JMAT 7303

Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods II

TRINITY TERM 2009
Thursday, 23rd April 2009, 9:30 a.m. – 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Applied Partial Differential Equations

Question 1

Consider the first order partial differential equation for $u(x, y)$:

$$F(x, y, u, p, q) = 0, \text{ where } p := u_x,\ q := u_y.$$

Here F is C^2 in its arguments, and initial data

$$\{x_0(s), y_0(s), u_0(s) \text{ for } 0 \le s \le 1\}$$

is given.

(a) State Charpit's equations for solving this problem, and show that along their solution $F = 0$ (you should specify their initial data clearly).

(8 marks)

(b) If

$$\frac{\partial u}{\partial s} = p\frac{\partial x}{\partial s} + q\frac{\partial y}{\partial s}$$

holds along these solutions, explain why these solutions provide a parametric solution on $F = 0$.

(3 marks)

(c) Show that $u = -y$ is a solution of

$$\frac{\partial u}{\partial x} + (1 - y)\left(\frac{\partial u}{\partial y}\right)^2 - u - 1 = 0,$$

$$u(x, 0) = 0 \text{ for } 0 \le x \le 1,$$

and find the region in the $x - y$ plane where this solution is determined by the initial data. Find the other solution that is determined by the initial data.

(14 marks)

Question 2

The continuously differentiable functions $u(x, y)$, $v(x, y)$ satisfy

$$u_x + uu_x + 4vv_y = 1,$$

$$v_x + vu_y + uv_y = \frac{1}{2}.$$

(a) Find where the system is hyperbolic, and give the characteristic directions in the $x-y$ plane. (6 marks)

(b) For $x > 0$ the initial data

$$u(x, 0) = x + 1,\ v(x, 0) = 1 + \frac{1}{2}x,$$

determines solutions in a region $\mathcal{D}$ of the $x - y$ plane.

Find $u(x, y)$, $v(x, y)$ and $\mathcal{D}$; illustrate $\mathcal{D}$ on a sketch. (12 marks)

(c) For $y > 0$ the initial data is

$$u(1, y) = \frac{1}{2}(1 + y),\ \ v(1, y) = \frac{1}{4}(1 + y).$$

Find $u(x, y)$, $v(x, y)$ and show where in the $x - y$ plane these solutions are determined by the intial data. (7 marks)

Question 3

(a) The twice continuously differentiable function $u(x,t)$ satisfies

$$\begin{array}{ll} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + f(x,t,u) & \text{for } x,t > 0, \\ u(x,t) \to 0 \text{ as } x \to \infty & \text{for fixed } t > 0, \\ u(x,0) = 0,\ u(0,t) = 1 & \text{for } x,t > 0, \\ u(x,t) \text{ bounded} & \text{for } x,t \geq 0, \end{array}$$

where the continuous function f satisfies $f(x,t,u) < 0$ for $x,t > 0$, and $f \to 0$ as $u \to 0$ or $x \to \infty$.

Show that $u(x,t)$ cannot have a maximum in $x,t > 0$.

State conditions that a Green's function $G(x,t,\xi,\eta)$ for the above problem must satisfy in order that $u(\xi,\eta)$ may be expressed in terms of integrals involving G and f. (10 marks)

(b) Determine the function $f \in C^2$ and real constants α and β so that, for $y > 0$,

$$u(x,y) = y^{\alpha} f\big(\frac{x}{y^{\beta}}\big),$$

satisfies the problem

$$u_y = (uu_x)_x \text{ for } 0 < x < y^{\beta},$$

$$u_x(0,y) = 0 \ \text{ and } \ \int_0^{y^{\beta}} u(x,y)\mathrm{d}x = 1.$$

(15 marks)

Question 4

Let $\mathcal{D}$ be a closed, bounded region of the $x - y$ plane with smooth boundary $\partial\mathcal{D}$ and outward pointing unit normal n.

