Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Linear Algebra & Finite Element Methods

TRINITY TERM 2009
Thursday, 23rd April 2009, 2:00 p.m. – 4:00 p.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Part A — Numerical Linear Algebra

Question 1

Let Π_k denote the set of real polynomials of degree less than or equal to k.

(a) Suppose that $A \in \mathbb{R}^{n \times n}$ is a nonsingular matrix and it is desired to solve $Ax = b$ for some given b; what advantage is there in employing a Krylov subspace method for a sparse matrix A, which does not exist if A is a general full matrix? (2 marks)

(b) Define the Krylov subspace $\mathcal{K}_k(A, b)$ and show that $y \in \mathcal{K}_k(A, b)$ if and only if $y = p(A)b$ for some $p \in \Pi_{k-1}$. Explain why the mathematical definition of $\mathcal{K}_k(A, b)$ does not provide a good basis to use for computation; you may assume that A is diagonalisable and has a simple (i.e. non-multiple) largest eigenvalue. (7 marks)

(c) Now take $x_0 = 0$. Give an algorithm for the Arnoldi method and say what it computes. Further describe how the GMRES algorithm uses the Arnoldi method and the solution of linear least squares problems to compute iterates $x_k \in \mathcal{K}_k(A, b), k = 1, 2, \ldots$ with corresponding residuals which are minimal in the 2-norm. (12 marks)

(d) What would happen if b happened to be an eigenvector of A? (4 marks)

Question 2

In this question $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite.

(a) Show that $\|x\|_A := \sqrt{x^TAx}$ defines a norm on $\mathbb{R}^n$. (5 marks)

(b) You are given the Conjugate Gradient method :
choose x_0 , $r_0 = b - Ax_0 = p_0$ and for $k = 0, 1, 2, \ldots$

$$\begin{aligned} \alpha_k &= p_k^T r_k / p_k^T A p_k \\ x_{k+1} &= x_k + \alpha_k p_k \\ r_{k+1} &= b - Ax_{k+1} \qquad (\dagger) \\ \beta_k &= -p_k^T A r_{k+1} / p_k^T A p_k \\ p_{k+1} &= r_{k+1} + \beta_k p_k. \end{aligned}$$

Show that $r_{k+1} = r_k - \alpha_k A p_k$ is an alternative to the statement $(\dagger)$. Prove that $r_{k+1}^T p_k = 0$ and that $p_{k+1}^T A p_k = 0$. Prove also that $r_{k+1}^T r_k = 0$. (12 marks)

(c) Show how preconditioning with preconditioner $P = HH^T$ can be applied whilst preserving symmetry and positive definiteness in the preconditioned system. (You do not need to derive the preconditioned conjugate gradient algorithm).

Suppose that

$$A = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & \frac{1}{2} \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & 1 \\ \frac{1}{2} & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}.$$

Can you suggest an efficient preconditioning strategy for this matrix, explaining how the efficiency might be achieved? (8 marks)

Section B — Finite Element Methods

Question 3

(a) Derive the weak form for the boundary value problem

$$-(xu')' + 3u = f, \quad x \in (1,2), \quad u(1) = 0, u'(2) = 0,$$

which requires the weak solution and test functions to have a square integrable first derivative. Express the weak form as $a(u,v) = \ell(v)$, In precisely what space is any weak solution to be found?

(7 marks)

