Adaptive attacks against FESTA without input validation or constant-time implementation

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PQCRYPTO 2024

Result

FESTA: An isogeny-based PKE proposed by Basso, Maino, and Pope
"*FESTA: Fast Encryption from Supersingular Torsion Attacks*" (ASIACRYPT 2023)

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Proposed adaptive attacks for FESTA variants in which there are no

1. input validation
2. constant-time implementation

Background

Elliptic curve

p: a prime
k: a field of characteristic p
$E: y^2 = x^3 + ax + b \ /k$
E is an elliptic curve $/k \Leftrightarrow 4a^3 + 27b^2 \neq 0$

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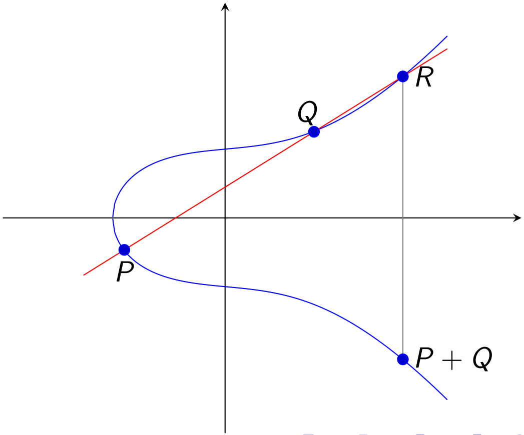
- E has a commutative group structure.

- $E[N] := \{P \in E \mid [N]P = \overbrace{P + \cdots + P}^{N} = 0\}$
  $E[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$ for N with $\gcd(N, p) = 1$.

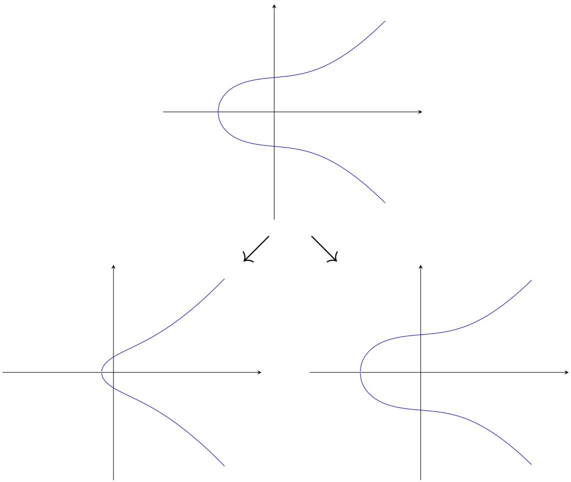


Isogeny

E_1, E_2: elliptic curves

$\phi\colon E_1 \to E_2$ is an isogeny $\Leftrightarrow$ ϕ is

- a morpihsm (rational function),
- a group morphism,
- surjective
- with a finite kernel.



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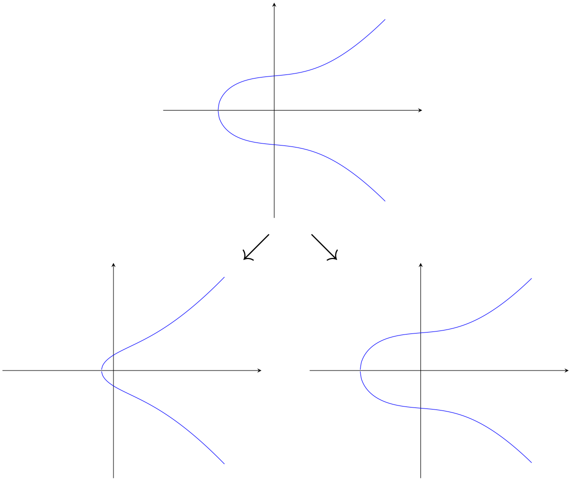
(Suppose that ϕ is separable.)

- $\deg \phi := \# \ker \phi$
- $\hat{\phi}\colon E_2 \to E_1$ is the dual isogeny of ϕ
  $\Leftrightarrow$ $\phi \circ \hat{\phi} = [\deg \phi]$ and $\hat{\phi} \circ \phi = [\deg \phi]$.



