Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods I

HILARY TERM 2009

Thursday 15th January, 9:30 a.m.– 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Mathematical Methods

Question 1

Assume that $0 < \epsilon \ll 1$.

(a) Suppose that x satisfies

$$x \sin x = \epsilon.$$

Sketch the graph of $x \sin x$ for $0 \leq x \leq 6\pi$. Find the leading order approximation as $\epsilon \to 0$ for the root of this equation that is closest to zero, and find the first two terms in the approximation for the remaining roots.

[12 marks]

(b) Consider the two-point boundary value problem

$$\epsilon y'' + xy' + y = 1, \quad 1 < x < 2, \qquad y(1) = 1, \quad y(2) = 2.$$

Explain why there is a boundary layer near $x = 1$ but not near $x = 2$. Find the leading-order inner and outer expansions of the solution. Use these to write down a composite expansion, valid on the whole interval. Sketch the leading-order inner, leading-order outer and composite solutions on the same graph.

[13 marks]

TURN OVER

Question 2

The function $x(t)$ satisfies the initial value problem

$$\ddot{x} + x - \epsilon x^3 = 0, \qquad x(0) = 1, \quad \dot{x}(0) = 0. \tag{1}$$

Find the first two terms in a regular expansion as $\epsilon \to 0$ and comment on any limits to its validity.
[You may find useful the identity $\cos^3 t = \frac{1}{4}(\cos 3t + 3\cos t)$*.]*

Use the method of multiple time scales, with the variables t and $\tau = \epsilon t$, to find an approximately periodic solution of (1) in the form $A(\tau)\cos\big(t + \phi(\tau)\big)$ with $A(0) = 1$ and $\phi(0) = 0$. Deduce that the period of oscillation of the solution is approximately $2\pi\left(1 + \frac{3}{8}\epsilon + O(\epsilon^2)\right)$.

[25 marks]

Question 3

The operator L is defined by

$$Ly(x) = \int_0^1 k(x,t)y(t)\,\mathrm{d}t.$$

(a) Find the adjoint operator L^* (with respect to the usual inner product over $(0,1)$). Show that $L = L^*$ if $k(x,t) = k(t,x)$.

[3 marks]

(b) In the self-adjoint case, show that the eigenvalues λ_k and eigenfunctions $y_k(x)$ of L, satisfying

$$Ly_k = \lambda_k y_k,$$

are real.

[4 marks]

(c) Find the eigenvalues of finite multiplicity, and the corresponding eigenfunctions, when $k(x,t) = x+t$. Give an example of an eigenfunction corresponding to $\lambda = 0$.

[10 marks]

(d) Suppose that $y(x)$ satisfies $Ly = f(x)$. For each of the cases (i) $f(x) = x$; (ii) $f(x) = x^2$, state with reasons whether the problem has a solution; if so, state whether it is unique.

[4 marks]

(e) Suppose that $Ly = y + x^2$. Write down the form of $y(x)$ (you need not compute any unknown constants involved) and explain briefly why it is uniquely determined.

[4 marks]

Question 4

The function $\rho(x,t)$ represents the density of a conserved quantity whose flux is $q(\rho)$.

(a) Explain why, for any fixed a and b,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho(x,t)\,\mathrm{d}x = -q(\rho)\big|_a^b.$$

Deduce that ρ satisfies the conservation-law partial differential equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0$$

away from shocks, and derive the shock (jump) conditions that hold at shocks.

[8 marks]

(b) Suppose that

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \rho}{\partial x} = 0$$

with

$$\rho(x,0) = \begin{cases} x^{1/2} & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\int_{-\infty}^{\infty} \rho(x,t)\,\mathrm{d}x = 2/3$ for all t. Find $\rho(x,t)$, showing that there is a shock, across the curve $x = (1+t)^{1/3}$, satisfying the conditions derived above.

[17 marks]

Section B — Further Mathematical Methods

Question 5

(a) What does it mean to say that a function $f(x)$ is equal to the delta function $\delta(x)$ in the distributional sense?

The functions $f_n(x)$ are defined by

$$f_n(x) = \begin{cases} 0 & |x| > 1/n, \\ n/2 & |x| \leq 1/n. \end{cases}$$

Show that, as $n \to \infty$, $f_n(x) \to \delta(x)$ in the distributional sense.

[8 marks]

(b) If $f(x)$ is a distribution, define its distributional derivative $f'(x)$.

Show that

$$g(x,t) = \begin{cases} -\frac{1}{2}\mathrm{e}^{x-t} & x < t, \\ -\frac{1}{2}\mathrm{e}^{t-x} & x > t \end{cases}$$

satifies the differential equation

$$g'' - g = \delta(x-t)$$

in the distributional sense.

Assuming that the solution of

$$y'' - y = h(x), \qquad y \to 0 \quad \text{as} \quad x \to \pm\infty,$$

(where $h(x)$ is given) exists, show that it is given by

$$y(x) = \int_{-\infty}^{\infty} g(x,t)h(t)\,\mathrm{d}t.$$

[17 marks]

 TURN OVER

Question 6

The twice continuously differentiable function $y(x)$ minimises the functional

$$J = \int_a^b f\big(x, y(x), y'(x)\big)\,\mathrm{d}x, \qquad y(a) = A, \quad y(b) = B$$

given the constraint

$$\int_a^b g\big(x, y(x), y'(x)\big)\,\mathrm{d}x = 1,$$

where a, b, A and B are prescribed constants. By varying $y(x)$ to $y(x) + \epsilon\big(\eta_1(x) + \eta_2(x)\big)$, show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} - \lambda\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial g}{\partial y'}\right) - \frac{\partial g}{\partial y}\right] = 0,$$

where λ is a Lagrange multiplier.

A uniform heavy string of length $L > 2$ hangs under gravity between the points $(-1, 0)$ and $(1, 0)$ and its shape is given by $y = y(x)$, the y axis being vertically upwards. Explain why $y(x)$ can be found by minimising

$$\int_{-1}^{1} y\sqrt{1 + (y')^2}\,\mathrm{d}x, \qquad \text{given} \quad \int_{-1}^{1} \sqrt{1 + (y')^2}\,\mathrm{d}x = L.$$

Show that

$$(y')^2 = \left(\frac{y - \lambda}{a}\right)^2 - 1$$

where a and λ are constants, and deduce that

$$y(x) = a\left(\cosh\frac{x}{a} - \cosh\frac{1}{a}\right),$$

where a is such that

$$2a\sinh\frac{1}{a} = L.$$

[25 marks]