JMAT 7304

Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Linear Algebra & Finite Element Methods

TRINITY TERM 2013
Friday 19th April 2013, 9.30 a.m. – 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Part A — Numerical Linear Algebra

Question 1

What is a Krylov subspace? Show that if $A \in \mathbb{R}^{n \times n}$ is nonsingular then any Krylov subspace method for the solution of a linear system, $Ax = b$, computes iterates $x_k, k = 1, 2, \ldots$, from a starting guess x_0 such that the residuals $r_k = b - Ax_k, k = 0, 1, 2, \ldots$, satisfy

$$r_k = p(A)r_0, \qquad (\star)$$

where p is a polynomial. Exactly what conditions does p satisfy?

[4 marks]

What is Arnoldi's method? Show that it can be written in the form

$$AV_k = V_{k+1}\widehat{H}_k,$$

where you should describe the exact form of the matrices V_k and $\widehat{H}_k$. Hence show how the residuals and iterates of the GMRES method, which minimizes $\|r_k\|_2$ for each k, can be computed via the solution of a linear least squares problem involving $\widehat{H}_k$. If the vector y is the solution of this linear least squares problem, show that

$$y = V_k^T q(A) r_0$$

for some polynomial q, which you should express in terms of the polynomial p in $(\star)$.

[17 marks]

After how many GMRES iterations should termination occur with the correct solution for the problem

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}?$$

[4 marks]

Question 2

Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite and denote the set of real polynomials of degree less than or equal to j by Π_j.

The Conjugate Gradient method is applied to find the solution, x, of the linear system $Ax = b$; it generates iterates $x_k, k = 1, 2, \ldots$, from a starting guess x_0. In what sense is the error vector, $x - x_k$, minimized? Show that

$$\|x - x_k\|_A \le \min_{p\in\Pi_k, p(0)=1} \max_j |p(\lambda_j)| \, \|x - x_0\|_A,$$

where $\lambda_j, j = 1, 2, \ldots, n$, are the eigenvalues of A. If

$$\min_j \lambda_j = a \quad \text{and} \quad \max_j \lambda_j = b,$$

deduce that

$$\frac{\|x - x_k\|_A}{\|x - x_0\|_A} \le 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k, \quad \text{where } \kappa = \frac{b}{a}.$$

[You might find it helpful to introduce a shifted and scaled Chebyshev polynomial.]

[19 marks]

What is the idea of preconditioning? Why might $P = \mathrm{diag}(A)$ be a good preconditioner when

$$A = \begin{bmatrix} 10 & 1 & & & & \\ 1 & 10^2 & 1 & & & \\ & 1 & 10^3 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 10^n \end{bmatrix}?$$

(You may use Gershgorin Theorems without proof).

[6 marks]

Section B — Finite Element Methods

Question 3

(i) Define the Sobolev space $\mathrm{H}^1(0,1)$ and the Sobolev norm $\|\cdot\|_{\mathrm{H}^1(0,1)}$.

What is meant by saying that u is a weak solution in $\mathrm{H}^1(0,1)$ of the boundary-value problem

$$-u'' + (\exp x)u = f(x), \quad x \in (0,1); \qquad u'(0) = 0, \quad u(1) + u'(1) = 1,$$

where $f \in \mathrm{L}^2(0,1)$?

[6 marks]

(ii) Show that the bilinear form associated with the weak formulation of this problem is coercive on $\mathrm{H}^1(0,1)$.

Consider the continuous piecewise linear basis functions φ_i, $i = 0,1,\ldots,N$, defined by $\varphi_i(x) = (1 - |x - x_i|/h)_+$ on the uniform mesh of size $h = 1/N$, $N \geq 2$, with mesh-points $x_i = ih$, $i = 0,1,\ldots,N$. Using the basis functions φ_i, $i = 0,1,\ldots,N$, define the finite element approximation of the boundary-value problem and show that it has a unique solution u_h.

[6 marks]

(iii) Expand u_h in terms of the basis functions φ_i, $i = 0,1,\ldots,N$, by writing

$$u_h(x) = \sum_{i=0}^{N} U_i \varphi_i(x),$$

where $\mathbf{U} = (U_0, U_1, \ldots, U_N)^{\mathrm{T}} \in \mathbb{R}^{N+1}$, to obtain a system of linear algebraic equations for the vector of unknowns $\mathbf{U}$. Show that the matrix A of this linear system is symmetric (i.e. $A^{\mathrm{T}} = A$) and positive definite (i.e. $\mathbf{V}^{\mathrm{T}} A \mathbf{V} > 0$ for all $\mathbf{V} \in \mathbb{R}^{N+1}$, $\mathbf{V} \neq \mathbf{0}$).

