EXTENDED ESSAYS:
Option BEE or BOE in Part B of the Final
Honour School of Mathematics

DISSERTATIONS:
Option CCD or COD in Part C of the Final
Honour School of Mathematics

SOME IDEAS FOR PROJECTS

May 23, 2016

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1 Logic

1.1 Model Theory — Strongly minimal structures

Part C dissertation.

Strongly minimal structures and their combinatorial geometries are important notions of modern model theory. These notions are also central for understanding principles of a model-theoretic classification of classical mathematical structures. There is an extensive discussion of these in graduate textbooks.

The aims of the project are:
1. to present and prove basic facts about strongly minimal structures;
2. to present classical examples of strongly minimal structures and to discuss the existence of non-classical ones;
3. to discuss generalisations of strongly minimal structures.

1.2 Model Theory — The model theory of cyclic groups.

Part C dissertation.

The aim of this project is to understand the model-theoretic limit of a sequence of finite cyclic groups and relate it to the theory of the infinite cyclic group.

The main intended result would be the following.

**Theorem** Given a positive sentence \( \varphi \) in the standard language of Abelian groups, \( \mathbb{Z}/n\mathbb{Z} \models \varphi \) for all \( n \in \mathbb{N} \), if and only if \( \mathbb{Z} \models \varphi \).

The proof of the right-to-left implication is relatively easy and is based on one of the basic facts of model theory, the preservation of positive formulas under homomorphisms.

The converse requires much more work and reading. Textbook material on the theory of abelian groups and the paper *The elementary theory of abelian...*

1.3 Model theory of the real numbers

Part C dissertation.

According to Gödel’s theorem and its developments, the first order theory of the natural numbers is undecidable: there is no decision procedure for deciding whether statements are true or not. However, some other natural mathematical structures are decidable, including the theory of the complex numbers as an algebraically closed field of characteristic zero, and the theory of the real numbers as an ordered field. How are such results established and can they be extended to more complicated structures?


1.4 O-minimal structures

Part C dissertation.

An o-minimal structure is a model-theoretic structure $\mathcal{M}$ whose underlying domain $M$ possesses a dense linear order such that the definable subsets of $M$ are “as simple as possible” : they are just the finite unions of points and open intervals. This simple requirement has very strong consequences. For example, definable functions in such a structure are continuous (in the order topology) except at finitely many points of their domain. Further, if $M$ is a field, then definable functions are differentiable except at finitely many points in their domain. A good deal of real (and complex) analysis can be developed in this setting, where the underlying field may be very different from the real or complex numbers.

1.5 Local equivalents of the Axiom of Choice

The famous equivalence of the Axiom of Choice and the Well-Order Principle can be proved ‘locally’: a set $X$ has a choice function if and only if $X$ is well orderable. When we come to examine the equivalence of the Axiom of Choice with other assertions of Set Theory we often find that mismatches appear in the local versions. In Zorn’s Lemma, for example, if $X$ has a choice function then in every inductive partial ordering on $X$ there are maximal elements; on the other hand, the usual argument requires that there should be maximal elements in every inductive partial ordering of the power set of $X^2$ to yield that there is a choice function on $X$. Investigate such ‘gaps’ in local equivalents of the Axiom of choice.

1.6 Theories of the real numbers

The system of real numbers may be defined as a complete linearly ordered field. What this is may be defined in many ways. In particular, many different versions of the completeness axiom have been proposed and used. Collect, compare and contrast these various theories.

2 Algebra

2.1 Quantum groups and crystal basis

Part C dissertation

Quantum groups and crystal basis: Quantum groups are deformations of classical groups. Using them it is possible to get bases with very good properties for representations of reductive algebraic groups. This project will cover Hopf algebras, comodules, quantum groups, crystal basis and canonical basis. The emphasis will be on the $sl_2$ case.

Contact: Prof. Kobi Kremnitzer (kobi.kremnitzer@oriel.ox.ac.uk)
2.2 Affine algebraic groups schemes

Part C dissertation

Affine algebraic groups schemes: Affine algebraic groups schemes are central objects in algebraic geometry and in representation theory. This project aim at introducing Hopf algebras, their categories of comodules, different examples of commutative Hopf algebras (affine algebraic group schemes), their Lie algebras and descent theory.

