JMAT 7303

Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods II

TRINITY TERM 2011

Thursday, 18th April 2013, 9:30 a.m. – 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Applied Partial Differential Equations

Question 1

Consider the first-order partial differential equation for $u(x, y)$:

$$F(x, y, u, p, q) = 0,$$

where $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$ and F is C^2 in its arguments. Suppose that initial data is specified on a curve in the (x, y)-plane so that

$$x = x_0(s), \quad y = y_0(s), \quad u = u_0(s) \quad \text{for} \quad s_1 \leqslant s \leqslant s_2.$$

(a) State Charpit's equations for this problem, together with appropriate initial data for their solution. Show that $F = 0$ along their solutions. **[8 marks]**

(b) Find a solution in parametric form for

$$p^2 x + qy = u, \tag{1}$$

with $x = x_0(s),\ y = y_0(s)$ and $u = u_0(s)$ for $s_1 \leqslant s \leqslant s_2$. Is your solution unique? **[8 marks]**

(c) Determine all solutions to (1) when the initial data reads

$$u(1, s) = 1 \quad \text{for} \quad s_1 \leqslant s \leqslant s_2.$$

Explain clearly where in the (x, y)-plane your solutions are defined. **[9 marks]**

Question 2

(a) You are given that $F(t, x, u) = \text{constant}$ and $G(t, x, u) = \text{constant}$ are linearly independent solutions of the ordinary differential equations

$$\frac{\mathrm{d}t}{a(t,x,u)} = \frac{\mathrm{d}x}{b(t,x,u)} = \frac{\mathrm{d}u}{c(t,x,u)}.$$

Explain why F and G satisfy the partial differential equations

$$a\frac{\partial F}{\partial t} + b\frac{\partial F}{\partial x} + c\frac{\partial F}{\partial u} = 0,$$

$$a\frac{\partial G}{\partial t} + b\frac{\partial G}{\partial x} + c\frac{\partial G}{\partial u} = 0.$$

Show further that if $u(t, x)$ is determined implicitly by the relation

$$F(t, x, u) = \Lambda(G(t, x, u))$$

where $\Lambda(.)$ is any suitably smooth function, then $u(t, x)$ satisfies the partial differential equation

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = c.$$

[13 marks]

(b) Hence, or otherwise, determine the general solution of the following partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(u^2) = u^2. \tag{2}$$

[6 marks]

(c) Determine an explicit solution to equation (2) when $u(0, x) = e^{-x}$ for $0 \leqslant x < \infty$. Where is your solution valid? **[6 marks]**

Question 3

(a) Suppose that $T(x,t) > 0$ satisfies the partial differential equation

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} - T, \tag{3}$$

By taking a Fourier transform in x, or otherwise, show that the solution of (3) that satisfies $T(x,0) = \delta(x)$ and $T \to 0$ as $x \to \pm\infty$ is

$$T(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left\{-t - \frac{(x-t)^2}{4t}\right\}.$$

[6 marks]

[You may use, without proof, the result that

$$\int_{-\infty}^{\infty} \exp\left\{-(y-\theta)^2\right\} \mathrm{d}y = \sqrt{\pi}$$

for all $\theta \in \mathbb{C}$.]

(b) Suppose now that $T(x,t) > 0$ satisfies equation (3) in $x_1 < x < x_2$ and $0 < t < \tau$, with

$$\left.\begin{array}{lll} T(x,0) & = T_0(x), & x_1 < x < x_2, \\ T(x_1,t) & = T_1(t), & 0 < t < \tau, \\ T(x_2,t) & = T_2(t), & 0 < t < \tau. \end{array}\right\} \tag{4}$$

Show that, if a solution exists, then $T(x,t)$ attains its maximum value on $x = x_1, x_2$ or at $t = 0$. Show further that such a solution is unique. **[8 marks]**

(c) Suppose that $T(x,t)$ satisfies equations (3) and (4). State the conditions satisfied by the associated Green's function $G(x,t;\xi,\tau)$. Derive an expression for $T(\xi,\tau)$ in terms of the Green's function and the boundary and initial conditions. **[6 marks]**