Isogeny graph

Vertices: Elliptic curves
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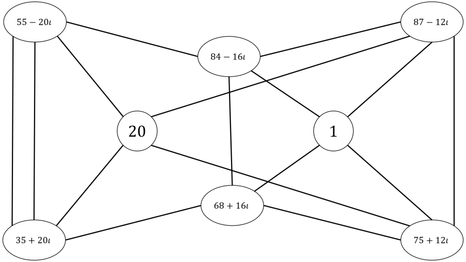

Figure: $p = 97$, isogenies of degree 3

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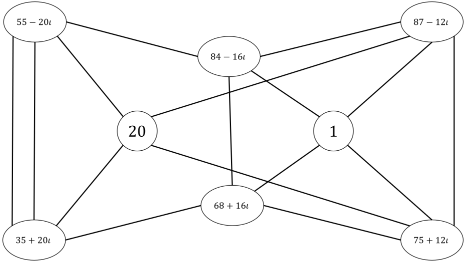

Figure: $p = 97$, isogenies of degree 3

isogeny	$\longleftrightarrow$	path
$\deg \phi$	$\longleftrightarrow$	"length" of the path
$\hat{\phi}$	$\longleftrightarrow$	the backtracking path of ϕ

Isogeny computation (random walking):

Isogeny computation and Isogeny Problem

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Isogeny Problem (path-finding):

An isogeny $\phi\colon E \to E'$ or $G := \ker \phi$

$\xleftarrow[\text{hard}]{\text{Isogeny Problem}}$

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Isogeny Problem with torsion points information (1/2)

Remind that $E[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$.
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CSSI Problem [Jao and De Feo (PQCRYPTO 2011)]

(E, P, Q) and $(E', \phi(P), \phi(Q)) \rightsquigarrow \phi$

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This problem can be solved in polynomial time. (**the SIDH attacks**)

- Castryck and Decru "*An Efficient Key Recovery Attack on SIDH*" (EUROCRYPT 2023)
- Maino, Martindale, Panny, Pope and Wesolowski "*A Direct Key Recovery Attack on SIDH*" (EUROCRYPT 2023)
- Robert "*Breaking SIDH in polynomial time*" (EUROCRYPT 2023)

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CIST Problem [Basso, Maino and Pope (ASIACRYPT 2023)]

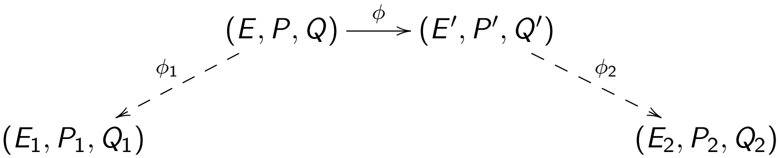
$(E, E', P, Q, P', Q', \mathcal{M}_b) \rightsquigarrow \phi$

$\mathcal{M}_b$: a commutative subgroup of $\mathrm{GL}_2(\mathbb{Z}/2^b\mathbb{Z})$

P', Q': points such that, for $\mathbf{A} \in \mathcal{M}_b$,

$$\begin{pmatrix} P' \\ Q' \end{pmatrix} = \mathbf{A} \begin{pmatrix} \phi(P) \\ \phi(Q) \end{pmatrix}$$

FESTA trapdoor function (1/3)



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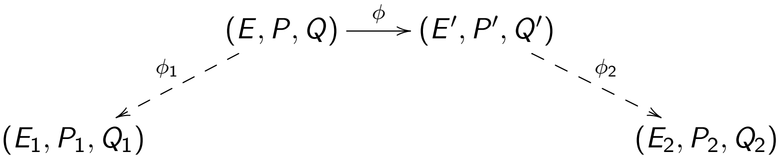

(P', Q'): masked by $\mathbf{A} \in \mathcal{M}_b$
(P_1, Q_1) and (P_2, Q_2): masked by $\mathbf{B} \in \mathcal{M}_b$

$$\begin{pmatrix} P' \\ Q' \end{pmatrix} = \mathbf{A} \begin{pmatrix} \phi(P) \\ \phi(Q) \end{pmatrix}, \quad \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} = \mathbf{B} \begin{pmatrix} \phi_1(P) \\ \phi_1(Q) \end{pmatrix}, \quad \begin{pmatrix} P_2 \\ Q_2 \end{pmatrix} = \mathbf{B} \begin{pmatrix} \phi_2(P') \\ \phi_2(Q') \end{pmatrix}.$$

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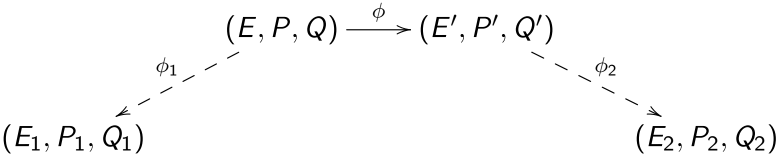