[6 marks]

(iv) Show also that $\|u - u_h\|_{\mathrm{H}^1(0,1)} = \mathcal{O}(h)$ as $h \to 0$.

[You may assume, without proof, that $u \in \mathrm{H}^2(0,1)$.

Any bound on the error between u and its finite element interpolant $\mathcal{I}_h u$ may be used without proof, but must be stated carefully.]

[7 marks]

Question 4

(i) Let $\psi \in \mathrm{L}^2(0,1)$ and let $a(\cdot,\cdot)$ be the bilinear form on $\mathrm{H}^1_0(0,1) \times \mathrm{H}^1_0(0,1)$ defined by

$$a(w,v) = \int_0^1 (w'(x)\,v'(x) + c(x)\,w(x)\,v(x))\,\mathrm{d}x,$$

where $c \in \mathrm{C}^2[0,1]$ is such that $c(x) \geq 1$ and $c''(x) \leq -2$ for all $x \in [0,1]$. Suppose, further, that $z \in \mathrm{H}^1_0(0,1)$ is such that

$$a(w,z) = \int_0^1 w(x)\,\psi(x)\,\mathrm{d}x \qquad \forall w \in \mathrm{H}^1_0(0,1).$$

Show that $z \in \mathrm{H}^2(0,1) \cap \mathrm{H}^1_0(0,1)$, and $\|z\|_{\mathrm{H}^2(0,1)} \leq \|\psi\|_{\mathrm{L}^2(0,1)}$.

[5 marks]

(ii) Suppose that $f \in \mathrm{L}^2(0,1)$ and let $u \in \mathrm{H}^1_0(0,1)$ be the weak solution of the problem

$$a(u,v) = \int_0^1 f(x)\,v(x)\,\mathrm{d}x \qquad \forall v \in \mathrm{H}^1_0(0,1).$$

Let, further, u_h denote the piecewise linear finite element approximation to u on the subdivision $\mathcal{S}_h = \{[x_{i-1}, x_i] \,:\, i = 1,2,\ldots,N\}$, where $x_i - x_{i-1} = h_i$, $i = 1,2,\ldots,N$.

[6 marks]

Show that

$$\int_0^1 (u - u_h)\,\psi\,\mathrm{d}x = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} R(u_h)\,(z - \mathcal{I}_h z)\,\mathrm{d}x,$$

where $\mathcal{I}_h z$ is the continuous piecewise linear finite element interpolant of z on the subdivision $\mathcal{S}_h$, and $R(u_h)$ is the *residual* that you should carefully define in terms of f, c and u_h.

(iii) Show that

$$\left|\int_0^1 (u - u_h)\,\psi\,\mathrm{d}x\right| \leq \frac{1}{\pi^2}\left(\sum_{i=1}^N \|R(u_h)\|^2_{\mathrm{L}^2(x_{i-1},x_i)} h_i^4\right)^{1/2} \|\psi\|_{\mathrm{L}^2(0,1)},$$

and deduce the *a posteriori* error bound

$$\|u - u_h\|_{\mathrm{L}^2(0,1)} \leq \frac{1}{\pi^2}\left(\sum_{i=1}^N \|R(u_h)\|^2_{\mathrm{L}^2(x_{i-1},x_i)} h_i^4\right)^{1/2}.$$

[Any bound on the error between z and its finite element interpolant $\mathcal{I}_h z$ may be used without proof, but must be stated carefully.]

[7 marks]

(iv) Discuss, briefly, how this *a posteriori* error bound could be implemented into an adaptive mesh-refinement algorithm to compute, for a prescribed tolerance $\mathtt{TOL} > 0$, an approximation u_h to u such that $\|u - u_h\|_{\mathrm{L}^2(0,1)} \leq \mathtt{TOL}$.