Contact: Prof. Kobi Kremnitzer (kobi.kremnitzer@oriel.ox.ac.uk)

2.3 Homotopy type theory

Part C dissertation

Homotopy type theory: Homotopy type theory is a new foundational language for mathematics. In it basic notion from homotopy theory are taken as primitive notions. This allows for very elegant and simple presentation of homotopy theory and the theory of homotopy types. The aim of this project is to introduce the homotopy category, introduce the language of homotopy type theory, develop homotopy theory in this language and compute some homotopy types.

Contact: Prof. Kobi Kremnitzer (kobi.kremnitzer@oriel.ox.ac.uk)

2.4 Relative algebraic geometry

Part C dissertation

Relative algebraic geometry: Relative algebraic geometry is an approach to algebraic geometry using category theory. This allows to generalise algebraic geometry to many different settings. This project will cover basic notions from category theory, symmetric monoidal categories, Grothendieck topologies, algebraic geometry relative to a symmetric monoidal category and the example of usual algebraic geometry and monoid algebraic geometry which is a version of the field with one element.

Contact: Prof. Kobi Kremnitzer (kobi.kremnitzer@oriel.ox.ac.uk)
2.5 Beilinson-Bernstein localisation for $sl_2$

Part C dissertation

Beilinson-Bernstein localisation for $sl_2$: The Beilinson-Bernstein localisation theorem is one of the most important tools in the representation theory of semi-simple Lie algebras. This project will cover the basics of category theory, the category of representations of $sl_2$, the Weyl algebra, the categories of O-modules and D-modules on the projective line and the Beilinson-Bernstein localisation theorem for $sl_2$.

Contact: Prof. Kobi Kremnitzer (kobi.kremnitzer@oriel.ox.ac.uk)

2.6 Convereses of Lagrange’s Theorem

Lagrange’s Theorem states that if $G$ is a finite group and $H$ is a subgroup then $|H|$ divides $|G|$. In general, if $m$ is a divisor of $|G|$ there need not be a subgroup of order $m$. Investigate those finite groups $G$ with the property that for every divisor $m$ of their order there exists at least one subgroup of order $m$.

2.7 Alhazen’s Problem

Alhazen’s Problem asks for the point $P$ on a given spherical mirror at which a ray of light is reflected from a source at $A$ to an observer at $B$. See John D. Smith ‘The remarkable Ibn al-Haytham’, Math. Gazette, 76 (1992), 189–198 for a good account of the problem. It has been proved [Peter M. Neumann, ‘Reflections on reflection in a spherical mirror’, Amer. Math. Monthly, 105 (1998)] that there is in general no ruler-and-compass construction for $P$. Also, Michael Drexler & Martin J. Gander, in their paper ‘Circular Billiard’ in SIAM Review 1999, investigate for which configurations there are four reflection points, for which only two. Several further questions of a similar nature suggest themselves, and should be of a suitable standard for an undergraduate project.

2.8 Reduction of quadratic forms

One of the very useful theorems of algebra states that any real quadratic form $\sum a_{i,j}x_i x_j$ can be changed to the form $y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2$ by suitable
non-singular linear change of variables, and that moreover the numbers $p, q$ are independent of the particular method used (‘Sylvester’s Law of Inertia’). This theorem depends heavily on the fact that the coefficients come from $\mathbb{R}$. How far can analogous normal form theorems be proved for quadratic forms over $\mathbb{C}$, over $\mathbb{Q}$, or over other fields $F$, such as finite fields?

2.9 Differential Galois Theory

Part C dissertation

Just as Galois theory involves extensions of algebraic fields and the resulting study of finite groups, the differential Galois theory [1,2] involves extensions of differential fields (differential fields are fields equipped with a derivation) and the study of resulting Lie groups. See [3] for a brief summary, or [4] for a longer summary. While the theory for linear differential equations is rather well developed, exploration of these tools in nonlinear [5] and infinite-dimensional [6] cases continues. The nonlinear case relates to Liouvillian integrability of nonlinear differential equations (see Theorem 4.9 of [7]), and this would be an interesting direction to take. This would be a nice project for a pure mathematics student looking to broaden their horizons and use their algebraic knowledge differently than they might be used to.

Prerequisites: B3.1: Galois Theory and knowledge of elementary differential equations and complex analysis. Depending on the direction taken, knowledge of C3.5 Lie Groups could prove useful.

