Impossible Maths I

Sheet 3 — MT18

First topic questions

1. (a) Define what it means for a finite subset of a vector space to be *linearly independent*, a *spanning set*, and a *basis*.

   State the *Steinitz Exchange Lemma*.

   Prove that if a vector space V has a finite basis, then every linearly independent subset of V may be extended to a basis.

   Prove that any two finite bases of a vector space have the same number of elements. Define the *dimension* of a finite-dimensional vector space.

   (b) Let V be a finite-dimensional vector space.

   Suppose that X and Y are subspaces of V.

   Prove that

$$\dim(X + Y) + \dim(X \cap Y) = \dim X + \dim Y.$$

   (c) Suppose V is a finite-dimensional vector space, and $T : V \to V$ is a linear transformation. Suppose that for all $v \in V$, if $T(T(v)) = 0$, then $T(v) = 0$. Prove that $V = \ker T \oplus \operatorname{im} T$, and that the restriction, $T{\restriction}\operatorname{im} T$, of T to $\operatorname{im} T$ is a bijection from $\operatorname{im} T$ to itself.

   [*You may assume the Rank-Nullity Theorem.*]

   [*If V and W are vector spaces, $T : V \to W$ is a linear transformation, and U is a subspace of V, then we define the* restriction $T{\restriction}U$ *of T to U to be a linear transformation from U to W such that for all $u \in U$, $(T{\restriction}U)(u) = T(u)$.*]

2. (a) Define an *elementary row operation*, and say what it means for a matrix to be in *row-reduced echelon form*. Describe how any matrix may be reduced to row-reduced echelon form.

   (b) For which values of λ is the following system of equations solvable? In each case calculate the solutions.

$$\begin{array}{rrrcl} x & +2y & -3z & = & 5 \\ x & -2y & -5z & = & 7 \\ 2x & +8y & +(-\lambda-6)z & = & 8 \\ x & -2y & +(\lambda-3)z & = & \lambda^2+3 \end{array}$$

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3. (a) (i) Define what it means to say that a square matrix with real entries is *diagonalisable* over $\mathbb{R}$.

   (ii) Show that if A is a square matrix with real entries and $\mathbf{u}$ and $\mathbf{v}$ are eigenvalues of A corresponding to different eigenvalues, then $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

   (b) (i) Consider the 2×2 real square matrix

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

   Show that A has at least one real eigenvalue; and that if $a \neq d$ or $b \neq 0$, then it has two distinct real eigenvalues.
   Deduce that A is diagonalisable.

   (ii) Determine the values of α and β for which the real matrix

$$A = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}$$

   is diagonalisable.

   (iii) Determine when the 2×2 real matrix

$$A = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

   is diagonalisable.

4. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$.

   (a) (i) Prove that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent if and only if $\mathbf{u}.(\mathbf{v} \wedge \mathbf{w}) \neq \mathbf{0}$.

   (ii) Establish the identity

$$\mathbf{u} \wedge (\mathbf{w} \wedge \mathbf{v}) = (\mathbf{u}.\mathbf{v})\mathbf{w} - (\mathbf{u}.\mathbf{w})\mathbf{v}.$$

   (b) Prove that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent if and only if $\mathbf{v} \wedge \mathbf{w}$, $\mathbf{w} \wedge \mathbf{u}$, and $\mathbf{u} \wedge \mathbf{v}$ are linearly independent.

   (c) Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, and that

$$\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

   Find coefficients α, β and γ such that

$$\mathbf{r} = \alpha\mathbf{v} \wedge \mathbf{w} + \beta\mathbf{w} \wedge \mathbf{u} + \gamma\mathbf{u} \wedge \mathbf{v}.$$

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