

# Impossible Maths I

## Sheet 3 — MT18

### First topic questions

1. (a) Define what it means for a finite subset of a vector space to be *linearly independent*, a *spanning set*, and a *basis*.

State the *Steinitz Exchange Lemma*.

Prove that if a vector space  $V$  has a finite basis, then every linearly independent subset of  $V$  may be extended to a basis.

Prove that any two finite bases of a vector space have the same number of elements. Define the *dimension* of a finite-dimensional vector space.

- (b) Let  $V$  be a finite-dimensional vector space.

Suppose that  $X$  and  $Y$  are subspaces of  $V$ .

Prove that

$$\dim(X + Y) + \dim(X \cap Y) = \dim X + \dim Y.$$

- (c) Suppose  $V$  is a finite-dimensional vector space, and  $T : V \rightarrow V$  is a linear transformation. Suppose that for all  $v \in V$ , if  $T(T(v)) = 0$ , then  $T(v) = 0$ . Prove that  $V = \ker T \oplus \operatorname{im} T$ , and that the restriction,  $T|_{\operatorname{im} T}$ , of  $T$  to  $\operatorname{im} T$  is a bijection from  $\operatorname{im} T$  to itself.

[You may assume the Rank-Nullity Theorem.]

[If  $V$  and  $W$  are vector spaces,  $T : V \rightarrow W$  is a linear transformation, and  $U$  is a subspace of  $V$ , then we define the restriction  $T|_U$  of  $T$  to  $U$  to be a linear transformation from  $U$  to  $W$  such that for all  $u \in U$ ,  $(T|_U)(u) = T(u)$ .]

**Solution:** The solution would go here

2. (a) Define an *elementary row operation*, and say what it means for a matrix to be in *row-reduced echelon form*. Describe how any matrix may be reduced to row-reduced echelon form.
- (b) For which values of  $\lambda$  is the following system of equations solvable? In each case calculate the solutions.

$$\begin{array}{rrcrcl} x & + & 2y & - & 3z & = & 5 \\ x & - & 2y & - & 5z & = & 7 \\ 2x & + & 8y & + & (-\lambda - 6)z & = & 8 \\ x & - & 2y & + & (\lambda - 3)z & = & \lambda^2 + 3 \end{array}$$

**Solution:** The solution would go here

3. (a) (i) Define what it means to say that a square matrix with real entries is *diagonalisable* over  $\mathbb{R}$ .
- (ii) Show that if  $A$  is a square matrix with real entries and  $\mathbf{u}$  and  $\mathbf{v}$  are eigenvalues of  $A$  corresponding to different eigenvalues, then  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent.
- (b) (i) Consider the  $2 \times 2$  real square matrix

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

Show that  $A$  has at least one real eigenvalue; and that if  $a \neq d$  or  $b \neq 0$ , then it has two distinct real eigenvalues.

Deduce that  $A$  is diagonalisable.

- (ii) Determine the values of  $\alpha$  and  $\beta$  for which the real matrix

$$A = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}$$

is diagonalisable.

- (iii) Determine when the  $2 \times 2$  real matrix

$$A = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

is diagonalisable.

**Solution:** The solution would go here

4. Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be vectors in  $\mathbb{R}^3$ .

- (a) (i) Prove that  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  are linearly independent if and only if  $\mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}) \neq 0$ .  
 (ii) Establish the identity

$$\mathbf{u} \wedge (\mathbf{w} \wedge \mathbf{v}) = (\mathbf{u} \cdot \mathbf{v})\mathbf{w} - (\mathbf{u} \cdot \mathbf{w})\mathbf{v}.$$

- (b) Prove that  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  are linearly independent if and only if  $\mathbf{v} \wedge \mathbf{w}$ ,  $\mathbf{w} \wedge \mathbf{u}$ , and  $\mathbf{u} \wedge \mathbf{v}$  are linearly independent.  
 (c) Suppose that  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  are linearly independent, and that

$$\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

Find coefficients  $\alpha$ ,  $\beta$  and  $\gamma$  such that

$$\mathbf{r} = \alpha \mathbf{v} \wedge \mathbf{w} + \beta \mathbf{w} \wedge \mathbf{u} + \gamma \mathbf{u} \wedge \mathbf{v}.$$

**Solution:** The solution would go here