References:


Contact: Dr Robert A Van Gorder (Robert.VanGorder@maths.ox.ac.uk)

3 Geometry and Number Theory

3.1 Simple singularities and the McKay correspondence

Part B extended essay or Part C dissertation

Finite subgroups $G$ of the group $SU(2)$ can be classified into types governed by one of the most ubiquitous patterns in mathematics: the ADE pattern. Geometrically, one can consider the quotient of the complex plane $\mathbb{C}^2$ by the group $G$, obtaining what is known as a simple singularity. Algebraically, a study of the representation theory of the finite group $G$ leads to an oriented graph called the McKay quiver. There are fascinating relationships between the geometry of the quotient, as well as its resolution of singularities, on the one hand, and the McKay quiver on the other. A deeper study also brings in another object classified by ADE patterns, the corresponding simple Lie algebra; one can also look at the situation in metric geometry, leading to a highly symmetric metric on these geometries called a hyperkahler metric. There are many avenues this project can take, looking at the big picture or concentrating on one particular corner, and on some of a large variety of possible approaches into this rich field of ideas.

Ito-Nakamura: *Hilbert schemes and simple singularities*, downloadable from http://homepages.warwick.ac.uk/staff/Miles.Reid/McKay/

Miles Reid, *La correspondance de McKay*. http://xxx.lanl.gov/abs/math/9911165

Contact: Prof. Balazs Szendroi (szendroi@maths.ox.ac.uk)

3.2 Recognizing the unknot

Given a diagram of a knot, how do we decide whether it is the unknot? This is a surprisingly difficult question to answer, although there are now several methods for doing so. The first to solve the problem was Wolfgang Haken
[2], using his theory of normal surfaces. In your dissertation, you might explain this theory [5], and then perhaps go on to explore its consequences and extensions, for example the upper bound on Reidemeister moves by Joel Hass and Jeffrey Lagarias [3], or the existence of an NP algorithm to solve the problem [4]. Another more recent solution to the problem was given by Ivan Dynnikov [1], who used arc presentations of knots. You might choose to focus on this method instead.

References:


Contact: Prof. Marc Lackenby (lackenby@maths.ox.ac.uk)

### 3.3 Geometrisation of 3-manifolds

In the late 1970s, Bill Thurston revolutionised the study of 3-manifolds with the introduction of his geometrisation conjecture [3]. Roughly speaking, this proposed that any compact orientable 3-manifold has a “canonical decomposition into geometric pieces”. This conjecture was a far-reaching generalisation of the Poincare conjecture, which asserted that any 3-manifold homotopy equivalent to the 3-sphere is homeomorphic to the 3-sphere. The geometrisation conjecture, and hence the Poincare conjecture, were proved by Perelman in 2003 [1]. Your dissertation should give a precise explanation of the statement (but not the proof) of the geometrisation conjecture. The canonical decomposition along spheres and tori should be discussed [2], as should the eight model geometries, particularly hyperbolic geometry which is at the heart of the conjecture. Your dissertation might then go on to describe some applications of the conjecture.
3.4 Crystallographic groups

A crystallographic group is a discrete, cocompact group of Euclidean isometries. These groups are of interest to chemists as well as mathematicians, because the symmetry group of any crystalline arrangement of atoms is a crystallographic group. In dimension 2, crystallographic groups are known as “wallpaper” groups, and it is a famous theorem that there are 17 of them (up to a suitable notion of equivalence). A fairly simple proof of this was given by John Conway and Bill Thurston, using the theory of orbifolds. Your dissertation might explain their proof [2, 3], and then go on to examine more advanced material. This might include Bieberbach’s theorem, which states that any crystallographic group has a finite index subgroup that is an abelian group of translations [1, 3]. You might explain why, in each dimension, there are only finitely many crystallographic groups, and you might investigate how the number of crystallographic groups grows as the dimension increases [1]. You could also consider non-Euclidean geometries, such as spherical or hyperbolic geometry.

3.5 Gromov’s theorem and groups of polynomial growth

The growth function $f(n)$ of a finitely generated group $G$ is defined to be the number of vertices in a ball of radius $n$ of a Cayley graph of $G$. Gromov in a seminal paper showed that if this growth function is bounded by a polynomial then $G$ has a finite index nilpotent subgroup. Gromov’s original proof relies on the solution of Hilbert’s 5th problem.

Recently Kleiner gave a different proof of Gromov’s theorem using analysis on groups and Shalom-Tao used this to strengthen Gromov’s theorem giving an effective version whereby it is enough to have a bound for $f(n)$ for a single sufficiently large $n$. The project should introduce the subject of group growth and give an exposition of a proof and remaining open questions related to this.


