JMAT 7302
JACM 7C65
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Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Analysis

Friday, 22nd April 2005, 9:30 a.m. – 12:30 p.m.

Candidates may attempt as many questions as they wish.

JACM 7C65

Degree Master of Science in Applied & Computational Mathematics

Numerical Solution of Differential Equations, Numerical Linear Algebra

& Finite Element Methods

Friday, 22nd April 2005, 9:30 a.m. – 12:30 p.m.

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JACM 7C63

Degree Master of Science in Applied & Computational Mathematics

Numerical Solution of Differential Equations & Numerical Linear Algebra

Friday, 22nd April 2005, 9:30 a.m. – 11:30 a.m.

Candidates may attempt questions 1,2,3 only.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Numerical Solution of Differential Equations

Question 1

Consider the initial value problem $y' = f(y)$, $y(0) = 1$, where $f(y) = \tanh(\sin y)$.

[You may assume that this problem has a unique solution $x \mapsto y(x)$, defined for all $x \in \mathbb{R}$, such that the functions $x \mapsto y'(x)$ and $x \mapsto y''(x)$ are defined and continuous for all $x \in \mathbb{R}$.]

(i) Show that $|y''(x)| \leq 1$ for all $x \in \mathbb{R}$.

Show further that the function f satisfies the following Lipschitz condition:

$$|f(u) - f(v)| \leq |u - v| \qquad \forall u, v \in \mathbb{R}\,.$$

[5 marks]

(ii) The implicit Euler approximation y_n to $y(x_n)$, on the mesh $\{x_n \,:\, x_n = nh,\ n = 0, 1, \dots\}$ of uniform spacing $h \in (0, 1)$, is obtained from the formula

$$\frac{y_n - y_{n-1}}{h} = f(y_n)\,, \quad n = 1, 2, \dots\,, \qquad y_0 = 1.$$

Let $g(y) = y - hf(y)$. Show that the function $y \mapsto g(y)$ is strictly monotonic increasing and $\lim_{y \to \pm\infty} g(y) = \pm\infty$. By rewriting Euler's method as $g(y_n) = y_{n-1}$, deduce that, given $y_{n-1} \in \mathbb{R}$, the Euler approximation y_n is uniquely defined in $\mathbb{R}$. [8 marks]

(iii) Show that the truncation error T_n of the implicit Euler method applied to the initial value problem under consideration satisfies

$$|T_n| \leq \tfrac{1}{2}h\,, \qquad n = 1, 2, \dots\,.$$

Show further that

$$|y(x_n) - y_n| \leq \frac{1}{1-h}|y(x_{n-1}) - y_{n-1}| + \frac{h}{1-h}|T_n|, \qquad n = 1, 2, \dots\,,$$

and deduce that

$$|y(x_n) - y_n| \leq \frac{h}{2}\left[\left(1 + \frac{h}{1-h}\right)^n - 1\right], \qquad n = 1, 2, \dots\,.$$

Show that there exists $h_0 \in (0, 1)$ such that if $h \leq h_0$, then y_n approximates $y(x_n)$ to within 10^{-2} for all $x_n \in [0, 1]$. [12 marks]

[You may use without proof the result that, for any constant c, $\lim\limits_{n\to\infty}(1 + c/n)^n = \exp(c)$.]

Question 2

Consider the initial value problem

$$\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad 0 < t \leq T,$$

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty,$$

where T is a fixed real number, and u_0 is a real-valued, bounded and continuous function of $x \in (-\infty, \infty)$.

(i) Formulate the θ scheme for the numerical solution of this initial value problem on a mesh with uniform spacings $\Delta x > 0$ and $\Delta t = T/M$ in the x and t co-ordinate directions, respectively, where M is a positive integer. You should state the scheme so that U_j^m denotes the θ-scheme-approximation to $u(j\Delta x, m\Delta t)$, $0 \leq m \leq M$, $j \in \mathbb{Z}$, and $\theta = 0$ corresponds to the explicit (forward) Euler scheme. [5 marks]

(ii) Suppose that $\|U^0\|_{\ell_\infty} = \max_{j\in\mathbb{Z}} |U_j^0|$ is finite. Show that if $\theta \in [0,1]$, then

$$\|U^m\|_{\ell_\infty} \leq \left(\frac{1-(1-\theta)\Delta t}{1+\theta\Delta t}\right)^m \|U^0\|_{\ell_\infty}$$

for all m, $1 \leq m \leq M$, provided that $(1-\theta)\Delta t \leq \frac{(\Delta x)^2}{2+(\Delta x)^2}$.