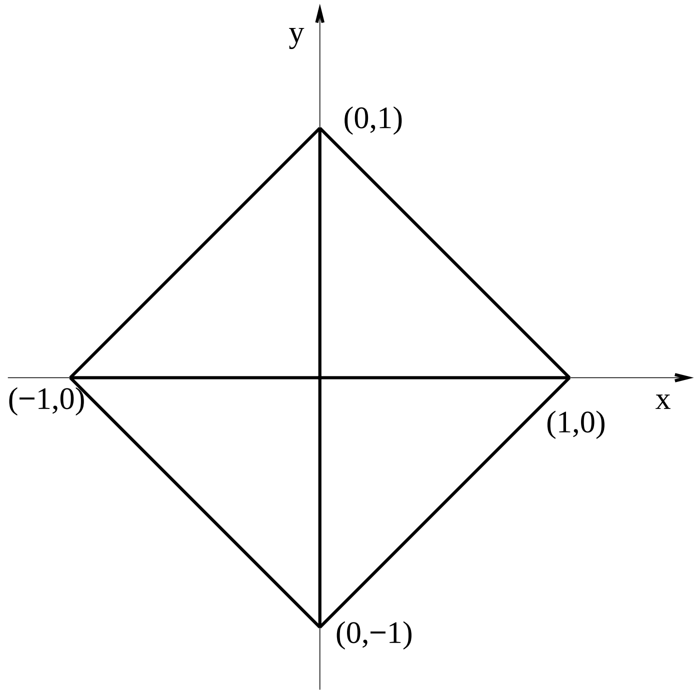
(b) Prove that the weak solution is unique; you must prove any results that you need. (3 marks)

(c) The piecewise quadratic (P2) Galerkin finite element solution for this problem is to be computed on elements which form a partition of the interval $(1,2)$. Sketch the basis functions to be used and write down precisely the integrals required in one row of an element matrix for $a(\cdot,\cdot)$; you do not need to evaluate any of these integrals. Will this element matrix be positive definite? Give reasons for your answer.

(15 marks)

Question 4

(a) Calculate the P1 Galerkin finite element solution to the homogeneous Dirichlet problem for the differential equation $-\nabla^2 u = 2$ on the square domain of side $\sqrt{2}$ as illustrated, using the four triangular elements shown.



(17 marks)

(b) Suppose now that the boundary condition is changed so that $u = x$ on the boundary for $x \geq 0$ whilst it stays that $u = 0$ on the boundary for $x < 0$. Calculate the Galerkin finite element solution for this problem.
(8 marks)

Question 5

(a) Define the Sobolev space $\mathcal{H}^1(\Omega)$ where $\Omega \subset \mathbb{R}^2$. Denote by $\partial\Omega$ the boundary of Ω. For what value of the constant $\alpha \in \mathbb{R}$ is

$$\{f \in \mathcal{H}^1(\Omega) | f = \alpha \text{ on } \partial\Omega\}$$

a subspace of $\mathcal{H}^1(\Omega)$? For this particular value of α we denote this space by $V(\Omega)$. If $\Omega = (0,1) \times (0,1)$, state and prove the Poincaré inequality for functions in $V(\Omega)$. (12 marks)

(b) Prove that the bilinear form

$$a(u,v) := \int_\Omega (1 + x^2 + y^2) \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y$$

is coercive on $V(\Omega)$ in the L_2 norm. Show whether or not $a(\cdot,\cdot)$ is continuous on $V(\Omega)$ in the L_2 norm. (7 marks)

(c) Using $a(\cdot,\cdot)$ defined in part (b), let $\ell(v)$ be any linear form defined for $v \in V(\Omega)$, prove that any solution of the problem: find $u \in V(\Omega)$ such that

$$a(u,v) = \ell(v) \quad \text{for all } v \in V(\Omega)$$

is necessarily unique. Determine values of the constant $\beta \in \mathbb{R}$ for which the solution of

$$a(u,v) + \beta\langle u,v \rangle = \ell(v) \quad \text{for all } v \in V(\Omega)$$

is unique. Here $\langle \cdot,\cdot \rangle$ is the $L_2(\Omega)$ inner product. (6 marks)

Question 6

(a) It is desired to solve the diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ (x,t) \in (0,1) \times [0,T],\ u(0,t) = 0, \frac{\partial u}{\partial x}(1,t) = 0, u(x,0) = \sin \pi x/2$$

using Galerkins method with piecewise linear finite elements in x and forward Euler timestepping in t. A uniform spatial partition $x_j = jh, j = 0, 1, \ldots, n$ with $nh = 1$ is to be used.

Derive the fully discrete equations that need to be solved at each time step. Explicitly define the two matrices which arise and calculate their entries.

(21 marks)

(b) What is special about the choice of time step $\Delta t = \frac{1}{3}h^2$? (4 marks)