3.6 Asymptotic Approaches for the Riemann zeta Function

Part C dissertation.

The asymptotic study of the Riemann zeta function has been of historical [1] and continuing [2] interest. There have been many approaches developed for the asymptotic analysis of differential equations which permit one to recover properties of a solution, even if no exact solution representation is known. Therefore, starting with an infinite-order differential equation for the Riemann zeta function [3], one may wonder if it is possible to recover some
nice asymptotic results obtained previously from the various series or product representations for the Riemann zeta function. One could also explore using such approaches to study properties of more general zeta functions (e.g., the Lerch generalization).

Prerequisites: C3.8: Analytic Number Theory (and prerequisites thereof). Any experience with differential equations will be a plus.

References:


Contact: Dr Robert A Van Gorder (Robert.VanGorder@maths.ox.ac.uk)

3.7 Arithmetic progressions of length 3

What is $r_3(N)$, the largest subset of $\{1, \ldots, N\}$ not containing 3 distinct elements in arithmetic progression? The answer to this question is unknown. The aim of this essay is to explore some results that have been obtained on this, and related, questions. Topics might include

- Roth’s proof that $r_3(N)/N \to 0$;
- The Behrend Construction, showing that if $\epsilon > 0$ then $r_3(N) > N^{1-\epsilon}$, provided $N$ is sufficiently large;
- The work of Croot–Lev–Pach and Ellenberg–Gijswijt, obtaining very strong upper bounds for the finite field version of the problem.

The third of these items is extremely recent (May 2016), whilst the first is quite classical (1953). A google search will easily reveal several appropriate references.

Contact: Prof. Ben Green (ben.green@maths.ox.ac.uk)
3.8 Freiman’s theorem

If $A \subset \mathbb{Z}$ is a finite set then we define its sumset $A + A$ to be the set of all pairs \{$a_1 + a_2 : a_1, a_2 \in A$\}. If $A + A$ is not much larger in size than $A$, what can be said about the structure of $A$? Freiman’s theorem is an answer to this question: it states that $A$ is efficiently contained in a “grid”. The aim of this essay is to explore the statement and proof of the theorem, which involves tools from combinatorics, discrete geometry and (quite elementary) Fourier analysis. The essay might then go on to explore some other aspect of the theory, for example finite field analogues, quantitative issues, or other types of structural result in additive number theory.

References: Freiman’s theorem is quite widely discussed on the internet. A decent published source for the result is the book “Additive number theory: Inverse Problems and the geometry of sumsets” by M. Nathanson, Springer GTM 165.

Contact: Prof. Ben Green (ben.green@maths.ox.ac.uk)

4 Analysis

4.1 Abelian locally compact groups

Part C dissertation.

Abstract: The Fourier transform defined on the Hilbert space of square-integrable functions on the real-line can be readily extended to all locally compact abelian groups by the introduction of the dual group. An essay could look at what locally compact groups are, what the Haar measure is, how to define the Fourier transform, and what Pontryagin duality is. Ideas from functional analysis, group theory and measure (integration) theory are all involved.

4.2 Kazhdan’s property (T)

Part C dissertation.
Kazhdan’s property (T) was introduced in a short paper, defined in terms of the Fell topology on the space of representations of a group. However, we can state the property, informally, in an easy way: a group has property (T) if whenever, given a (unitary) representation, there is a unit vector which is only perturbed slightly by the group action, then there is actually a unit vector fixed by the group action. Such groups hence have very “rigid” actions. Recently property (T) has found its way into a diverse collection of applications, from number theory, measure theory, and operator algebras, to graph theory and computer science. An essay could either explore some of these applications: for example, the applications to graph theory and computer science can be stated using a minimal amount of representation theory. Alternatively, one could look at the theory of representing topological groups, and how classical representation theory can be used to show that some concrete groups have property (T). A longer essay could touch upon both.

4.3 Analysis in a rational world

The basic concepts of real analysis make perfectly good sense for functions $f : \mathbb{Q} \to \mathbb{Q}$ and in a world where only rational numbers are contemplated. Thus, for example, continuity and differentiability are defined in exactly the same way as for functions $f : \mathbb{R} \to \mathbb{R}$ (but note that $f'(x)$ has to be rational for all $x \in \mathbb{Q}$). In this rational world we find that basic theorems such as the Intermediate Value Theorem, Rolle’s Theorem, and the Mean Value Theorem fail. How badly do they fail? What can be rescued? What can one say about solutions to differential equations such as $f' = f$, or $f'' = f$?

