JMAT 7303

Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods II

TRINITY TERM 2015

Thursday, 23rd April 2015, 9:30 a.m. – 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Applied Partial Differential Equations

Question 1

(a) Suppose that $T(x,t) > 0$ satisfies

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - Tf, \tag{1}$$

in the rectangular domain $\mathcal{D} = \{(x,t) : 0 < x < L, 0 < t < \tau\}$. Suppose also that $f(x,t) > 0$ and

$$T(x,0) = T_{in}(x), \qquad 0 < x < L, \tag{2}$$
$$T(0,t) = T_0(t), \qquad 0 < t < \tau, \tag{3}$$
$$T(L,t) = T_L(t), \qquad 0 < t < \tau, \tag{4}$$

where $T_{in}(x), T_0(t)$ and $T_L(t)$ are known smooth functions. Show that if T has a positive maximum then it must be attained on either $x = 0$, $x = L$ or at $t = 0$. Show further that such a solution is unique.

[8 marks]

(b) Let $A(x,t)$ and $B(x,t)$ be (2×2) matrices and let $c(x,t) = (c_1(x,t), c_2(x,t))^T$ and $\mathbf{u} = (u(x,t), v(x,t))^T$ be continuously differentiable. Suppose that for $0 < x$ and $0 < t$

$$A\frac{\partial u}{\partial t} + B\frac{\partial u}{\partial x} = c \text{ for } (x,t) \in R^2. \tag{5}$$

State conditions on A and B that make this system of partial differential equations parabolic.

[4 marks]

(c) Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{6}$$

and suppose further than $u(0,t) = 1/t$ and $v(0,t) = V^*/t$ for $t > 0$, where V^* is a positive constant. Use the result from part (b) to show that this system is parabolic.

[5 marks]

(d) Determine the real constants α and β and the functions $f(.)$ and $g(.)$ for which equation (5), with A, B and C defined by equation (6), admits solutions of the form $u(x,t) = t^\alpha f(x/t^\beta)$ and $v(x,t) = t^\alpha g(x/t^\beta)$. Derive analytical expressions for $u(x,t)$ and $v(x,t)$. Comment briefly on where you solution is valid.

[8 marks]

Question 2

Consider the first order partial differential equation for $u(x, y)$

$$F(x, y, u, p, q) = 0$$

where

$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y},$$

and F is C^2 in its arguments. Suppose that $u = u_0(x, y)$ is specified on a curve in the (x, y)-plane on which $x = x_0(s)$, $y = y_0(s)$ and $s_1 \leqslant s \leqslant s_2$.

(a) State Charpit's equations for this problem, together with appropriate initial data for their solution. Show that $F = 0$ along their solution.

[8 marks]

(b) Use Charpit's equations to find a solution in parametric form for

$$(p^2 + q^2)u = 1,$$

with $u = u_0(x, y)$ on $x = x_0(s)$, $y = y_0(s)$, $s_1 \leqslant s \leqslant s_2$.

[8 marks]

(c) Suppose that $u = 1$ on $x^2 + y^2 = 1$. Obtain an explicit solution for $u = u(x, y)$ in the interior of the circle and state clearly where your solution is defined. Show that the maximum value of u is $(5/2)^{2/3}$. Is your solution unique?

[9 marks]

Question 3

Consider the system

$$\left.\begin{aligned} 0 &= n_t + (nu)_x, \\ 0 &= u_t + uu_x + \tfrac{1}{n}\, f_x, \end{aligned}\right\} \tag{7}$$

where $f = f(n) = n^{\alpha}$ and $\alpha > 1$ is a constant.

(a) Determine the characteristics of equations (7) and the associated Riemann invariants.

[12 marks]

(b) Consider the case for which $\alpha = 3$. Suppose that u and n are everywhere smooth, except on a smooth curve $x = S(t)$ which splits the upper half plane ($t > 0, -\infty < x < \infty$) into two regions, D_+ and D_- say. For $(x, t) \in D_\pm$, we define

$$[u] = u_+ - u_- \neq 0, \quad [n] = n_+ - n_- \neq 0,$$

where

$$u_\pm = \lim_{x \to S(t)} u(x,t), \quad n_\pm = \lim_{x \to S(t)} n(x,t).$$

Show that the curve $x = S(t)$ satisfies

$$\begin{bmatrix} n \\ u \end{bmatrix} \frac{dS}{dt} = \begin{bmatrix} nu \\ \frac{u^2}{2} + 3n^2 \end{bmatrix}.$$

Determine the conditions for the shock to be causal.

[7 marks]

(c) Suppose further that $u_+ = 0$ and $n_- = 1$. Determine u_- and dS/dt in terms of n_+. For what values of n_+ is the shock causal?

