Properties of Lattice Isomorphism as a Cryptographic Group Action

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Outline

Equivalence Problems in Cryptography

Several protocols have been proposed using hard problems as underlying assumption consisting of finding the *equivalence/isomorphism* between two algebraic/geometrical objects.

Equivalence Problems in Cryptography

Several protocols have been proposed using hard problems as underlying assumption consisting of finding the *equivalence/isomorphism* between two algebraic/geometrical objects.

In the NIST Standardization of Additional Digital Signature Schemes we find:

- LESS [1] $\leftarrow$ equivalence of linear codes
- MEDS [7] $\leftarrow$ equivalence of matrix codes
- ALTEQ [4] $\leftarrow$ equivalence of alternating trilinear forms
- HAWK [5] $\leftarrow$ isomorphism of lattices
- SQIsign [6] $\leftarrow$ isogenies between supersingular elliptic curves

Equivalence Problems in Cryptography

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Equivalence Problems in Cryptography

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This framework brings the following benefit. It allows us to

1. Define a cryptographic primitive in general for group actions,
2. Instantiate the primitive with a specific group action.

Examples: (Linkable) Ring Signatures [3], Updatable Encryption [10], Threshold signatures [2], MPCiTH [9].

Contributions

• In this work, we formalize Lattice Isomorphism as a group action, and study its cryptographic properties.

• Our study highlights that certain group actions-based primitives cannot be instantiated securely with Lattice Isomorphism.

• We introduce two new hard problems (and prove them to be) equivalent to LIP - one of which appeared already for isogenies [8].

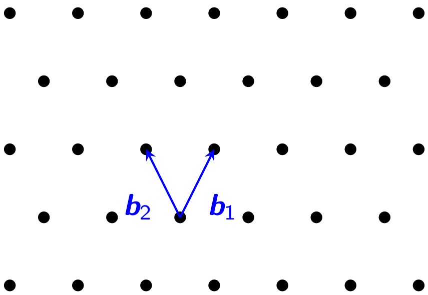
Outline

1. Introduction
2. **Preliminaries**
3. Lattice Isomorphism as a Group Action
4. Cryptographic Properties of LIGA
5. Two New Hard Problems
6. Discussion

Lattices

A lattice is the set of all **integer** linear combinations of a basis $B = \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n, \in \mathbb{R}^n$

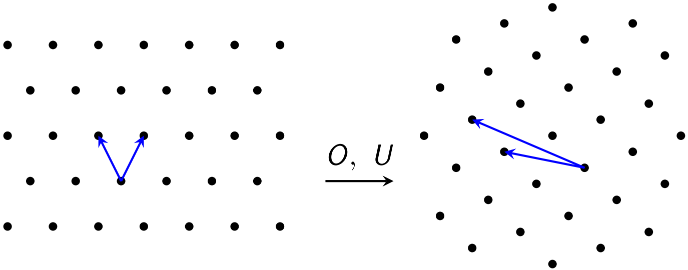
$$\mathcal{L}(B) = \left\{ \sum_{i=1}^{n} \alpha_i \boldsymbol{b}_i, \quad \alpha_i \in \mathbb{Z} \right\}.$$



Lattice Isomorphism

Two lattices $\mathcal{L}_1(B)$ and $\mathcal{L}_2(B')$ are *isomorphic* if there exist an ortonormal matrix O and an invertible integer matrix U such that

$$B' = OBU$$



Lattice Isomorphism Problem

Definition (LIP)

Given two basis B, B', find (if they exist) an orthonormal matrix $O \in \mathcal{O}_n(\mathbb{R})$ and an invertible integer matrix $U \in \mathsf{GL}_n(\mathbb{Z})$ such that

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The orthonormal matrix O has, in general, entries in $\mathbb{R}$. For this reason, in practice, one uses quadratic forms as follows

$$Q = B^\top B \in \mathcal{S}_n^{>0}$$

$$Q' = B'^\top B' = U^\top B^\top O^\top OBU = U^\top QU \in \mathcal{S}_n^{>0}$$

Lattice Isomorphism Problem Reformulated

We can reformulate LIP in terms of Quadratic Forms

Definition (LIP - Quadratic Forms)

Given two quadratic forms Q, Q', find (if it exists) an invertible integer matrix $U \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$Q' = U^\top Q U.$$

We denote with $[Q]$ the equivalence class of all quadratic forms Q' equivalent to Q.