4.4 Analysis of holomorphic functions with special values

We denote by $\mathbb{Q}[i]$ the set of complex numbers whose real and imaginary parts are rational. Investigate holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ with the property that $z \in \mathbb{Q}[i] \Rightarrow f(z) \in \mathbb{Q}[i]$.

4.5 Integration

Compare and contrast Lebesgue integration and Riemann-Stieltjes integration. [Caution: the material for such a project should cite but not repeat
4.6 Measure Theory

The development of abstract measure theory yields rich rewards with a variety of generalisations and applications, for example, fractals, stochastic integration. References include: Robert Strichartz *The Way of Analysis* (Jones and Bartlett Publishers, 2000); H. L. Royden *Real Analysis* (MacMillan, 1963); Kenneth Falconer *Fractal Geometry* (Wiley 1990). [Caution: the material for such a project may cite but should not repeat that given in the third-year lecture course ‘Martingales Through Measure Theory’].

4.7 Univalent functions

A univalent map is a one-to-one conformal map $z = f(\zeta)$ from the unit disc to a domain $\Omega$ in the complex plane. It is usual to normalize $f$ by assuming that $f(0) = 0$ and $f'(0) = 1$. Among many interesting results in the field are the Koebe $\frac{1}{4}$-Theorem (the distance from 0 to the boundary of $\Omega$ is never less than $\frac{1}{4}$, with equality for $f(\zeta) = \zeta/(1 - \zeta)^2$) and the celebrated Bieberbach conjecture (proved in 1985 by De Branges) that if $f$ is univalent and has the Taylor series $f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n$, then $|a_n| \leq n$, with equality for the same map. Proofs of the conjecture for small values of $n$ are not hard.

4.8 Distributions

Investigate the theory of distributions and their applications. Probably the most immediate starting point is the delta function (cf point masses, sources, or charges) and its relation to the derivative of a function with a jump discontinuity, but a proper theoretical development is not hard to set out. Further topics include the relation with Fourier transforms.

4.9 Special functions

There are many possible project topics here, for example: Bessel functions and their applications, hypergeometric functions (and the connection with complex differential equations and conformal maps), the Riemann zeta function, the Gamma function.
4.10 Riemann surfaces

Explore the idea of the Riemann surface associated with a multi-valued conformal map.

4.11 The Schwarzian derivative

The Schwarzian derivative has many interesting properties. Find out about them.

4.12 Chaos in nonlinear ordinary differential equations

Explore the connection between the various kinds of homoclinic bifurcations and the onset of chaos in ordinary differential equations.

Investigate the occurrence of stochasticity in Hamiltonian systems, as an integrable system is perturbed more and more strongly.

The period doubling sequence for unimodal (one-humped) maps is well known. What happens for other maps, e.g. cubics?

4.13 Algebro-Geometric Approaches for Integrable Systems

Part C dissertation

Algebro-Geometric tools [1] have been employed to obtain solutions to a variety of integrable systems; see [2,3,4,5]. In this project, the student would review some of these methods, and attempt to apply them to an integrable system. One logical application would be to 3x3 systems such as that arising in the nonlinear three-wave interaction [6] or its generalizations [7], as this would effectively extend the work in [2]. This would be a nice project for a pure mathematics student looking to broaden their horizons and apply what they know to integrable systems theory.

Prerequisites: C3.4: Algebraic Geometry (as a co-requisite), otherwise B3.3: Algebraic Curves and some readings. C3.7: Elliptic Curves might be helpful, depending on the direction the project takes. Knowledge of partial differential equations (from either an analysis or application point of view) will be useful for intuition.
References:


Contact: Dr Robert A Van Gorder (Robert.VanGorder@maths.ox.ac.uk)