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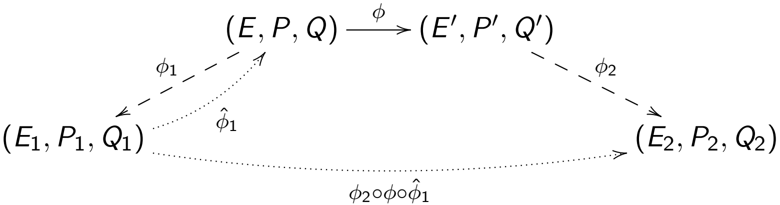

Trapdoor:

Let ${}^t(P_2', Q_2') = (\deg \phi_1) \cdot \mathbf{A}^{-1} \cdot {}^t(P_2, Q_2)$.
Since $\mathbf{AB} = \mathbf{BA}$, we have

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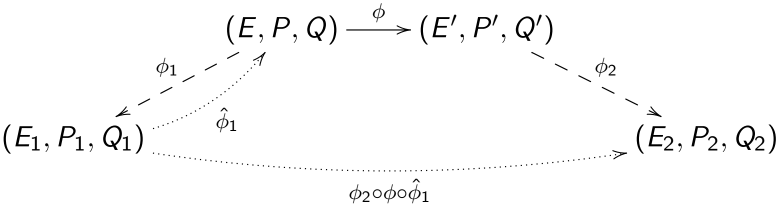

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$\longrightarrow$ One who knows $\mathbf{A}$ can obtain ϕ_1, ϕ_2 and $\mathbf{B}$.

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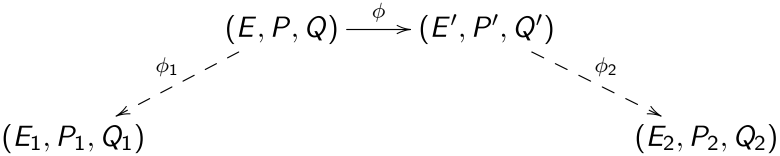

Trapdoor function

Public key: (E, P, Q, E', P', Q')
Secret key: $\mathbf{A}$
Input: $\phi_1, \phi_2, \mathbf{B}$
Output: $(E_1, P_1, Q_1, E_2, P_2, Q_2)$

Inverse map: Hard to be computed without $\mathbf{A}$

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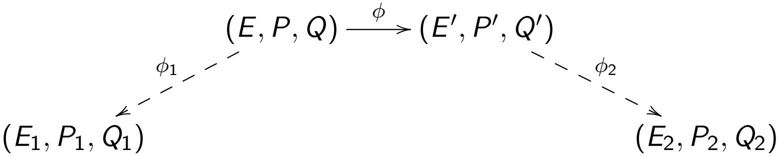

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Inverse map: Hard to be computed without $\mathbf{A}$

Input validation: If $\mathbf{B}$ does not belong to $\mathcal{M}_b$, then the recipient rejects $\phi_1, \phi_2, \mathbf{B}$.

Adaptive attack against FESTA variants

Attack scenario

Bob (sender) tries to obtain the secret key **A** of Alice (recipient).

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$\rightarrow$ Assume that Bob can distinguish the above two cases.

$$O'(\mathrm{ot}) = \begin{cases} 1 & \text{(if Alice succeeds in using the SIDH attacks)} \\ 0 & \text{(if Alice fails to use the SIDH attacks)} \end{cases}$$

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Main theorem

Theorem

Consider the CIST problem.
If there is an oracle O such that, for $\mathbf{B_1}, \mathbf{B_2} \in \mathrm{GL}_2(\mathbb{Z}/2^b\mathbb{Z}) \neq \mathcal{M}_b$,

$$O(\mathbf{B_1}, \mathbf{B_2}) = \begin{cases} 1 & (\textit{if } \mathbf{AB_1} = \mathbf{B_2A}) \\ 0 & (\textit{if } \mathbf{AB_1} \neq \mathbf{B_2A}) \end{cases},$$

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It is easy to see that

$$O(\mathbf{B_1}, \mathbf{B_2}) = O'((E_1, E_2, \mathbf{B_1} \cdot {}^t(\phi_1(P), \phi_1(Q)), \mathbf{B_2} \cdot {}^t(\phi_2(P'), \phi_2(Q')))).$$

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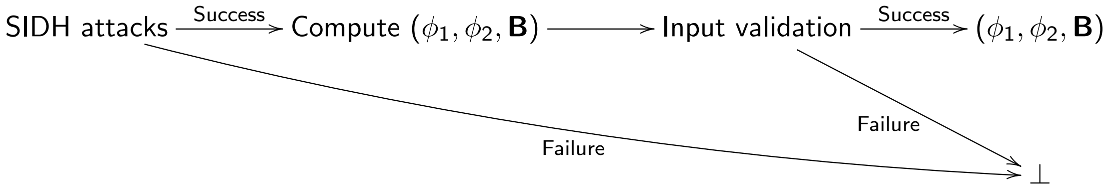
How do we construct the oracle O'?
Note: O' is NOT a decryption oracle because of the input validation.

