Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods I

Thursday, 14th January 2010, 9:30 a.m.- 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Mathematical Methods

Question 1

(a) Find the general solution of the differential equation ($' = \mathrm{d}/\mathrm{d}x$):

$$Ly(x) \equiv y''(x) - \frac{2}{1+x} y'(x) + \frac{2}{(1+x)^2} y(x) = 0. \qquad (1)$$

[*Hint: Find two linearly independent solutions of the form* $(1+x)^n$ *(i.e. with two different integer values for* n*).*]

(b) Explicitly determine the Green's function $g(x,\xi)$ for the boundary value problem

$$Ly(x) = f(x) \quad \text{on } 0 < x < 1, \qquad y(0) = 0, \quad y(1) = 0, \qquad (2)$$

(for Ly as in (1)) and use it to write down the solution $y(x)$ for a general, continuous function $f(x)$.

TURN OVER

Question 2

We consider here the **polynomial solutions** $y(x) = H_n(x)$ of *Hermite's equation* ($' = \mathrm{d}/\mathrm{d}x$)

$$y'' - 2xy' + 2ny = 0 \tag{3}$$

for an integer $n \geqslant 0$.

You may use, without proof, that these polynomials H_n have degree n and are given by a Rodrigues' formula

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}. \tag{4}$$

(a) Obtain H_0, H_1, H_2, H_3 explicitly from (4). If $\alpha_n x^n$ denotes the leading term of H_n, formulate a conjecture expressing α_n in terms of n.

(b) Derive a recurrence relation of the form

$$H_{n+1}(x) = 2xH_n(x) + c_n H_{n-1}(x), \tag{5}$$

giving an expression for c_n in terms of n such that the relation holds for all $n \geqslant 1$. Use this recurrence relation to prove by induction the conjecture you made in (a).

(c) Determine the value of

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)\, e^{-x^2} \mathrm{d}x$$

for the case of general integers $n \geqslant 0$, $m \geqslant 0$.
[Hint: You may use, without proof, that $\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$.]

Question 3

We consider here the *Chebyshev equation*

$$(1 - x^2)y'' - xy' + m^2 y = 0 \tag{6}$$

($' = \mathrm{d}/\mathrm{d}x$) for integers $m \geqslant 0$.

(a) Classify the four points $x = 0$, $x = \pm 1$ and $x = \infty$ as ordinary, regular singular or irregular singular points. For the regular singular points, determine the indicial equation and the indicial exponents.

(b) Find the recursion relation for the coefficients of the Taylor expansion

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

about $x = 0$ for the solution $y(x)$ of (6). Show that $y(x)$ may be decomposed into two solutions that are even and odd, respectively, about $x = 0$. Show that when m is **odd**, the **odd** solution is a polynomial of exact degree m, while when m is **even**, the **even** solution is a polynomial of exact degree m.
[Hint: Remember that $y(x)$ is called an odd function, if $y(-x) = -y(x)$ for all x, and an even function, if $y(-x) = y(x)$ for all x. Also remember what this means for the coefficients of the function's Taylor expansion about $x = 0$.]

(c) Use the substitution $x = \cos z$ in (6) and determine the general real-valued solution for the resulting equation in z. Use the answer to give the general solution of (6) in the interval $-1 < x < 1$ for all $m \geqslant 0$.

TURN OVER

Question 4

We consider here the integral operator

$$Ly(x) = \int_0^1 g(x,\xi)y(\xi)\mathrm{d}\xi, \tag{7}$$

where

$$g(x,\xi) = \begin{cases} (1-\xi)x & \text{for } x < \xi, \\ (1-x)\xi & \text{for } x \geqslant \xi. \end{cases} \tag{8}$$

(a) Is the operator L symmetric? Give reasons for your answer.

(b) Determine all $\alpha > 0$ for which $\sin(\alpha x)$ is an eigenfunction of L, i.e. $Ly = \lambda y$ with real λ, and determine the corresponding eigenvalues λ.
[*Hint: Split the integral at* $\xi = x$ *and use integration by parts.*]

In the following, you may assume, without proof, that all eigenfunctions of L are found in this way.

(c) Find all real A, B for which the integral equation

$$Ly - \frac{1}{\pi^2}y = A\sin x + B\cos x$$

has solutions $y(x)$.

Section B — Further Mathematical Methods

Question 5

1. It is required to minimise the cost function

$$C[x,u,T] = \int_0^T h\big(t, x(t), u(t)\big)\,\mathrm{d}t$$

over all process times T and control functions u which evolve the solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f\big(t, x(t), u(t)\big)$$

from $x(0) = a$ to $x(T) = 1$, where a is a positive constant. Assuming that $\partial f/\partial u \neq 0$, show that the optimal control satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial h/\partial u}{\partial f/\partial u}\right) = \frac{\partial h}{\partial x} - \left(\frac{\partial h/\partial u}{\partial f/\partial u}\right)\frac{\partial f}{\partial x},$$

and the free boundary condition

$$H = \left(\frac{\partial h/\partial u}{\partial f/\partial u}\right) f - h = 0 \quad \text{at } t = T.$$

Deduce that, if f and h do not depend explicitly on t, then $H \equiv 0$ for all t.

If $f = ux - x^3$ and $h = u^2 + x^2$, show that the optimal process time is given by

$$T = \left|\frac{\sqrt{1+a^2}}{a} - \sqrt{2}\right|$$

and sketch a graph of T versus a.

[*Hint: you may find the identity* $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\sqrt{1+x^2}}{x}\right) \equiv -\dfrac{1}{x^2\sqrt{1+x^2}}$ *helpful.*]

Question 6

1. Find and classify the critical points of the plane autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ay + y^2 - x^2 y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - y,$$

   determining how they depend on the parameter a. Plot a response diagram of the equilibrium values of x versus a, and identify any bifurcations that occur.

   [11 marks]

2. Explain why the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x + \epsilon x y^2,$$

   where ϵ is a parameter, must have periodic orbits in a neighbourhood of the origin. Use the Poincaré–Linstedt method to approximate the orbit satisfying the initial conditions $x(0) = A$, $y(0) = 0$ when ϵ is small. Show that the frequency is given asymptotically by

$$\omega \sim 1 + \frac{\epsilon A^2}{8} + \cdots \quad \text{as } \epsilon \to 0.$$

   [14 marks]