JMAT 7304

Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Linear Algebra & Finite Element Methods

TRINITY TERM 2011
Thursday, 28th April 2011, 2:00 p.m. – 4:00 p.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Part A — Numerical Linear Algebra

Question 1

(a) Define what it means that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction mapping. State, without proof, Banach's fixed point theorem. **[5 marks]**

(b) Let $C \in \mathbb{R}^{n\times n}$ and $\mathbf{c} \in \mathbb{R}^n$ be given. Let $\mathbf{x}^{(0)} \in \mathbb{R}^n$. Define an iterative method by

$$\mathbf{x}^{(j+1)} = C\mathbf{x}^{(j)} + \mathbf{c}, \qquad j \geq 0.$$

(i) Assume that there is an induced matrix norm $\|\cdot\|$ such that $\|C\| < 1$. Show that the method converges for every starting vector $\mathbf{x}^{(0)}$.

(ii) Show that the spectral radius $\rho(C)$ satisfies $\rho(C) < 1$ if the method converges for every starting vector. In this case, determine the limit in terms of C and $\mathbf{c}$. **[9 marks]**

(c) Consider the Jacobi method for solving linear systems $A\mathbf{x} = \mathbf{b}$ iteratively. The scheme is given by picking a starting vector $\mathbf{x}^{(0)}$ and then computing $\mathbf{x}^{(j+1)}$ iteratively via

$$x_i^{(j+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{k=1\\k\neq i}}^{n} a_{ik}x_k^{(j)}\right), \qquad 1 \leq i \leq n,$$

for $j = 0, 1, 2, 3, \ldots$

(i) Rewrite the Jacobi method in the form $\mathbf{x}^{(j+1)} = C\mathbf{x}^{(j)} + \mathbf{c}$, i.e, determine the iteration matrix $C \in \mathbb{R}^{n\times n}$ and the vector $\mathbf{c} \in \mathbb{R}^n$. Do this by decomposing A into $A = L + D + R$ and explain the matrices involved in this decomposition.

(ii) Assume that the matrix A satisfies

$$q := \max_{1\leq j\leq n} \sum_{\substack{i=1\\i\neq j}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| < 1.$$

Show that the Jacobi method converges for every starting vector $\mathbf{x}^{(0)}$ to the solution $\mathbf{x}^*$ of $A\mathbf{x} = \mathbf{b}$.

(iii) Assume now that the matrix A and the right-hand side $\mathbf{b}$ are given by

$$A = \begin{pmatrix} 10 & 1 & 0 \\ 4 & 100 & 5 \\ -6 & 0 & 100 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 50 \\ 20 \end{pmatrix}.$$

Assume that $\mathbf{x}^{(0)} = \mathbf{0}$. How many iterations j are at most necessary so that the error satisfies $\|\mathbf{x}^* - \mathbf{x}^{(j)}\|_1 \leq 10^{-10}$? **[11 marks]**

Question 2

Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite and $\mathbf{b} \in \mathbb{R}^n$. For any $\mathbf{y} \in \mathbb{R}^n$ define

$$F(\mathbf{y}) := \frac{1}{2}\mathbf{y}^T A\mathbf{y} - \mathbf{y}^T\mathbf{b}.$$

(a) Given that $\mathbf{x}^*$ solves $A\mathbf{x} = \mathbf{b}$, show that

$$F(\mathbf{y}) = F(\mathbf{x}^*) + \frac{1}{2}(\mathbf{y} - \mathbf{x}^*)^T A(\mathbf{y} - \mathbf{x}^*).$$

Deduce that $\mathbf{x}^*$ is a minimiser of F. **[5 marks]**

(b) Let $\mathbf{x}_j \in \mathbb{R}^n$ be a current location and $\mathbf{p}_j \neq \mathbf{0}$ a current direction. Show that minimising F along the line $t \mapsto \mathbf{x}_j + t\mathbf{p}_j$ leads to the next location

$$\mathbf{x}_{j+1} = \mathbf{x}_j + \frac{\mathbf{p}_j^T(\mathbf{b} - A\mathbf{x}_j)}{\mathbf{p}_j^T A\mathbf{p}_j}\mathbf{p}_j.$$

Hence, we have the following generic algorithm to compute a minimum of F, which we will investigate further later on:

- Choose $\mathbf{x}_1$ and $\mathbf{p}_1$.
- For $j = 1, 2, \ldots,$ do
  - $\mathbf{r}_j = \mathbf{b} - A\mathbf{x}_j$
  - $\alpha_j = \frac{\mathbf{p}_j^T\mathbf{r}_j}{\mathbf{p}_j^T A\mathbf{p}_j}$
  - $\mathbf{x}_{j+1} = \mathbf{x}_j + \alpha_j\mathbf{p}_j$
  - Choose next direction $\mathbf{p}_{j+1}$.

[5 marks]

(c) Give the definition for A-conjugate directions. Consider the matrix

$$A = \begin{pmatrix} 5 & 2 \\ -1 & 2 \end{pmatrix}$$

Compute an A-conjugate direction to $\mathbf{p}_1 = (1, 1)^T$.

[4 marks]

(d) Show that A-conjugate directions are linearly independent.

