Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods I

Thursday, 10th January 2008, 9:30 a.m.- 11:30 a.m.

Candidates may attempt as many questions as they wish. The best four solutions will count.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Question 1

(i) Find leading-order asymptotic approximations to each of the four roots of the equation

$$\epsilon x^4 - x^3 + \epsilon = 0$$

in the limit $\epsilon \to 0$.

(ii) Find the leading-order solution, in the limit $\epsilon \to 0$, of the differential equation

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \left(3 + x^2\right) \frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 0$$

subject to the boundary conditions $y(0) = 1$, $y(1) = 0$. Show that there is a boundary layer at $x = 1$ and find the leading-order value of $\mathrm{d}y/\mathrm{d}x(1)$.

(iii) The second-order differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - \epsilon y^2 = 0$$

is to be solved in the limit $\epsilon \to 0$, subject to the boundary conditions

$y(0) = 1$ and $y \to 0$ as $x \to \infty$.

Show that a naïve asymptotic expansion fails to satisfy both boundary conditions. Show also how the difficulty is resolved by matching to a "boundary layer at infinity".

Question 2

Define a fundamental matrix $Y(t)$ for the system

$$\dot{\mathbf{x}} = A(t)\mathbf{x}.$$

If $A(t+2) = A(t)$, show that $Y(t+2) = Y(t)C$, where C is a constant matrix.

The function $x(t)$ satisfies

$$\frac{d^2x}{dt^2} + f(t)x = 0,$$

where

$$f(t) = \begin{cases} -\alpha^2 & \text{for } 2n < t < 2n+1 \\ \alpha^2 & \text{for } 2n+1 < t < 2n+2 \end{cases}$$

Write the equation for $x(t)$ in the form $\dot{\mathbf{x}} = A(t)\mathbf{x}$ and show that

$$x_1 = \begin{cases} \cosh \alpha t & 0 \le t \le 1 \\ \cosh \alpha \cos \alpha(t-1) + \sinh \alpha \sin \alpha(t-1) & 1 \le t \le 2 \end{cases}$$

and

$$x_2 = \begin{cases} \frac{1}{\alpha} \sinh \alpha t & 0 < t < 1 \\ \frac{1}{\alpha} \sinh \alpha \cos \alpha(t-1) + \frac{1}{\alpha} \cosh \alpha \sin \alpha(t-1) & 1 < t < 2 \end{cases}$$

are linearly independent solutions of $\frac{d^2x}{dt^2} + f(t)x = 0$. Hence find the monodromy matrix C.

Show that the eigenvalues of C satisfy

$$\varrho^2 - 2\varrho \cos \alpha \cosh \alpha + 1 = 0$$

and deduce the condition for $\frac{d^2x}{dt^2} + f(t)x = 0$ to possess solutions with period 2.

Question 3

Show that the operator

$$Ly = \frac{d^2y}{dx^2} + y$$

subject to $y(0) = 0$ and $y(a) = 0$ is self-adjoint and that the eigenvalues are $\lambda_n = 1 - \frac{n^2\pi^2}{a^2}$ and that $y_n = \sin\frac{n\pi x}{a}$ are corresponding eigenfunctions for $n = 1, 2, 3...$

Assuming that the solution of the problem $Ly = f(x), \quad y(0) = y(a) = 0$ can be written in the form

$$y = \sum_{n=1}^{\infty} c_n y_n(x),$$

show that, provided $a \neq \pi$,

$$c_n = \frac{2}{a\lambda_n}\int_0^a f(s)y_n(s)ds.$$

Hence show that the Green's function is

$$G(x, s) = \sum_{n=1}^{\infty} \frac{2y_n(s)y_n(x)}{a\lambda_n}.$$

When $a = \pi$, show that there is a solution to $Ly = f(x)$, subject to $y(0) = y(a) = 0$, only if $f(x)$ satisfies a certain condition, which you should state, and find the general solution when this condition is satisfied. Explain how this result is an example of the Fredholm Alternative.

Question 4

(i) Give a brief explanation of why, if $f(x)$ is well-behaved and tends to zero as $|x| \to \infty$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(k) e^{-ikx} dk \text{ where } \bar{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx.$$

(ii) Suppose $f(x) = 0$ for $x < 0$ and f grows no faster than $e^{(\alpha - \varepsilon)x}$ as $x \to +\infty$, where α is real and $\varepsilon > 0$. Show that

$$f(x) = \frac{1}{2\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} \tilde{f}(p) e^{px} dp \text{ where } \tilde{f}(p) = \int_{0}^{\infty} f(x) e^{-px} dx.$$

(iii) Suppose $\frac{df}{dx} = xf$. For what Γ and what $\bar{f}(k)$ will

$$f(x) = \frac{1}{2\pi} \int_{\Gamma} \bar{f}(k) e^{-ikx} dk?$$

Question 5

For integrable $f(x)$, define $f(x)$ to be the linear functional that takes test functions $\varphi(x)$ into $\int_{-\infty}^{\infty} f(x)\varphi(x)dx$. Show that

$$f^{'}(x) : \varphi(x) \to -\int_{-\infty}^{\infty} f(x)\varphi^{'}(x)dx.$$

What is the delta function $\delta(x)$?
By writing

$$\frac{1}{x} = \frac{d}{dx}\log|x|,$$

show that

$$\frac{1}{x} : \varphi(x) \to \fint_{-\infty}^{\infty} \frac{1}{x}\varphi(x)dx = \lim_{\varepsilon\downarrow 0}\left(\int_{\varepsilon}^{\infty} + \int_{-\infty}^{-\varepsilon}\right)\frac{\varphi(x)dx}{x}.$$

Show also that the right-hand side is

$$\lim_{y\downarrow 0}\int_{-\infty}^{\infty}\frac{x\varphi(x)dx}{x^2+y^2}.$$

TURN OVER

Question 6

(i) Prove that if $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, and if there exists a differentiable $V(\mathbf{x})$ such that $V(\mathbf{0}) = 0$, and $V > 0$ is a neighbourhood of $\mathbf{0}$ with $\nabla V.\dot{\mathbf{x}} \leq 0$, then $\mathbf{x} = \mathbf{0}$ is a stable solution. State, without proof, the condition for $\mathbf{x} = \mathbf{0}$ to be asymptotically stable. Show also that if $\mathbf{f}(\mathbf{x}) = \nabla F(\mathbf{x}),$ with $\nabla F\,|_{\mathbf{0}} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$ is asymptotically stable if F has a minimum at $\mathbf{0}$.

(ii) Suppose

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \text{ where } H = H(p_i, q_i).$$

Show that if Π is a region of phase space (p_i, q_i), points of whose boundary $\partial\Pi$ move with velocity $\mathbf{u} = \left(\frac{dp_i}{dt}, \frac{dqi}{dt}\right)$, then

$$\frac{d}{dt}\int_{\Pi} dV = \int_{d\Pi} (\mathbf{u}.\mathbf{n}) dS = \int_{\Pi} \operatorname{div} \mathbf{u}\, dV = 0$$