(d) Use the results from (b) to determine an analytical expression for $G(x,t;\xi,\tau)$ when $x_1 \to -\infty$ and $x_2 \to \infty$. **[5 marks]**

Question 4

Consider the system

$$\left.\begin{aligned} 0 &= \rho_t + (\rho u)_x, \\ 0 &= u_t + u u_x + 2\rho x. \end{aligned}\right\} \tag{5}$$

(a) Determine the characteristics of equations (2) and show that $u \pm 2\sqrt{2\rho}$ are Riemann invariants.

[12 marks]

(b) Suppose that u and ρ are everywhere smooth, except on a smooth curve $x = S(t)$, which splits the upper half plane ($t > 0, -\infty < x < \infty$) into two regions, D_+ and D_-, say. For $(x, t) \in D_\pm$, we define

$$[u] = u_+ - u_- \neq 0, \quad [\rho] = \rho_+ - \rho_- \neq 0,$$

where

$$u_\pm = \lim_{x \to S(t)} u(x, t), \quad \rho_\pm = \lim_{x \to S(t)} \rho(x, t).$$

Show that the curve $x = S(t)$ satisfies

$$\begin{bmatrix} \rho \\ u \end{bmatrix} \frac{\mathrm{d}S}{\mathrm{d}t} = \begin{bmatrix} \rho u \\ \frac{u^2}{2} + 2\rho \end{bmatrix}.$$

What are the conditions for the shock to be causal? **[7 marks]**

(c) Suppose further that $u_+ = 0$ and $\rho_- = 1$. Determine u_- and $\mathrm{d}S/\mathrm{d}t$ in terms of ρ_+. For what values of ρ_+ is the shock causal? **[6 marks]**

Section B — Further Applied Partial Differential Equations

Question 5

The Legendre polynomials $P_n(x)$ for $n = 0, 1, 2, \ldots$ are solutions of

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_n(x)\right) + n(n+1)P_n(x) = 0$$

on the interval $|x| \leq 1$, normalised by $P_n(1) = 1$. Their generating function is

$$\phi(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x).$$

(a) Show that

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad \text{for } n \geq 2.$$

[6 marks]

(b) Show that

$$\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x = 0 \text{ for } n \neq m.$$

[4 marks]

(c) Hence show that

$$\int_{-1}^{1} P_n(x)P_m(x)\,\mathrm{d}x = \frac{2}{2n+1}\delta_{nm}.$$

[6 marks]

(d) Hence find the Green's function for

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda y = f(x),$$

where λ is not an eigenvalue of Legendre's equation.

Calculate y explicitly for $f(x) = 5x^3$ and $\lambda = 7$.

[9 marks]

Hint: The first few Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, $P_3(x) = \frac{1}{2}(5x^3-3x)$. *You may wish to consider the integral*

$$\int_{-1}^{1} \frac{1}{1-2xt+t^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{2}{2n+1}t^{2n}.$$

Question 6

(a) Write down the forward Hankel transform of a function $u(r)$ with respect to r, and the corresponding inverse transformation.

[3 marks]

(b) Derive the expression for the forward Hankel transform from the two-dimensional Fourier transform.

[7 marks]

(c) Small displacements u of an infinite elastic membrane are governed by

$$\frac{\partial^2 u}{\partial t^2} + \nabla^4 u = 0,$$

where ∇^4 is the two-dimensional biharmonic operator. Show that the solution for the axisymmetric initial conditions $u(r, 0) = f(r)$, $u_t(r, 0) = 0$ may be written as

$$u(r,t) = \int_0^\infty \xi f(\xi) \left[\int_0^\infty k J_0(k\xi) J_0(kr) \cos(k^2 t) \, \mathrm{d}k \right] \mathrm{d}\xi.$$

[15 marks]

Hint: You may use the integral representation $J_n(x) = (2\pi)^{-1} \int_0^{2\pi} \exp\{i(n\phi - x\cos\phi)\} \mathrm{d}\phi$.