The function $u(x, y)$ satisfies

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \qquad (x,y) \in \mathcal{D}, \\ \alpha(x,y)u + \beta(x,y)\frac{\partial u}{\partial n} &= g(x,y), \quad (x,y) \in \partial\mathcal{D}. \end{aligned} \tag{1}$$

(a) Show that if a solution to (1) exists, then it is unique provided $\alpha > 0$ and $\beta \geq 0$. (7 marks)

(b) For the case $\alpha \equiv 0$, $\beta \equiv 1$, show that (1) has no solution unless a *solvability condition* (which you should find) is satisfied, in which case the solution is nonunique. (5 marks)

(c) Now suppose that $\beta \equiv 1$ and $\mathcal{D}$ is the unit circle $x^2 + y^2 \leq 1$. Show that nonzero solutions to the homogeneous problem (with $g \equiv 0$) exist if $\alpha \equiv -1$. [*Hint: look for a separable solution in polar coordinates.*] (5 marks)

(d) For the case $\alpha > 0, \beta \geq 0$, show that, if the function $G(x, y, \xi, \eta)$ is defined to satisfy, for $(x, y) \neq (\xi, \eta)$,

$$\begin{aligned} \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} &= 0, \quad (x,y) \in \mathcal{D}, \\ \alpha(x,y)G + \beta(x,y)\frac{\partial G}{\partial n} &= 0, \quad (x,y) \in \partial\mathcal{D}, \end{aligned}$$

and

$$G \sim -\frac{1}{2\pi}\log|(x,y) - (\xi,\eta)| \ \text{ as } \ (x,y) \to (\xi,\eta),$$

then the solution of (1) is

$$u(\xi,\eta) = -\oint_{\partial\mathcal{D}} \left(\frac{g}{\alpha}\right)\frac{\partial G}{\partial n}\mathrm{d}s.$$

(8 marks)

Section B — Further Applied Partial Differential Equations

Question 5

(a) Use the language of distributions to rigorously define the delta function $\delta_\xi \equiv \delta(x-\xi)$ with singularity $\xi = (\xi_1, \xi_2, \xi_3)$, where $\xi \in \Omega$ and Ω is a subset of 3-dimensional Euclidean space $\mathbb{R}^3$. Be sure to explicitly specify and describe the proper space for any function you define.

(5 marks)

(b) Give a rigorous definition of what it means for a sequence of locally integrable scalar functions $f_n(x)$ ($x \in \mathbb{R}$) to converge weakly as $n \to \infty$.

(2 marks)

(c) Show that $u = H(x-t)$ is a weak solution of

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0\,.$$

(8 marks)

(d) Give a rigorous definition of what it means for a sequence of scalar functions $f_n(x)$ to converge uniformly as $n \to \infty$. Consider

$$f(x) = 2H(x) - 1$$

for $x \in [-\pi, \pi]$ and $f(x+2\pi) = f(x)$. Derive the Fourier series of f and show that the sum of the first $(2n+1)$ terms is given by

$$\frac{2}{\pi} \int_0^x \frac{\sin[2(n+1)y]}{y} \frac{y}{\sin y} \mathrm{d}y\,.$$

Does this sum converge uniformly? (You must either prove uniform convergence or show precisely why it does not hold.)

(10 marks)

You may use without proof the fact that the function $\mathrm{Si}(z)$, *defined by*

$$\mathrm{Si}[z] := \frac{2}{\pi} \int_0^z \frac{\sin u}{u} \mathrm{d}u\,,$$

satisfies $\mathrm{Si}(\pi) \approx 1.18$.

Question 6

(a) Must the product of two test functions still be a test function? Prove that this is true or use a counterexample to show that it need not be true.

(3 marks)

(b) Must the product of two distributions still be a distribution? Prove that this is true or use a counterexample to show that it need not be true.

(7 marks)

(c) Consider the second-order differential operator

$$L = a_2(x)D^2 + a_1(x)D + a_0(x)\,,$$

where $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and the coefficients $a_k(x) \in C^2(R_1)$ are twice differentiable functions defined on a closed interval $R_1 \subseteq \mathbb{R}^1$. Derive an expression for the formal adjoint L^* as well as the conjunct. What conditions must the coefficients $a_k(x)$ satisfy in order for L to be formally self-adjoint?

(5 marks)

(d) Consider $f(x) = 1/|x|$, where $x \in \mathbb{R}^3$. Calculate the Laplacian of $f(x)$ in the distributional sense.

(10 marks)