**Degree Master of Science in Mathematical Modelling and Scientific Computing**

**Numerical Linear Algebra & Finite Element Methods**

**Thursday, 19th April 2007, 2:00 p.m. – 4:00 p.m.**

*Candidates may attempt as many questions as they wish. The best four solutions will count.*

Please start the answer to each question on a new page.

All questions will carry equal marks.

**Do not turn over until told that you may do so.**

# Finite Element Methods for Partial Differential Equations

## Question 1

(i) Define the Sobolev space $\mathrm{H}^1(0,1)$ and the Sobolev norm $\|\cdot\|_{\mathrm{H}^1(0,1)}$.

What is meant by saying that $u$ is a weak solution in $\mathrm{H}^1(0,1)$ of the boundary-value problem

$$-u'' + u = f(x), \quad x \in (0,1); \qquad u(0) - u'(0) = 1, \quad u(1) + u'(1) = 1,$$

where $f \in \mathrm{L}_2(0,1)$? [6 marks]

(ii) Show that the bilinear form associated with the weak formulation of this problem is coercive on $\mathrm{H}^1(0,1)$.

Consider the continuous piecewise linear basis functions $\varphi_i$, $i = 0, 1, \ldots, N$, defined by $\varphi_i(x) = (1 - |x - x_i|/h)_+$ on the uniform mesh of size $h = 1/N$, $N \geq 2$, with mesh-points $x_i = ih$, $i = 0, 1, \ldots, N$. Using the basis functions $\varphi_i$, $i = 0, 1, \ldots, N$, define the finite element approximation of the boundary-value problem and show that it has a unique solution $u_h$. [6 marks]

(iii) Expand $u_h$ in terms of the basis functions $\varphi_i$, $i = 0, 1, \ldots, N$, by writing

$$u_h(x) = \sum_{i=0}^{N} U_i \varphi_i(x)$$

where $\mathbf{U} = (U_0, U_1, \ldots, U_N)^{\mathrm{T}} \in I\!\!R^{N+1}$, to obtain a system of linear algebraic equations for the vector of unknowns $\mathbf{U}$. Show that the matrix $\mathcal{A}$ of this linear system is symmetric (*i.e.* $\mathcal{A}^{\mathrm{T}} = \mathcal{A}$) and positive definite (*i.e.* $\mathbf{V}^{\mathrm{T}}\mathcal{A}\mathbf{V} > 0$ for all $\mathbf{V} \in I\!\!R^{N+1}$, $\mathbf{V} \neq \mathbf{0}$). [6 marks]

(iv) Show also that $\|u - u_h\|_{\mathrm{H}^1(0,1)} = \mathcal{O}(h)$ as $h \to 0$.

[Any bound on the error between $u$ and its finite element interpolant $\mathcal{I}_h u$ may be used without proof, but must be stated carefully.] [7 marks]

## Question 2

(i) Let $\psi \in \mathrm{L}_2(0,1)$ and let $a(\cdot,\cdot)$ be the bilinear form on $\mathrm{H}^1(0,1) \times \mathrm{H}^1(0,1)$ defined by

$$a(w,v) = \int_0^1 (w'v' + wv)\mathrm{d}x.$$

Suppose, further, that $z \in \mathrm{H}^1(0,1)$ is such that

$$a(w,z) = \int_0^1 w\,\psi \mathrm{d}x \qquad \forall w \in \mathrm{H}^1(0,1).$$

Show that $\|z''\|_{\mathrm{L}_2(0,1)} \leq \|\psi\|_{\mathrm{L}_2(0,1)}$. [5 marks]

(ii) Suppose that $f \in \mathrm{L}_2(0,1)$ and let $u \in \mathrm{H}^1(0,1)$ be the weak solution of the problem

$$a(u,v) = \int_0^1 f\,v \mathrm{d}x \qquad \forall v \in \mathrm{H}^1(0,1).$$

Let, further, $u_h$ denote the piecewise linear finite element approximation to $u$ on the subdivision $\mathcal{S}_h = \{[x_{i-1}, x_i] \,:\, i = 1, 2, \ldots, N\}$, where $x_i - x_{i-1} = h_i$, $i = 1, 2, \ldots, N$.