How to construct O'



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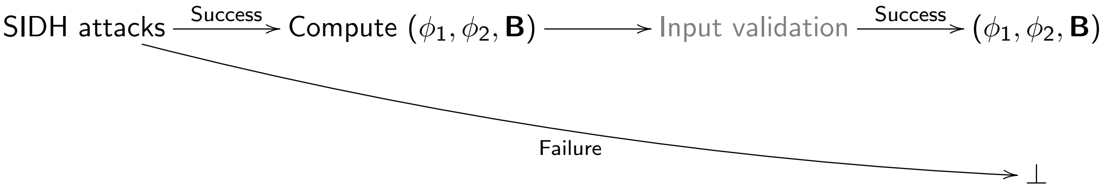

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   - Other schemes based on the CIST problem (e.g., IS-CUBE, LIT-SiGamal)

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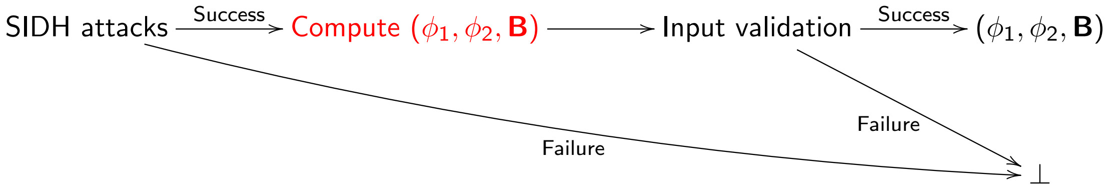

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Summary

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Thank you for listening! Any questions?

Sketch of the proof of the main theorem (1/3)

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We show the proof of this theorem in the case that $\mathcal{M}_b$ is the set of circulant matrices.

Sketch of the proof of the main theorem (2/3)

Put

$$\mathbf{A} = \begin{pmatrix} \gamma & \delta \\ \delta & \gamma \end{pmatrix}, \quad \gamma^2 - \delta^2 = 1, \quad \gamma = \sum_{i=0}^{b-1} \gamma_i 2^i, \quad \delta = \sum_{i=0}^{b-1} \delta_i 2^i.$$

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Define

$$O_{\mathrm{coeff}}(\varepsilon_1, \varepsilon_2) = O\left(\mathbf{B} + \begin{pmatrix} \varepsilon_1 & 0 \\ \varepsilon_2 & 0 \end{pmatrix}, \mathbf{B} + \begin{pmatrix} 0 & 0 \\ \varepsilon_1 & \varepsilon_2 \end{pmatrix}\right).$$

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Lemma

$$O_{coeff}(\varepsilon_1, \varepsilon_2) = \begin{cases} 1 & (\textit{if } \varepsilon_1\gamma + \varepsilon_2\delta = 0) \\ 0 & (\textit{if } \varepsilon_1\gamma + \varepsilon_2\delta \neq 0) \end{cases}$$

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- $1 - {\gamma^{(k-1)}}^2 + {\delta^{(k-1)}}^2 = \gamma^2 - \delta^2 - {\gamma^{(k-1)}}^2 + {\delta^{(k-1)}}^2 \equiv \gamma_k 2^{k+1} \pmod{2^{k+2}}$

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- $-2^{b-k-1}\delta^{(k-1)} \cdot \gamma + 2^{b-k-1}\gamma^{(k-1)} \cdot \delta = \delta_k 2^{b-1}$

  $\longrightarrow$ $O_{coeff}(-2^{b-k-1}\delta^{(k-1)}, 2^{b-k-1}\gamma^{(k-1)})$ reveals δ_k.

- $1 - \gamma^{(k-1)^2} + \delta^{(k-1)^2} = \gamma^2 - \delta^2 - \gamma^{(k-1)^2} + \delta^{(k-1)^2} \equiv \gamma_k 2^{k+1} \pmod{2^{k+2}}$

  $\longrightarrow$ $1 - \gamma^{(k-1)^2} + \delta^{(k-1)^2} \mod 2^{k+1}$ reveals γ_k.