JMAT 7302
JACM 7C65
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Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Analysis

Friday, 23rd April 2004, 9:30 a.m. – 12:30 p.m.

Candidates may attempt as many questions as they wish.

JACM 7C65

Degree Master of Science in Applied & Computational Mathematics

Numerical Solution of Differential Equations, Numerical Linear Algebra

& Finite Element Methods

Friday, 23rd April 2004, 9:30 a.m. – 12:30 p.m.

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JACM 7C63

Degree Master of Science in Applied & Computational Mathematics

Numerical Solution of Differential Equations & Numerical Linear Algebra

Friday, 23rd April 2004, 9:30 a.m. – 11:30 a.m.

Candidates may attempt questions 1,2,3 only.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Numerical Solution of Differential Equations

Question 1

(i) State the general form of a linear k-step method for the numerical solution of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ on the mesh $\{x_j : x_j = x_0 + jh\}$ of uniform spacing $h > 0$. [2 marks]

(ii) Define the *truncation error* of a linear k-step method. What is meant by saying that a linear k-step method is *consistent*? [4 marks]

(iii) What is meant by saying that a linear k-step method is *zero-stable*? Formulate an equivalent characterisation of zero-stability in terms of the roots of a certain polynomial of degree k. [6 marks]

(iv) Consider the linear three-step method defined by

$$y_{n+3} + ay_{n+2} - ay_{n+1} - y_n = hb(f_{n+2} + f_{n+1}),$$

where $f_j = f(x_j, y_j)$, and a and b are real parameters such that $a + 3 = 2b$. Find all values of a such that the method is zero-stable. Show that the maximum order of a zero-stable method of this form is 2.

Show further that there exists a unique choice of a and b such that the order of the method is 4. Is this fourth-order method convergent? Justify your answer. [13 marks]

Question 2

Consider the initial-value problem for the scalar nonlinear hyperbolic equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u(x,t)) = 0, \qquad -\infty < x < \infty, \quad 0 < t \leq T,$$
$$u(x, 0) = u_0(x), \qquad -\infty < x < \infty,$$

where $T > 0$ is a positive real number, u_0 is a real-valued, bounded, monotonic increasing and continuously differentiable function of $x \in (-\infty, \infty)$, and f is a real-valued, twice continuously differentiable function on $\mathbb{R}$ whose second derivative is nonnegative on $\mathbb{R}$.

(i) By using the Chain Rule, or otherwise, verify that the continuously differentiable function u, defined implicitly by the equation

$$u(x,t) = u_0\left(x - t\, f'(u(x,t))\right),$$

is a solution to the initial value problem. [12 marks]

(ii) Suppose that $f(u) = \frac{1}{2}u^2 - \tan^{-1} u$. Show that $f''(u) \geq 0$ for all $u \in \mathbb{R}$. Formulate the first-order upwind scheme for the numerical solution of the initial value problem on a uniform mesh of size $\Delta x > 0$ in the x-direction and size $\Delta t = T/M$ in the t-direction, where $M \geq 1$, denoting by U_j^m the approximation to $u(j\Delta x, m\Delta t)$.

Show that if $(\Delta t/\Delta x) \max_{x \in \mathbb{R}} |f'(u_0(x))| \leq 1$, then $\max_{0 \leq m \leq M} \max_{j \in \mathbb{Z}} |U_j^m| \leq \max_{j \in \mathbb{Z}} |U_j^0|$.

[13 marks]

Numerical Linear Algebra

Question 3

(a) If $A \in \mathbb{R}^{m\times n}, m \geq n$, what is the Singular Value Decomposition (SVD) of A? If A has $r \leq n$ non-zero singular values, show that dim(Range(A))$= r$ and dim(Ker(A))$= n - r$ and in each case describe a basis in terms of the SVD. [4+4 marks]

From the SVD in the case $m = n$, deduce that A admits a factorisation $A = HQ$ where H is a symmetric and positive semi-definite matrix (i.e. it is symmetric and has non-negative eigenvalues) and Q is an orthogonal matrix. If $\{\sigma_i\}$ are the singular values of A, deduce that

$$\|A - Q\|_2 = \max_i |\sigma_i - 1|,$$

proving any results that you need. [4+6 marks]

(b) If a numerical method used to solve a linear system of equations $Ax = b$ in fact finds the solution to a perturbed system $A(x + \delta x) = b + \delta b$, prove that

$$\frac{\|\delta x\|}{\|x\|} \leq \|A\| \, \|A^{-1}\| \, \frac{\|\delta b\|}{\|b\|}$$

for any operator norm. What is the important practical interpretation of this result? [4+3 marks]

Question 4

(a) What does it mean to say that $\langle \cdot, \cdot \rangle$ is an inner product on a linear space S? How is a norm $\| \cdot \|$ defined in terms of this inner product? [3+1 marks]

Suppose that $V \subset S$ is a finite dimensional vector space and $f \in S$. Prove that $p \in V$ satisfies $\|f - p\| \leq \|f - q\|$ for all $q \in V$ if and only if

$$\langle f - p, r \rangle = 0 \quad \text{for all } r \in V.$$

You should prove any results that you need. [7 marks]

Let U be a fixed $n \times n$ real orthogonal matrix and let

$$S = \{A \in \mathbb{R}^{n \times n} : A = U \Lambda U^T, \Lambda \text{ diagonal}\}.$$

Does

$$\langle A, B \rangle = \max\{\lambda : \lambda \text{ is an eigenvalue of } AB\}$$

define an inner product on S? Verify or provide a counterexample for each of the axioms. [6 marks]

(b) Describe briefly the concept of preconditioning for symmetric and positive definite matrix equation systems with reference to the convergence of the Conjugate Gradient method.