Deduce that the implicit (backward) Euler scheme is *unconditionally stable* in the $\|\cdot\|_{\ell_\infty}$ norm. Show, further, that the Crank–Nicolson scheme is *conditionally stable* in the $\|\cdot\|_{\ell_\infty}$ norm and state the condition on Δt and Δx that ensures stability. [10 marks]

(iii) Suppose that $\|U^0\|_{\ell_2} = \left(\Delta x \sum_{j\in\mathbb{Z}} |U_j^0|^2\right)^{1/2}$ is finite. Show that if $\theta \in [\frac{1}{2}, 1]$, then

$$\|U^m\|_{\ell_2} \leq \|U^0\|_{\ell_2}$$

for all m, $1 \leq m \leq M$, for any Δt and Δx.

Now, suppose that $\theta \in [0, \frac{1}{2})$. Show that $\|U^m\|_{\ell_2} \leq \|U^0\|_{\ell_2}$ for all m, $1 \leq m \leq M$, provided that $(1-2\theta)\Delta t \leq \frac{2(\Delta x)^2}{4+(\Delta x)^2}$.

Deduce that both the implicit (backward) Euler scheme and the Crank–Nicolson scheme are *unconditionally stable* in the $\|\cdot\|_{\ell_2}$ norm. [10 marks]

Numerical Linear Algebra

Question 3

Throughout this question A is an $m \times n$ real matrix with $m \geq n$, and norms and condition numbers are defined with respect to the usual 2-norm $\|x\| = (\sum_j x_j^2)^{1/2}$.

(i) Give algebraic definitions of a reduced (or "skinny" or "thin") QR factorization and also a reduced SVD of A, making it clear what kinds of matrices are involved and what their dimensions are. (Do not explain how these factorizations are computed.) [6 marks]

(ii) Using the SVD, define the condition number $\kappa(A)$. Give an interpretation of this number in terms of norms $\|Ax\|$ for various vectors $x \in \mathbf{R}^n$. [6 marks]

(iii) Let B be an $m \times (n+1)$ matrix consisting of A with a new $(n+1)$st column added. Show that $\kappa(B) \geq \kappa(A)$. [6 marks]

(iv) Let C be an $(m+1) \times n$ matrix consisting of A with a new $(m+1)$st row added. Give examples to show that both $\kappa(C) > \kappa(A)$ and $\kappa(C) < \kappa(A)$ are possible. [7 marks]

Question 4

(i) Let A and B be two $n \times n$ matrices. Derive *exact* formulae for how many additions and multiplications are involved in computing AB by the standard method. [5 marks]

(ii) Now assume A is symmetric. Describe an algorithm for computing A^{1024} in approximately $10n^3$ flops. (A flop is an addition or a multiplication; thus, one multiplication followed by one addition is 2 flops.) [5 marks]

(iii) For the same symmetric A, the eigenvalue decomposition can be computed with approximately $9n^3$ flops: approximately $(4/3)n^3$ flops for the first phase of the standard algorithm and approximately $8n^3$ flops for the second phase. State what is meant by the *eigenvalue decomposition*. Describe what these two phases of computation are and name the standard algorithms for them, but you do not need to describe these algorithms. [5 marks]

(iv) Describe a new algorithm based on eigenvalues, also requiring about $10n^3$ flops, for computing A^{1024}. [5 marks]

(v) Prove that if A is nonsingular, then A^{1024} is also nonsingular in theory. In practice, on a computer with 16-digit precision, A^{1024} will almost always be numerically singular. Explain this phenomenon. [5 marks]

Finite Element Methods

Question 5

(i) Given that (a, b) is a bounded open interval of the real line, define the Sobolev space $\mathrm{H}^1(a, b)$ and the Sobolev norm $\|\cdot\|_{\mathrm{H}^1(a,b)}$. [2 marks]

(ii) What is meant by saying that u is a weak solution in $\mathrm{H}^1(a, b)$ of the boundary value problem

$$-u'' + (x^2 + 1)u = f(x), \quad x \in (a, b); \qquad -u'(a) = A, \quad u'(b) = B,$$

where $f \in \mathrm{L}_2(a, b)$?

