Degree Master of Science in Mathematical Modelling and Scientific Computing

Numerical Solution of Differential Equations & Numerical Linear Algebra

Friday 11th January 2013, 9:30 p.m. – 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A —Numerical Solution of Differential Equations

Question 1

The function $u(t)$, $t \geq 0$, with $u(0) = u_0$, is determined by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u), \quad t > 0,$$

where f satisfies a Lipschitz condition

$$|f(u) - f(v)| \leq L|u - v| \ \ \forall u, v \in \mathbb{R}.$$

The differential equation is discretised at points $t_n = n\Delta t$, $n = 0, 1, \ldots$, where $\Delta t > 0$, by the third-order scheme

$$U_{n+1} = U_n + \frac{1}{4}\Delta t(K_1 + 3K_3),$$

where $U_0 = u_0$, and

$$\begin{aligned} K_1 &= f(U_n), \\ K_2 &= f(U_n + \frac{1}{3}\Delta t K_1), \\ K_3 &= f(U_n + \frac{2}{3}\Delta t K_2). \end{aligned}$$

a) Estimate the error $e_n = u(t_n) - U_n$ after n steps and confirm that the error vanishes as $\Delta t \to 0$.

[12 marks]

b) The scheme is applied to the case $f(u) = -\lambda u$, where λ is a positive real number, with $u(0) = 1$. Show that

$$U_{n+1} = p(\lambda\Delta t)U_n,$$

where p is a polynomial that you should determine. From a rough sketch of p, or otherwise, deduce that the sequence $(U_n)_{n=1}^{\infty}$ will converge provided $\lambda\Delta t \leq 2.5$ and that the sequence $(U_n)_{n=1}^{\infty}$ will converge monotonically provided $\lambda\Delta t \leq 1.5$.

Note that you are not required to determine the full range for convergence.

[8 marks]

c) Show that when $\lambda\Delta t \leq 2.5$, the error in the case (b) satisfies

$$|u(t_n) - U_n| \leq \frac{1}{24}\lambda^4 t_n \Delta t^3.$$

[5 marks]

Question 2

The function $u(t)$, $t \geq 0$, is determined by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(t, u), \quad t > 0,$$

where f is a uniformly continuous function of its arguments and $u(0) = u_0$.

A linear multistep method for numerical approximation of this equation at the points $t_r = r\Delta t,\ r = 0, 1, 2, \ldots$, with $\Delta t > 0$, is defined for integer $k > 0$ by

$$\sum_{r=0}^{k} \alpha_r U_{n+r} = \Delta t \sum_{r=0}^{k} \beta_r F_{n+r},\ n = 0, 1, \ldots,$$

where U_n is an approximation to $u_n = u(t_n)$, $F_n = f(t_n, U_n)$, $\alpha_k \neq 0$ and $\beta_0 \neq 0$. The polynomials $\rho(z)$ and $\sigma(z)$ are given by

$$\rho(z) = \sum_{r=0}^{k} \alpha_r z^r, \quad \sigma(z) = \sum_{r=0}^{k} \beta_r z^r.$$

a) Define *consistency*, *zero stability* and the *root condition* for a linear multistep method.

[2+2+2 marks]

b) Prove that consistency is a necessary condition for convergence.

[8 marks]

c) Show that the scheme

$$11U_{n+3} - 18U_{n+2} + 9U_{n+1} - 2U_n = 6\Delta t F_{n+3}$$

is zero stable and determine the order of the scheme.

[6 marks]

d) Define *absolute stability* for the case $f(u) = \lambda u$. Show that the scheme in (c) is absolutely stable as $\mathrm{Re}[\Delta t\lambda] \to -\infty$.

[5 marks]

Question 3

The function $u(x,t)$, defined for $x \in \mathbb{R}$ and $t \geq 0$, satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0,$$

and the initial condition $u(x,0) = u_0(x) \geq 0$, where $u_0 \to 0$ as $|x| \to \infty$.

The partial differential equation is discretised on a uniform mesh $x_r = rh$, $r = 0, \pm 1, \pm 2, \ldots$, and $t_n = n\Delta t$, $n = 1, 2, \ldots$ with $h > 0$ and $\Delta t > 0$ such that U_r^n is an approximation for $u_r^n = u(x_r, t_n)$ and $U_r^0 = u_0(x_r)$, by:

$$(U_{r+1}^{n+1} + 4U_r^{n+1} + U_{r-1}^{n+1}) - (U_{r+1}^n + 4U_r^n + U_{r-1}^n) = 6\mu(U_{r+1}^{n+1} - 2U_r^{n+1} + U_{r-1}^{n+1}),$$

where $\mu = \Delta t/h^2$.

a) Define the *truncation error* at (x_r, t_{n+1}), denoted T_r^{n+1}, determine the leading term of the truncation error when $\mu \neq \frac{1}{6}$ and hence show that the scheme is consistent.

[12 marks]

b) Define *practical stability* for a finite difference approximation of the above initial-value problem in terms of the l_2-norm

$$||U^n||_{l_2} = \big(h \sum_{r=-\infty}^{\infty} |U_r^n|^2\big)^{1/2}.$$

Determine the amplification factor $\lambda(k)$ for this scheme such that

$$\hat{U}^{n+1}(k) = \lambda(k)\hat{U}^n(k),$$

where $\hat{U}^n(k)$ is the semi-discrete Fourier transform of $(U_r^n)_{r=0,\pm1,\pm2,\ldots}$ defined by

$$\hat{U}^n(k) = h \sum_{r=-\infty}^{\infty} e^{ikrh} U_r^n, \qquad k \in \big(-\frac{\pi}{h}, \frac{\pi}{h}\big).$$

Show that this scheme is practically stable.