Show that

$$\int_0^1 (u - u_h)\,\psi \mathrm{d}x = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} R(u_h)\,(z - \mathcal{I}_h z)\mathrm{d}x,$$

where $\mathcal{I}_h z$ is the continuous piecewise linear finite element interpolant of $z$ on the subdivision $\mathcal{S}_h$, and $R(u_h)$ is the *residual* which you should carefully define in terms of $f$ and $u_h$. [6 marks]

(iii) Show that

$$\int_0^1 (u - u_h)\,\psi \mathrm{d}x \leq \frac{1}{\pi^2} \left( \sum_{i=1}^N \|R(u_h)\|^2_{\mathrm{L}_2(x_{i-1},x_i)} h_i^4 \right)^{1/2} \|\psi\|_{\mathrm{L}_2(0,1)},$$

and deduce the *a posteriori* error bound

$$\|u - u_h\|_{\mathrm{L}_2(0,1)} \leq \frac{1}{\pi^2} \left( \sum_{i=1}^N \|R(u_h)\|^2_{\mathrm{L}_2(x_{i-1},x_i)} h_i^4 \right)^{1/2}.$$

[7 marks]

(iv) Discuss, briefly, how this *a posteriori* error bound could be implemented in an adaptive mesh-refinement algorithm to compute, for a prescribed tolerance $\mathtt{TOL} > 0$, an approximation $u_h$ to $u$ such that $\|u - u_h\|_{\mathrm{L}_2(0,1)} \leq \mathtt{TOL}$. [7 marks]

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## Question 3

Suppose that $\Omega = (0,1)^2$ and $f \in \mathrm{L}_2(\Omega)$. Consider the quadratic energy-functional $J : \mathrm{H}^1(\Omega) \to I\!R$ defined by

$$J(v) = \frac{1}{2}a(v,v) - \ell(v),$$

where

$$a(w,v) = \int_\Omega \left[ \frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial y} + wv \right] \mathrm{d}x\mathrm{d}y \quad \text{and} \quad \ell(v) = \int_\Omega f v \mathrm{d}x\mathrm{d}y.$$

(i) Show that $u$ is a minimiser of $J$ over $\mathrm{H}^1(\Omega)$ (i.e. $J(u) \leq J(v)$ for all $v \in \mathrm{H}^1(\Omega)$) if, and only if,

$$a(u,v) = \ell(v) \qquad \text{for all } v \in \mathrm{H}^1(\Omega).$$

[5+5 marks]

(ii) Show that $a(\cdot,\cdot)$ and $\ell(\cdot)$ satisfy the hypotheses of the Lax–Milgram Theorem. Hence deduce the existence and uniqueness of the minimiser of $J$ in $\mathrm{H}^1(\Omega)$. [7 marks]

(iii) Consider a triangulation of $\bar{\Omega}$ which has been obtained from a square mesh of spacing $h = 1/N$, $N \geq 2$, in both co-ordinate directions by subdividing each mesh-square into two triangles with the diagonal of negative slope. Denote by $V_h$ the finite-dimensional subspace of $\mathrm{H}^1(\Omega)$ consisting of continuous piecewise linear functions defined on this triangulation. Show that there exists a unique element $u_h$ in $V_h$ such that $J(u_h) \leq J(v_h)$ for all $v_h \in V_h$. Show further that

$$\|u - u_h\|_{\mathrm{H}^1(\Omega)} = \min_{v_h \in V_h} \|u - v_h\|_{\mathrm{H}^1(\Omega)}.$$

[4+4 marks]

## Question 4

Let $u(x,t)$ denote the solution to the initial-boundary-value problem

$$\begin{aligned} \frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad 0 < t \le T, \\ u'(0,t) = 0, \quad u'(1,t) = 0, \qquad 0 \le t \le T, \\ u(x,0) = u_0(x), \qquad 0 < x < 1, \end{aligned}$$

where $T > 0$, $u_0 \in \mathrm{L}_2(0,1)$.

