolic, and find its characteristics and Riemann invariants. Deduce that the characteristics are straight lines.

Write down the Rankine-Hugoniot conditions, corresponding to ($\star$), that must be satisfied across a shock. Show that, if the shock speed is $\mathrm{d}x/\mathrm{d}t = \dot{x}$ and $q = (v - \dot{x})\,u$, then

$$[q]_-^+ = \left[\frac{q^2}{u} + \frac{u^3}{3}\right]_-^+ = 0.$$

Question 6

(a) Verify that the function

$$F(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

satisfies the problem

$$\begin{aligned} \frac{\partial F}{\partial t} &= \frac{\partial^2 F}{\partial x^2} && t>0, \quad -\infty < x < \infty, \\ F &= \delta(x) && t = 0, \quad -\infty < x < \infty, \\ F &\to 0 && t>0, \quad x \to \pm\infty, \end{aligned}$$

where δ is the Dirac delta-function.

(b) Define the *Green's function* $G(x,t;\xi,\tau)$ for the problem

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} &= 0 && t>0, \quad 0<x<\infty, \\ u &= f(t) && t>0, \quad x=0, \\ u &\to 0 && t>0, \quad x\to\infty, \\ u &= 0 && t=0, \quad 0<x<\infty, \end{aligned}\right\} \qquad (\star)$$

and find $u(\xi,\tau)$ in terms of $f(t)$ and G.

(c) Use part (a) to find $G(x,t)$ and hence show that

$$u(\xi,\tau) = \frac{2}{\sqrt{\pi}} \int_{\xi/(2\sqrt{t})}^{\infty} f\left(\tau - \frac{\xi^2}{4s^2}\right) \mathrm{e}^{-s^2}\,\mathrm{d}s. \qquad (\dagger)$$

(d) For the case $f(t) \equiv 1$, find a similarity solution of ($\star$) and show that the solution agrees with ($\dagger$).