Group Actions

Definition

Let $(G, \circ)$ be a group, and X be a set. G is said to *act* on X if there exists a map

$$\star : \; G \times X \to X$$

satisfying the following properties:

- *identity:* $id \star x = x$, for every $x \in X$ and $id \in G$ identity,
- *compatibility:* $(g \circ h) \star x = g \star (h \star x)$, $\forall g, h \in G$, $x \in X$.

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Basic properties.

- *Transitive*, $\forall x_1, x_2 \in X$, $\exists g \in G \,:\, x_2 = g \star x_1$.
- *Faithful*, $x = g \star x, \forall x \in X \Rightarrow g = id$.
- *Free*, $x = g \star x$, for some $x \in X \Rightarrow g = id$.

Cryptographic Group Action

Properties for the use of group actions in Cryptography.

- **One-wayness**: Given $x, x' \in X$ such that

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  it is hard to find g.

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- **One-wayness**: Given $x, x' \in X$ such that

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- **Weak-unpredictability**: Given a polynomial number of pairs $(x_i, g \star x_i) \in X \times X$, and given $y \in X$, it is hard to compute $g \star y$.

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- **Weak-unpredictability**: Given a polynomial number of pairs $(x_i, g \star x_i) \in X \times X$, and given $y \in X$, it is hard to compute $g \star y$.

- **Weak-pseudorandomness**: It is hard to distinguish a polynomial number of pairs $(x_i, g \star x_i) \in X \times X$, from random pairs $(x_i, y_i) \in X \times X$.

Outline

Lattice Isomorphism Group Action (LIGA)

- Define as the base set $X = [Q]$, for a chosen quadratic form Q.
- Define the group as the quotient

$$G \;=\; \mathsf{GL}_n(\mathbb{Z})/\simeq_\pm \;=:\; \mathsf{GL}_n^\pm(\mathbb{Z})$$

  where

$$A \simeq_\pm B \iff A = \pm B,$$

  and operation $A \circ B = BA$, for $A, B \in \mathsf{GL}(\mathbb{Z})$.

- Define the action $\star$: $(\mathsf{GL}_n^\pm(\mathbb{Z}) \times [Q]) \to [Q]$

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$\to \star$ is compatible and the identity element $I_n \in \mathsf{GL}_n^{\pm}(\mathbb{Z})$ fixes any element of $[Q] \Rightarrow$ it is a group action

Basic Properties of LIGA

- **Transitivity.** ✓
- **Faithfulness.** ✓
- **Free.** $\iff$ Q has trivial automorphism group

Outline

Cryptographic Properties of LIGA

- **One-wayness.** ✓
(assuming LIP hard to solve $\Rightarrow$ LIGA is *one-way*)

- **Weak-unpredictability.** ?

- **Weak-pseudorandomness.** ?

Cryptographic Properties of LIGA

Theorem (informal)

Given $d = \frac{n(n-1)}{2} \in O(n^2)$ *independent LIP samples*[1]

$$Q_i' = U^\top Q_i U, \quad i = 1, \ldots, d,$$

then one can retrieve the secret U *in polynomial time* $O(n^{2\omega})$*, where* $\omega \in [2, 3]$*.*

[1] sampled according to a certain distribution.

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- LIGA is not weakly-unpredictable
- LIGA is not weakly-pseudorandom

[1] sampled according to a certain distribution.

Cryptographic Properties of LIGA

proof.(informal)

• Given one sample $Q' = U^\top QU$, write the d-dimensional linear system of equation in d^2 variables

$$Q'_{i,j} = \sum_{k=1}^{n} \sum_{l=1}^{n} Q_{k,l} \cdot X_{(i,k),(j,l)}$$

where $X_{(i,k),(j,l)} = U_{i,k} \cdot U_{j,l}$ for each $i, j, k, l \in \{1, \ldots, n\}$.