[3 marks]

(e) Assume that the directions in the generic algorithm above are A-conjugate. Show that in this case the algorithm terminates after at most n steps with the solution $\mathbf{x}^*$ of $A\mathbf{x} = \mathbf{b}$. **[8 marks]**

Section B — Finite Element Methods

Question 3

(a) Given a bounded domain $\Omega \subset \mathbb{R}^d, d = 1, 2$ or 3, define the Sobolev space $\mathcal{H}^1(\Omega)$. For functions satisfying $p(x) > 0, q(x) \geq 0, x \in [0, 1]$, derive a weak form for

$$-(pu')' + qu = f \text{ on } \Omega = (0,1), \;\; u'(0) = u(0), u(1) = 0$$

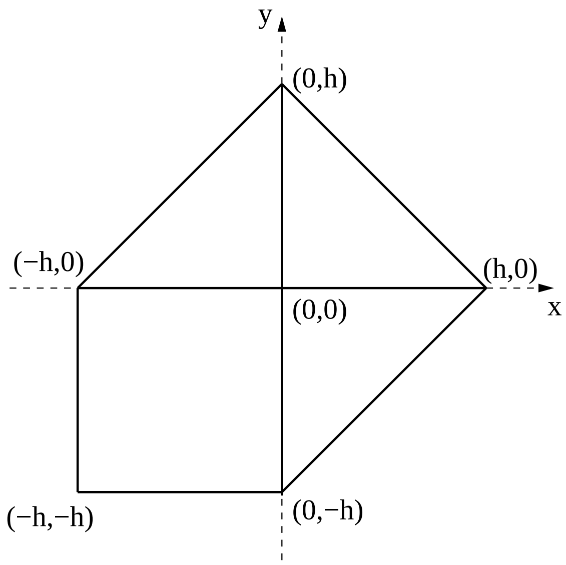
for which the smoothness requirement is only $u, v \in \mathcal{H}^1(0,1)$ for the weak solution u and test function v. Identify in precisely what sets u and v must be. **[2+6 marks]**

(b) Prove that if u is any such weak solution, then there exist a bounded constant L such that $\|u\| \leq L\|u'\|$ where $\|\cdot\|$ is the $L_2(0,1)$ norm. Hence or otherwise prove that any solution of the weak form must be unique. **[5+4 marks]**

(c) In the case that $p(x) = 1 + x^2$ and $q(x) = 0$, this problem is to be approximately solved by the Galerkin finite element method with $P1$ (piecewise linear) elements on an irregular mesh with nodes $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1$. Calculate a single row of the resulting coefficient matrix for a node not on the boundary. **[8 marks]**

Question 4

(a) Calculate the Galerkin finite element solution for the differential equation $-\nabla^2 u = 1$ on the domain illustrated and with homogeneous Dirichlet boundary conditions using the 3 $P1$ triangular elements and 1 $Q1$ square element as illustrated. Explain why the approximation space is conforming.



[20 marks]

(b) What issue arises if the node point at the origin is now moved to $(-h/3, -h/4)$ whilst the topology (connection) of the mesh remains the same? (You do not need to do any further calculations to explain this.) **[5 marks]**

Question 5

(a) Define coercivity and continuity of a bilinear form

$$a(\cdot,\cdot) : V \times V \to \mathbb{R}$$

where V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_V$. State and prove Cea's Lemma. Summarise in words what the outcome of Cea's lemma is. **[4+7+1 marks]**

(b) Now suppose that $V = \{v \in \mathcal{H}^1(\Omega); v = 0 \ \text{ on } \ \partial\Omega\}$ for the bounded domain $\Omega = (b, c) \times (\beta, \gamma) \subset \mathbb{R}^2$ with boundary $\partial\Omega$ and the inner product is

$$\langle u, v \rangle_V = \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v.$$

State but do not prove the Poincaré inequality for functions in V. **[2 marks]**

Prove that

$$a(u, v) = \int_\Omega e^{x+y} \, \nabla u \cdot \nabla v$$

is coercive and continuous on V with respect to the given inner product and associated norm $\|\cdot\|_V$. **[7 marks]**

(c) For $N \in \mathbb{N}$, let S be a $N-$dimensional vector subspace of V. Further suppose that for all $v \in V$ there exists $v_N \in S$ satisfying

$$\|v - v_N\|_V \leq CN^{-4}$$

for some constant C. Establish an error bound for the Galerkin approximation from S to the function $u \in V$ that satisfies

$$a(u, w) = \ell(w) \quad \text{for all } \ w \in V$$

for any given continuous linear form $\ell : V \to \mathbb{R}$. **[4 marks]**

Question 6

Given a bounded domain $\Omega \subset \mathbb{R}^2$, it is desried to solve the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x, y) \in \Omega, t \in [0, T]$$

with homogeneous Dirichlet boundary conditions and initial condition $u(x, y, 0) = g(x, y)$ by employing Galerkin finite element approximation in the spatial variables (x, y) only together with a finite difference time-stepping scheme.

(a) Derive the semi-discrete equations (i.e. ordinary differential equations) which result when a basis $\{\phi_1(x, y), \phi_2(x, y), \ldots, \phi_n(x, y)\}$ is used for the finite element approximating space. If these equations are written in the form

$$M\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = K\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{g},$$

precisely define the matrices M and K. **[10 marks]**

(b) Prove that M is positive definite. Is K definite? Explain your answer. **[4 marks]**

Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_h(x, y, t)^2 \mathrm{d}x\mathrm{d}y \leq 0$$

when u_h is the semi-discrete Galerkin finite element solution. **[4 marks]**

(c) Crank-Nicolson time-stepping for these equations is

$$M\frac{(\mathbf{u}_{k+1} - \mathbf{u}_k)}{\Delta t} = \frac{1}{2}K\ (\mathbf{u}_{k+1} + \mathbf{u}_k)$$

where $\mathbf{u}_k$ is associated with the time step $k\Delta t$. Prove that these fully discrete equations give a stable solution for any positive value of the time-step Δt. **[7 marks]**