You may use without proof Parseval's Identity

$$||U^n||_{l_2} = \frac{1}{\sqrt{2\pi}} ||\hat{U}^n||_{L_2},$$

where

$$||\hat{U}^n||_{L_2}^2 = \int_{-\pi/h}^{\pi/h} |\hat{U}^n(k)|^2 dk.$$

[9 marks]

c) When the space domain is restricted to $0 \leq x \leq 1$ with $u = 0$ at $x = 0$ and $x = 1$, $t > 0$, show that if $\mu \geq \frac{1}{6}$ then

$$||U^n||_\infty \leq ||U^0||_\infty$$

.

[4 marks]

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Question 4

The function $u(x,t)$, defined for $x \in \mathbb{R}$ and $t \geq 0$, satisfies

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad t > 0,$$

where $a > 0$ is constant, and the initial condition $u(x,0) = u_0(x)$, where u_0 is bounded on $\mathbb{R}$.

This hyperbolic equation is discretised on a uniform mesh $x_r = rh$, $r = 0, \pm 1, \pm 2, \ldots$, and $t_n = n\Delta t$, $n = 1, 2, \ldots$, with $h > 0$ and $\Delta t > 0$ such that U_r^n is an approximation for $u_r^n = u(x_r, t_n)$. Let $\nu = a\Delta t/h$.

a) Show that the *Box Scheme*

$$(1-\nu)U_{r-1}^{n+1} + (1+\nu)U_r^{n+1} - (1+\nu)U_{r-1}^n - (1-\nu)U_r^n = 0,$$

is consistent. *Hint: Consider the truncation error* $T_{r-\frac{1}{2}}^{n+\frac{1}{2}}$.

[10 marks]

b) Show that the scheme is unconditionally stable.

[7 marks]

c) The space domain is restricted to $x \geq 0$ so that $r = 0, 1, 2, \ldots$, and the boundary condition at $x = 0$ is set to $u(0,t) = 0$, with $u_0(0) = 0$.

Define

$$S^n = \sum_{r=1}^{\infty} \left[(U_r^n + U_{r-1}^n)^2 + \nu^2 (U_r^n - U_{r-1}^n)^2 \right].$$

Show that $S^{n+1} = S^n$ for all $n = 0, 1, \ldots$.

Deduce that $||U^n||_2$ is bounded independent of n, Δt and h, where

$$||U^n||_2^2 = \sum_{r=1}^{\infty} |U_r^n|^2,$$

provided that S^0 is bounded by a constant, independent of Δt and h, and $\nu \neq 0$ is held fixed.

[8 marks]

Section B — Numerical Linear Algebra

Question 5

What is $\|x\|_2$ for $x \in \mathbb{R}^n$? What is the *Frobenius norm*, $\|\cdot\|_F$, of a matrix? What is an *orthogonal matrix*? Using no more than the definitions you have provided in answering these three questions, prove that $\|Q\|_F = \sqrt{n}$ whenever $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix. Prove also that $\|Qx\|_2 = \|x\|_2$ for any $x \in \mathbb{R}^n$ whenever $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix.

[1+1+1+2+1 marks]

If $A \in \mathbb{R}^{m\times n}$ and $A = U\Sigma V^T$ is the Singular Value Decomposition (SVD) of A, give precise descriptions of U, Σ and V. In terms of the SVD, identify an orthonormal basis for

$$\mathrm{Ker}A = \{x \in \mathbb{R}^n : Ax = 0\}.$$

[3+4 marks]

Suppose now that $m > n$, A is of rank n and that for some $b \in \mathbb{R}^m$ it is desired to solve the linear least squares problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2$$

and that the SVD has already been computed. Show how the solution $\hat{x}$ and the least squares error $\|A\hat{x} - b\|_2$ can be computed without any further matrix factorization. What factorization is usually used for the solution of linear least squares problems? You should show how this factorization leads to a way of calculating the solution. (You do not need to show how this factorization can be computed).

[5+1+3 marks]

Suppose now that one is interested in solving the problem

$$\min_{X\in\mathbb{R}^{n\times k}} \|AX - B\|_F$$

for the given $A \in \mathbb{R}^{m\times n}$ and a given $B \in \mathbb{R}^{m\times k}, k \geq 2$. Describe an algorithm for computing the solution, X.

[3 marks]

Question 6

What is *Jacobi iteration* for the solution of a linear system of equations, $Ax = b$? What is the *iteration matrix* for this method? Prove that if

$$|a_{i,i}| > \sum_{j=1, j \neq i}^{n} |a_{i,j}| \quad \text{for } i = 1, 2, \ldots, n$$

then all of the eigenvalues of the Jacobi iteration matrix lie strictly inside the unit circle. Quoting but not proving any theorem that you use, explain the significance of this property of the eigenvalues of the iteration matrix.

[4+1+8+1 marks]

The five-point finite difference approximation of the Laplacian with homogeneous Dirichlet boundary conditions using a uniform grid with spacing h on a unit square gives rise to the linear system of equations

$$4U_{j,k} - U_{j+1,k} - U_{j-1,k} - U_{j,k+1} - U_{j,k-1} = b_{j,k}, \quad j, k = 1, \ldots, n,$$

where $U_{0,k}$ and $U_{n+1,k}$ are zero for each k, $U_{j,0}$ and $U_{j,n+1}$ are zero for each j and $b_{j,k}$ is known for all j and k.

Explicitly calculate the eigenvalues of the Jacobi itertion matrix for this linear system. Explain why the Jacobi method is not a good smoother for use with the multigrid method, whereas a relaxed Jacobi iteration (give a specific example) can be a very effective smoother.

[5+3 marks]

Outline the components of a multigrid algorithm for solution of linear systems like the five-point finite difference approximation of the Laplacian with Dirichlet boundary conditions.

[3 marks]