Show that the bilinear form associated with the weak formulation of this problem is coercive on $\mathrm{H}^1(a, b)$. [8 marks]

(iii) Consider the piecewise linear finite element basis functions φ_i, $i = 0, 1, \ldots, N$, defined by $\varphi_i(x) = (1 - |x - x_i|/h)_+$, $x \in [a, b]$, on the uniform mesh of size $h = (b - a)/N$, $N \geq 2$, with mesh-points $x_i = a + ih$, $i = 0, 1, \ldots, N$.

Using the basis functions φ_i, $i = 0, 1, \ldots, N$, define the finite element approximation of the boundary value problem.

Expand the finite element solution u_h in terms of the basis functions φ_i, $i = 0, 1, \ldots, N$, by writing

$$u_h(x) = \sum_{i=0}^{N} U_i \varphi_i(x)$$

where $\mathbf{U} = (U_0, U_1, \ldots, U_N)^\top \in \mathbb{R}^{N+1}$, to obtain a system of linear algebraic equations for the vector of unknowns $\mathbf{U}$.

Show that the matrix $\mathcal{A}$ of this linear system is symmetric (i.e. $\mathcal{A}^\top = \mathcal{A}$) and positive definite (i.e. $\mathbf{V}^\top \mathcal{A} \mathbf{V} > 0$ for all $\mathbf{V} \in \mathbb{R}^{N+1}$, $\mathbf{V} \neq \mathbf{0}$). Deduce that the solution is unique. [8 marks]

(iv) Show that $\|u - u_h\|_{\mathrm{H}^1(a,b)} = \mathcal{O}(h)$ as $h \to 0$. [7 marks]

[Any bound on the error between u and its finite element interpolant $\mathcal{I}_h u$ may be used without proof, but must be stated carefully.]

Question 6

Suppose that $\Omega = (0,1)^2$ and $f \in \mathrm{L}_2(\Omega)$. Consider the quadratic energy functional $J : \mathrm{H}^1(\Omega) \to \mathbb{R}$ defined by

$$J(v) = \tfrac{1}{2}\, a(v,v) - \ell(v),$$

where

$$a(w,v) = \int_\Omega \left[\frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial y} + w\,v \right] \mathrm{d}x\,\mathrm{d}y \quad \text{and} \quad \ell(v) = \int_\Omega f v \,\mathrm{d}x\,\mathrm{d}y.$$

(i) Show that u is a minimiser of J over $\mathrm{H}^1(\Omega)$ (i.e. $J(u) \le J(v)$ for all $v \in \mathrm{H}^1(\Omega)$) if, and only if,

$$a(u,v) = \ell(v) \qquad \text{for all } v \in \mathrm{H}^1(\Omega).$$

[8 marks]

(ii) Show that $a(\cdot,\cdot)$ and $\ell(\cdot)$ satisfy the hypotheses of the Lax–Milgram Theorem. Hence deduce the existence and uniqueness of the minimiser of J in $\mathrm{H}^1(\Omega)$. [9 marks]

(iii) Consider a triangulation of $\bar{\Omega}$ which has been obtained from a square mesh of spacing $h = 1/N$, $N \ge 2$, in both co-ordinate directions by subdividing each mesh-square into two triangles with the diagonal of negative slope. Denote by V_h the finite-dimensional subspace of $\mathrm{H}^1(\Omega)$ consisting of continuous piecewise linear functions defined on this triangulation.

Show that there exists a unique element u_h in V_h such that $J(u_h) \le J(v_h)$ for all $v_h \in V_h$.

Show further that $J(u) \le J(u_h)$ and that

$$\|u - u_h\|_{\mathrm{H}^1(\Omega)} = \min_{v_h \in V_h} \|u - v_h\|_{\mathrm{H}^1(\Omega)}.$$

[8 marks]