[7 marks]

Question 5

Suppose that Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary Γ, oriented in the anticlockwise direction. Suppose, further, that $f \in \mathrm{L}^2(\Omega)$ and $g \in \mathrm{L}^2(\Gamma)$ and consider the quadratic functional $J : v \in \mathrm{H}^1(\Omega) \mapsto J(v) \in \mathbb{R}$ defined by

$$J(v) = \tfrac{1}{2} \int_\Omega (|\nabla v|^2 + v^2)\,\mathrm{d}x - \int_\Omega f\,v\,\mathrm{d}x - \int_\Gamma g\,v\,\mathrm{d}s.$$

(i) Show that if $u \in \mathrm{H}^1(\Omega)$ is such that $J(u) \leq J(v)$ for all $v \in \mathrm{H}^1(\Omega)$, then there exist a bilinear functional $a(\cdot,\cdot)$ defined on $\mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$ and a linear functional $\ell(\cdot)$ defined on $\mathrm{H}^1(\Omega)$ such that

$$a(u, v) = \ell(v) \qquad \forall v \in \mathrm{H}^1(\Omega). \qquad \text{(P)}$$

Show further that if $u \in \mathrm{H}^1(\Omega)$ is such that (P) holds, then $J(u) \leq J(v)$ for all $v \in \mathrm{H}^1(\Omega)$.

[10 marks]

(ii) Show that (P) has a unique (weak) solution $u \in \mathrm{H}^1(\Omega)$; hence deduce that $u \in \mathrm{H}^1(\Omega)$ is the unique minimizer of J over the Sobolev space $\mathrm{H}^1(\Omega)$.

[7 marks]

[*You may use without proof that there exists a positive constant C_0 such that $\|v\|_{\mathrm{L}^2(\Gamma)} \leq C_0\|v\|_{\mathrm{H}^1(\Omega)}$ for all $v \in \mathrm{H}^1(\Omega)$.*]

(iii) Suppose that Ω is the unit square $(0,1) \times (0,1)$, and let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}$ constructed from a uniform square grid of spacing $h = 1/N$ by subdividing each grid-square by the diagonal of negative slope.

Formulate the piecewise linear finite element approximation (P_h) of problem (P) on the triangulation $\mathcal{T}_h$. Show that (P_h) has a unique solution u_h and that u_h is the unique minimizer of J over V_h. Show further that

$$\|u - u_h\|_{\mathrm{H}^1(\Omega)} = \min_{v_h \in V_h} \|u - v_h\|_{\mathrm{H}^1(\Omega)},$$

where V_h is the finite-dimensional vector space consisting of all continuous piecewise linear functions defined on the triangulation $\mathcal{T}_h$.

[8 marks]

Question 6

Let $\Omega := (0,1) \times (0,1)$ and denote by Γ the boundary of Ω, oriented in the anticlockwise direction. Let, further $T > 0$ and $u_0 \in \mathrm{L}^2(\Omega)$. Consider the initial-boundary-value problem

$$\begin{aligned} \frac{\partial u}{\partial t} + u &= \nabla^2 u, && \text{in } \Omega \times (0,T], \\ u(x,y,0) &= u_0(x,y), && \text{for } (x,y) \in \Omega, \\ \frac{\partial u}{\partial n} &= 0, && \text{on } \Gamma \times [0,T], \end{aligned}$$

where $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$, and $\mathbf{n}$ denotes the unit outward normal vector to Γ.

(i) Formulate the Crank–Nicolson scheme for the numerical solution of this initial-boundary-value problem, using continuous piecewise linear finite elements on a triangulation of Ω, and time step $\Delta t = T/M$, where M is a positive integer. The initial datum u_h^0 for the Crank–Nicolson scheme should be chosen as the orthogonal projection in $\mathrm{L}^2(\Omega)$ of the function u_0 onto the finite element space.

[7 marks]

(ii) Show that the Crank–Nicolson scheme is unconditionally stable in the $\mathrm{L}^2(\Omega)$ norm.

[7 marks]

(iii) Show that to compute the finite element approximation u_h^m to $u(\cdot, m\Delta t)$ at time level m, $1 \leq m \leq M$, one is required to solve a system of linear algebraic equations whose matrix is

$$A = K + \tfrac{1}{2}\,\Delta t\, K + \tfrac{1}{2}\,\Delta t\, S,$$

where the (i,j) entries of the matrices K and S are defined by the formulae

$$K_{ij} = \int_\Omega \varphi_i(x,y)\,\varphi_j(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{and} \quad S_{ij} = \int_\Omega \nabla\varphi_i(x,y) \cdot \nabla\varphi_j(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

respectively, and φ_i is the continuous piecewise linear finite element basis function, equal to 1 at node i and to 0 at all the other nodes.

Show that all eigenvalues of K are positive. Show further that, for all $\Delta t > 0$, the smallest eigenvalue of A is greater than the smallest eigenvalue of K.

[11 marks]