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• Given d samples, construct a determined linear system and solve it – Gaussian elimination.

• Retrieve U from the values $X_{(i,k),(j,l)}$.

Time/Samples Trade-off using Gröbner basis

In the system $Q' = U^\top Q U$

$$Q' = \begin{bmatrix} u_{1,1} & \cdots & u_{n,1} \\ \vdots & \ddots & \vdots \\ u_{1,n} & \cdots & u_{n,n} \end{bmatrix} \cdot Q \cdot \begin{bmatrix} u_{1,1} & \cdots & u_{1,n} \\ \vdots & \ddots & \vdots \\ u_{n,1} & \cdots & u_{n,n} \end{bmatrix}$$

we consider only *norm equations*, that is, equations in n variables of the form

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Proposition (informal)

For an index of regularity $i \geq 2$ and at least $m = O(\frac{n^2}{i^2})$ LIP samples, one can retrieve the secret U in time $O(n^{2+i\omega})$, where $\omega \in [2, 3]$.

Comparison

n	16	32	64	128	256	512	1024
LIN.	22.5	28.1	33.7	39.3	44.9	50.5	56.2
GB - $i_{\text{reg}} = 2$	30.5	38.1	45.7	53.3	60.9	68.5	76.2
GB - $i_{\text{reg}} = 3$	41.7	52.1	62.5	73.0	83.4	93.8	104.2
GB - $i_{\text{reg}} = 4$	52.9	66.2	79.4	92.6	105.8	119.1	132.3
GB - $i_{\text{reg}} = 5$	64.2	80.2	96.2	112.3	128.3	144.3	160.4

Estimated bit complexity comparison - Linearization vs. Gröbner basis approaches.

Experiments

n	16	20	24	28	32	36	40
LIN.	0.34	1.00	1.98	3.36	5.51	10.57	17.31
GB	2.04	5.64	13.40	31.59	67.72	130.52	252.16

Time in seconds for breaking weak-unpredictability - both with $m = \frac{n(n-1)}{2}$ samples. In the case of Gröbner basis, we considered the case of $i_{\text{reg}} = 2$.

Outline

Two New Hard Problems

We use our result to derive the following two new hard problems.

Definition (Transpose Quadratic Form Problem (TQFP))

Given Q and $Q' = U^\top QU$, find $\tilde{Q} = UQU^\top$.

Definition (Inverse Quadratic Form Problem (IQFP))

Given Q and $Q' = U^\top QU$, find $\tilde{Q} = (U^{-1})^\top Q(U^{-1})$.

We show that with $O(n^2)$ calls to an oracle that solves TQFP (or IQFP), one can solve LIP.

Two New Hard Problems

Sketch of the reduction. (LIP $\rightarrow$ TQFP).

Given an LIP instance $(Q, Q' = U^{\top} Q U)$, we give it as input to the TQFP oracle and get $(Q, \tilde{Q} = U Q U^{\top})$

Two New Hard Problems

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Given an LIP instance $(Q, Q' = U^\top QU)$, we give it as input to the TQFP oracle and get $(Q, \tilde{Q} = UQU^\top)$

- Sample a quadratic form $\bar{Q} = W^\top QW$ along with $W \in \mathrm{GL}_n(\mathbb{Z})$.
- Compute $Q'' = W\tilde{Q}W^\top = WUQU^\top W^\top$ and send (Q, Q'') to the TQFP oracle. Record its response as

$$\hat{Q} = U^\top W^\top QWU = U^\top \bar{Q} U.$$

This is a a new LIP sample with U as unknown unimodular matrix.

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Repeating the above two steps for $O(n^2)$ times, one obtains enough samples to retrieve the secret. (Same procedure for IQFP).

Outline

Implications of our work

What can/can't you do with LIP
(what can/can't you do with one-wayness only)

ID scheme/ signature	commitment	PRF	updatable encryption
✓	✓	✗	✗

Thanks for listening!

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