[6 marks]

Question 4

(a) You are given that $F(t, x, u) = \text{constant}$ and $G(t, x, u) = \text{constant}$ are linearly independent solutions of the ordinary differential equations

$$\frac{dt}{a(t,x,u)} = \frac{dx}{b(t,x,u)} = \frac{du}{c(t,x,u)}.$$

Explain why F and G satisfy the partial differential equations

$$a\frac{\partial F}{\partial t} + b\frac{\partial F}{\partial x} + c\frac{\partial F}{\partial u} = 0,$$

$$a\frac{\partial G}{\partial t} + b\frac{\partial G}{\partial x} + c\frac{\partial G}{\partial u} = 0.$$

Show further that if $u(t, x)$ is determined implicitly by the relation

$$F(t, x, u) = \Gamma(G(t, x, u))$$

where $\Gamma(.)$ is any suitably smooth function, then $u(t, x)$ satisifes the partial differential equation

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = c.$$

[13 marks]

(b) Hence, or otherwise, determine the general solution of the following partial differential equation:

$$\frac{\partial u}{\partial t} + tu\frac{\partial u}{\partial x} = tu^2. \tag{8}$$

[6 marks]

(c) Determine an explicit solution to equation (8) when $u(0, x) = 1 + e^{-x}$ for $0 \leqslant x < \infty$. Where is your solution valid?

[6 marks]

Section B — Further Applied Partial Differential Equations

Question 5

The Bessel functions $J_m(x)$ are the solutions of the equation

$$\left[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x\frac{\mathrm{d}}{\mathrm{d}x}+x^2-m^2\right]J_m(x)=0,$$

that are bounded as $x\to 0$.

(a) Use the generating function

$$\phi=\sum_{m=-\infty}^{\infty}t^m J_m(x)=\exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\}$$

to establish the results

$$J_{-m}(x)=(-1)^m J_m(x),\quad 2J_m'(x)=J_{m-1}(x)-J_{m+1}(x),\quad \frac{2m}{x}J_m(x)=J_{m-1}(x)+J_{m+1}(x).$$

Hence show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x^m J_m(x)\big)=x^m J_{m-1}(x).$$

[5 marks]

(b) Suppose that a function $f(r)$ on the interval $0\le r\le a$ satisfies the boundary condition $\partial f/\partial r=-\sigma f$ on $r=a$. Show that $f(r)$ may be expressed as the series

$$f(r)=\sum_{n=1}^{\infty}c_n J_0(k_n r),$$

with coefficients

$$c_n=\frac{2}{a^2(1+\sigma^2/k_n^2)J_0(k_n a)^2}\int_0^a f(r)J_0(k_n r)r\,\mathrm{d}r,$$

and explain how to determine the positive constants k_n.

[8 marks]

(c) The temperature $T(r,t)$ in an infinite cylinder of radius a surrounded by cold air satisfies:

$$\frac{\partial T}{\partial t}=\kappa\nabla^2 T \text{ in } 0\le r\le a,$$
$$\frac{\partial T}{\partial r}=-\sigma T \text{ on } r=a,$$
$$T(r,0)=T_0 \text{ in } 0\le r\le a.$$

[This question continues on the next page]

(d) Show that the temperature for $t > 0$ may be expressed as the series

$$T(r,t) = \frac{2T_0\sigma}{a}\sum_{n=1}^{\infty}\frac{J_0(k_n r)\exp(-\kappa k_n^2 t)}{(k_n^2+\sigma^2)J_0(k_n a)}.$$

[12 marks]

You may use the relations

$$\int_\alpha^\beta x\,J_m(kx)J_m(\ell x)\,\mathrm{d}x = \frac{1}{k^2-\ell^2}\Big[\ell x J_m(kx)J_m'(\ell x) - kxJ_m(\ell x)J_m'(kx)\Big]_\alpha^\beta,$$

and

$$\int_\alpha^\beta x\,(J_m(kx))^2\,\mathrm{d}x = \frac{1}{2}\left[\left(x^2-\frac{m^2}{k^2}\right)(J_m(kx))^2 + x^2(J_m'(kx))^2\right]_\alpha^\beta.$$

Question 6

(a) The Hermite polynomials are defined by

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \mathrm{e}^{-x^2}.$$

By considering the series expansion of e^{-z^2}, show that the generating function for the Hermite polynomials is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \mathrm{e}^{2xt-t^2} = \phi(x,t).$$

Hence show that

$$H_n'(x) = 2nH_{n-1}(x).$$

By considering the expression $\phi_{xx} - 2x\phi_x + 2t\phi_t$, show that each Hermite polynomial satisfies the differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

[7 marks]

(b) Show that

$$H_n(x) = 2^n x^n + \mathcal{O}(x^{n-1}).$$

Hence establish the orthogonality relation

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) \mathrm{e}^{-x^2}\, \mathrm{d}x = 2^n n! \sqrt{\pi}\, \delta_{nm}.$$

[8 marks]

(c) Show that the Hermite functions defined by

$$\psi_n(x) = \mathrm{e}^{-x^2/2} H_n(x)$$

are eigenfunctions of the Fourier transform, in the sense that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_n(x) \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x = (-\mathrm{i})^n \psi_n(k).$$

[Hint: consider the Fourier transform of their generating function and complete the square.]

[10 marks]