(i) Construct a finite element method for the numerical solution of this problem, based on the Crank–Nicolson scheme with time step $\Delta t = T/M$, $M \ge 2$, and continuous piecewise linear approximation in $x$ on a uniform subdivision of spacing $h = 1/N$, $N \ge 2$, of the interval $[0,1]$, denoting by $u_h^m$ the finite element approximation to $u(\cdot, t^m)$ where $t^m = m\Delta t$, $0 \le m \le M$. [9 marks]

(ii) Show that, for $0 \le m \le M-1$,

$$\frac{1}{2\Delta t}\left(\|u_h^{m+1}\|^2_{\mathrm{L}_2(0,1)} - \|u_h^m\|^2_{\mathrm{L}_2(0,1)}\right) + \left\|\frac{u_h^{m+1}+u_h^m}{2}\right\|^2_{\mathrm{L}_2(0,1)} + \left|\frac{u_h^{m+1}+u_h^m}{2}\right|^2_{\mathrm{H}^1(0,1)} = 0,$$

where $\|\cdot\|_{\mathrm{L}_2(0,1)}$ is the $\mathrm{L}_2$-norm on the interval $(0,1)$, and $|\cdot|_{\mathrm{H}^1(0,1)}$ is the seminorm of the Sobolev space $\mathrm{H}^1(0,1)$.

Hence deduce that the method is unconditionally stable in the $\mathrm{L}_2$-norm in the sense that, for any $\Delta t$, independent of the choice of $h$,

$$\|u_h^m\|_{\mathrm{L}_2(0,1)} \le \|u_h^0\|_{\mathrm{L}_2(0,1)}, \qquad 1 \le m \le M.$$

[9 marks]

(iii) Show that, for each $m$, $0 \le m \le M-1$, $u_h^{m+1}$ can be obtained from $u_h^m$ by solving a system of linear algebraic equations with a symmetric matrix $\mathcal{A}$ whose entries you should define in terms of the standard piecewise linear basis functions $\varphi_i$, $i = 0, \dots, N$. [7 marks]



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# Numerical Linear Algebra

## Question 5

(a) Arnoldi's method generates a set of orthonormal vectors $\{v_1, v_2, \ldots, v_k\}$ which are the columns of the matrix $V_k$ satisfying

$$AV_k = V_{k+1}\widehat{H}_k.$$

What is the structure of the matrix $\widehat{H}_k$? For what space does $\{v_1, v_2, \ldots, v_k\}$ form a basis? What iterative method for a linear system $Ax = b$ is based on use of Arnoldi's method and computes iterates $x_k$ for which $\|r_k\|_2$ is minimal where $r_k = b - Ax_k$? Give a brief outline of the algorithm for this iterative method. [2+3+2+6 marks]

(b) If the matrix $A$ happens to be symmetric, in what way does the Arnoldi method simplify? [4 marks]

(c) Compute the first two iterates, $x_1, x_2$ for the linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

when $x_0 = 0$. [8 marks]

## Question 6

Let $\Pi_m$ denote the set of real polynomials of degree $m$ or less.

(a) Given a square matrix $A \in I\!R^{n\times n}$ and a non-zero vector $r \in I\!R^n$, how do they define a family of Krylov subspaces $\mathcal{K}_k$ for $k = 1, 2, \ldots$? If $\{x_k \in I\!R^n, k = 0, 1, 2, \ldots\}$ is a sequence of vectors for which $r_j = b - Ax_j, j = 0, 1, 2, \ldots$ for a given vector $b \in I\!R^n$ and which satisfy

$$x_k \in x_0 + \mathcal{K}_k(A, r_0), \quad k = 1, 2, \ldots, \quad (\dagger)$$

show that $r_k = p_k(A)r_0$ for some $p_k \in \Pi_k$ with $p_k(0) = 1$. [4+6 marks]

(b) Assume now that $A$ is symmetric and positive definite and that the iterates $x_k, k = 1, 2, \ldots$ generated by the Conjugate Gradient method for the linear system $Ax = b$ satisfy $(\dagger)$ and

$$\|x - x_k\|_A \le \|x - y\|_A, \quad y \in x_0 + \mathcal{K}_k(A, r_0),$$

where $\|r\|_A^2 = r^T Ar$. Show that

$$\|x - x_k\|_A \le \min_{p\in\Pi_k, p(0)=1} \max_j |p(\lambda_j)| \; \|x - x_0\|_A$$

where $\{\lambda_j\}$ are the eigenvalues of $A$. Indicate why the Conjugate Gradient method will be a particularly effective method for the solution of the linear system is $A$ has few distinct eigenvalues. [8+2 marks]

(c) Explain the idea of preconditioning in connection with the Conjugate Gradient solution of a linear system of equations, clearly outlining what are desirable features of a good preconditioner. [5 marks]