JMAT 7303

Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods II

TRINITY TERM 2010

Thursday, 22nd April 2010, 9:30 a.m. – 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Applied Partial Differential Equations

Question 1

Consider the first order partial differential equation for $u(x, y)$:

$$F(x, y, u, p, q) = 0, \ \text{ where } \ p := u_x, \ q := u_y,$$

together with the initial data

$$\{x_0(s), y_0(s), u_0(s) \text{ for } 0 \leq s \leq 1\}.$$

(a) Let F be C^2 in its arguments. State Charpit's equations for solving this problem, and state appropriate initial data for their solutions. Show that along their solutions $F = 0$.

(8 marks)

(b) Find a solution, in parametric form, for

$$x\frac{\partial u}{\partial x} + \frac{1}{8}\Big(\frac{\partial u}{\partial y}\Big)^3 + u = 2,$$

$$u(x, 0) = 2 + \frac{1}{4}x^3 \ \text{ for } \ 0 \leq x \leq 1.$$

Find and sketch the region in the $x - y$ plane where your solution is determined by the initial data.

(17 marks)

Question 2

The continuously differentiable functions $u(x, y), v(x, y)$ satisfy, for $v \geq 0$,

$$\begin{aligned} u_x + uu_y + 2v_y &= 1, \\ v_x + 2vu_y + uv_y &= \sqrt{v}. \end{aligned}$$

(a) Find where this system is hyperbolic, and give the characteristic directions in the (x, y) plane. Hence show that $u - 2\sqrt{v}$ and $u + 2\sqrt{v} - 2x$ are constant (respectively) along characteristics defined by your previously found directions.

(10 marks)

(b) Find u and v for $x > 0$ when $u(0, y) = 2, v(0, y) = 1$ for $y > 0$. (10 marks)

(c) Find and sketch the characteristics passing through $(0, s)$ for $s > 0$ for the data of part (b). Find the region in the (x, y) plane where your solution in part (b) is uniquely defined by its initial data.

(5 marks)

Question 3

Consider a twice continuously differentiable solution $u(x, y)$ to the problem (P):

$$u_y = u_{xx} - \frac{x}{2t} u_x - u^3 \quad \text{for} \quad x, y > 0$$

where $u \to \frac{1}{\sqrt{2y}}$ as $x \to \infty$ for any fixed $y > 0$.

(a) Show that any maximum of $u(x, y)$ must occur on either $x = 0, y > 0$ or in the limit for fixed $x \geq 0$ when $y \to 0$. (6 marks)

(b) Solutions $u_1(x, y)$ and $u_2(x, y)$ of (P) satisfy $u_1(0, y) = u_2(0, y)$ for $y > 0$, and for fixed $x \geq 0$, $\lim_{y\to 0}(u_1(x, y) - u_2(x, y)) = 0$. Show that $u_1(x, y) = u_2(x, y)$ for $x, y > 0$. (5 marks)

(c) Show that (P) has a similarity solution of the form $u(x, y) = y^{-\frac{1}{2}} f(\eta), \eta = \frac{x}{y^\beta}$, where

$$f''(\eta) = f^3(\eta) - \frac{1}{2} f(\eta),$$

for some value of β. Hence show that, for some constant A,

$$u(x, y) = \frac{1}{\sqrt{2y}} \Big(\frac{1 - Ae^{-\frac{x}{\sqrt{y}}}}{1 + Ae^{-\frac{x}{\sqrt{y}}}} \Big).$$

(11 marks)

(d) Find A when, for $y > 0$,

$$\int_0^\infty \Big(\frac{1}{\sqrt{2y}} - u(x, y) \Big) \mathrm{d}x = \sqrt{2} \ln 2.$$

(3 marks)

Question 4

(a) State the problem satisfied by the Green's function for the boundary value problem

$$-\nabla^2 u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad \text{for} \quad x^2+y^2<1,$$

$$u(x,y) = 0 \quad \text{for} \quad x^2+y^2=1,$$

and use it to express $u(x,y)$ in terms of an area integral involving $f(x,y)$. (15 marks)

(b) If $f(x,-y) = f(x,y)$ for all (x,y), show that your solution for part (a) satisfies

$$\frac{\partial u}{\partial y} = 0 \quad \text{on} \quad \{-1 \le x \le 1, y=0\}.$$

(4 marks)

(c) In terms of the Green's function for part (a), find the Green's function for the problem

$$-\nabla^2 u = f(x,y) \quad \text{for} \quad x^2+y^2<1, y>0,$$

$$u(x,y) = g(x,y) \quad \text{for} \quad x^2+y^2=1, y>0,$$

$$\frac{\partial u}{\partial y} = 0 \quad \text{for} \quad -1 \le x \le 1, y=0.$$

(6 marks)

Section B — Further Applied Partial Differential Equations

Question 5

Define the Mellin transform and the inverse Mellin transform for a function $f(r)$.
Determine the Mellin transform of

$$r\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}f}{\mathrm{d}r}\Big),$$

providing certain conditions are satisfied which you should state. **(7 marks)**

Use the Mellin transform to find the potential function $u(r,\theta)$, satisfying Laplace's equation

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad \text{for} \quad 0 < r < \infty, 0 < \theta < \alpha,$$

subject to $u(r,\alpha) = u_0(r)$, $\frac{\partial u}{\partial \theta} = 0$ on $\theta = 0$, for $r > 0$. **(10 marks)**

Now suppose

$$u_0(r) = \begin{cases} 1 & \text{for} \quad 0 < r \leq 1 \\ 0 & \text{for} \quad 1 < r < \infty \end{cases}.$$

Show that

$$u(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{r^s}\frac{\cos s\theta}{s\cos(s\alpha)}\mathrm{d}s.$$

Hence show that

$$u(r,\theta) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \sin(k\log_e r)\frac{\cosh k\theta}{\cosh k\alpha}\mathrm{d}k.$$

(8 marks)

Question 6

Show that the general modified Korteweg de Vries equation

$$u_t + (n+1)(n+2)u^n u_x + n^2 u_{xxx} = 0 \quad (1)$$

for $n = 1, 2$, is invariant under a rescaling of t, x and u which you should find.

Show that (1) is invariant under the transformation $u \mapsto -u$ if and only if n is even. **(7 marks)**

Show that solitary wave solutions $u = f(x - ct)$ are possible with c positive but not with c negative.

Show further that, for n even, there are waves of both elevation and depression but, for n odd, there are only waves of elevation. **(10 marks)**

Show that

$$u(x,t) = A\,\text{cosech}^2(B(x - Vt))$$

is a singular solution of the Korteweg de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0$$

where A, B and V are to be determined. **(8 marks)**