[You may use the result that the k^{th} iterate in a Conjugate Gradient iteration to solve the system of equations $Ax = b$ for symmetric positive-definite A satisfies the convergence bound

$$\frac{\|x - x_k\|_A}{\|x - x_0\|_A} \leq 2 \left(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^k,$$

but you should define the condition number $\kappa(A)$.]

[8 marks]

Finite Element Methods

Question 5

(i) Suppose that $f \in \mathrm{L}^2(0,1)$. State the weak formulation of the boundary-value problem

$$\begin{aligned} -u'' + xu' + u = f(x), \qquad x \in (0,1), \\ u'(0) = 0, \quad u'(1) + u(1) = 0. \end{aligned}$$

[4 marks]

(ii) By using the Lax–Milgram Theorem, show that the boundary value-problem has a unique weak solution u in $\mathrm{H}^1(0,1)$. [The following inequality may be used without proof:

$$\max_{x\in[0,1]} w^2(x) \le \|w\|^2_{\mathrm{L}^2(0,1)} + 2\|w\|_{\mathrm{L}^2(0,1)}\|w'\|_{\mathrm{L}^2(0,1)}, \qquad w \in \mathrm{H}^1(0,1).]$$

[7 marks]

(iii) Let N be a positive integer and $h = 1/N$. Consider the uniform subdivision $\mathcal{S}_h = \{[x_{i-1}, x_i] \,:\, i = 1,\dots,N,\ x_0 = 0,\ x_N = 1\}$ of the interval $[0,1]$, where $x_i - x_{i-1} = h$ for $i = 1,\dots,N$. Using continuous piecewise linear basis functions on $\mathcal{S}_h$, formulate the finite element approximation of the boundary-value problem. [4 marks]

(iv) Show that the finite element method from part (c) has a unique solution u_h. Show further that there exists a positive constant C, independent of h, such that, for any continuous piecewise linear function v_h defined on the subdivision $\mathcal{S}_h$,

$$\|u - u_h\|_{\mathrm{H}^1(0,1)} \le C\|u - v_h\|_{\mathrm{H}^1(0,1)}.$$

Deduce that $\|u - u_h\|_{\mathrm{H}^1(0,1)} = \mathcal{O}(h)$ as $h \to 0$ [10 marks]

[Any bound on the error between u and its finite element interpolant $\mathcal{I}_h u$ may be used without proof, but must be stated carefully.]

Question 6

Suppose that Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary Γ, oriented in the anticlockwise direction. Suppose, further, that $f \in \mathrm{L}^2(\Omega)$ and consider the quadratic functional $J : v \in \mathrm{H}^1(\Omega) \mapsto J(v) \in \mathbb{R}$ defined by

$$J(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + v^2)\,\mathrm{d}x + \frac{1}{2}\int_\Gamma v^2\,\mathrm{d}s - \int_\Omega f \cdot v\,\mathrm{d}x.$$

(i) Show that if $u \in \mathrm{H}^1(\Omega)$ is such that $J(u) \leq J(v)$ for all $v \in \mathrm{H}^1(\Omega)$, then there exist a bilinear functional $a(\cdot,\cdot)$ defined on $\mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$ and a linear functional $\ell(\cdot)$ defined on $\mathrm{H}^1(\Omega)$ such that

$$a(u,v) = \ell(v) \qquad \forall v \in \mathrm{H}^1(\Omega). \qquad \text{(P)}$$

[7 marks]

(ii) Show that (P) is the weak formulation of the elliptic boundary value problem

$$-\nabla^2 u + u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + u = 0 \quad \text{on } \Gamma,$$

where $\dfrac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$ and $\mathbf{n}$ denotes the unit outward normal vector to Γ. [7 marks]

(iii) Suppose that Ω is the unit square $(0,1) \times (0,1)$, and let $\mathcal{T}_h$ be a triangulation of Ω constructed from a uniform square grid of spacing $h = 1/N$ by subdividing each grid-square by the diagonal of negative slope. Formulate the piecewise linear finite element approximation (P_h) of problem (P) on the triangulation $\mathcal{T}_h$. Show that (P_h) has a unique solution u_h and that $J(u) \leq J(u_h) \leq J(v_h)$ for any continuous piecewise linear function v_h defined on the triangulation $\mathcal{T}_h$. [11 marks]