5 Mathematical Methods and Applications

5.1 Mathematics and the environment

The analysis of low dimensional Plankton models

Part B extended essay or Part C dissertation

The increasing exploitation of marine resources has driven a demand for complex biogeochemical models of the oceans and the life they contain. The
current models are constructed from the ‘bottom up’, considering the bio-
chemistry of individual species or functional types, allowing them to interact
according to their position in the food web, and embedding the ecological sys-

tem in a physical model of ocean dynamics. The resulting ecology simulation
models typically have no conservation laws and the ecology often produces
‘emergent properties’, that is, surprising behaviours for which there is no
obvious explanation. Because realistic models have too many experimentally
poorly defined parameters (often in excess of 100), there is a need to analyse
simpler models. A recent approach by Cropp and Norbury (2007) involves
the construction of complex ecosystem models by imposing conservation of
mass with explicit resource limitation at all trophic levels (i.e. positions
occupied in a food chain). The project aims to analyse models containing
two ‘predators’ and two ‘prey’ with Michaelis-Menten kinematics. A system-
atic approach to elicit the bifurcation structure and routes to chaos using
parameter values, appropriate to different ocean areas would be adopted.
In particular the influence of nonlinearity in the functional (life) forms on
the stability properties of the system and the bifurcation properties of the
model will be comprehensively numerically enumerated and mathematically
analysed.

REFS:
(2014).

Contact: Prof. Irene Moroz (moroz@maths.ox.ac.uk)

Part C dissertations
Surging glaciers.
Observations and theories concerning glacier surges.

Waves on rivers
Formation of waves on rivers (roll waves, tidal bores).

Formation of aeolian and fluvial bedforms such as dunes.

5.2 Mathematical biology and physiology

There are many topics that could be explored in relation to the Part B and
Part C courses in the field mathematical biology and ecology. In each case,
the aim will be to explore a particular biological phenomenon using mathematical and computational techniques. The models used could consist of, for example, ordinary and partial differential equation, stochastic differential equation and individual-based approaches. Exploration of the models will be carried out using a suitable combination of analytical and computational methods, with the aim being the generation of experimentally testable predictions. Some possible areas for investigation include (but are not limited to):

- physiological/biological fluid mechanics;
- tissue engineering;
- the mechanics of growth processes;
- pattern formation;
- cell movement, signalling and interaction;
- multiscale modelling.

See also project 6.7 ‘Numerical simulation of multi-species interaction within porous tissues’ in the Numerical Analysis section below.

References


5.3 Fluid dynamics

There are many topics that could be done in parallel with the course for Paper B6; consult the lecturer for possibilities.

Describe some of the phenomena which occur in rotating flows, and their application in meteorology.

Linear and nonlinear stability theory of Rayleigh–Bénard convection.
Transition to turbulence in shear flows.

5.4 Mechanics of Solids

Write on the linear theory of elastic solids, including for instance some of the following topics: the stress–strain relation in an anisotropic crystalline material and its relation to the symmetries of the material; Saint-Venant torsion of a prismatic beam of isotropic material; wave motion in an isotropic elastic material, including waves in thin rods and thin plates.

5.5 Nonlinear Waves

Part C Dissertation

Nonlinear waves appear as solutions to nonlinear PDE models used throughout science and engineering, and it would be easy to find particular applications relevant to the interested student. Possible directions could be nonlinear waves in discrete dynamics (with applications including, e.g., granular media [1], optics or atomic physics [2]), continuous dynamics (e.g., nonlinear waves along vortex filaments [3], Bose-Einstein condensates [4]), or even network dynamics [5]. Particular specifications of the application and model studied, as well as the degree to which the project is theory or application driven, can be made based upon student interests.

Prerequisites: Some experience with solving PDEs: asymptotic, perturbation, numerical approaches. Particular courses like B5.6: Nonlinear Systems, B5.2: Applied Partial Differential Equations, B5.4: Waves and Compressible Flow, C4.3: Functional Analytic Methods for PDEs, C4.4: Hyperbolic Equations, and C5.5: Perturbation Methods are useful, but no one course is essential.


6 Numerical Analysis

There are many topics that could be addressed as a follow-up to the Part A Numerical Analysis course or in parallel with the third-year courses in this area. Here is a small sample of possible projects; the lecturer may be able to suggest some others.

6.1 Stiff ordinary differential equations

Part B extended essay

Consider an initial value problem for the differential equation $\epsilon y' = f(x, y)$, or a system of differential equations of the form $\epsilon y' = f(x, y)$, where $0 < \epsilon \ll 1$ is a small parameter. An interesting and practically relevant question concerns the construction and analysis of numerical methods for the accurate solution of such problems. How do standard one-step and linear multi-step methods behave when $\epsilon$ is very small? How would you improve the performance of these methods by adapting the computational mesh? Is it possible to design special methods which provide accurate approximations for such stiff initial value problems?


6.2 Numerical approximation of singular integrals

Part B extended essay

You are probably familiar with simple numerical integration rules such as the trapezium rule or Simpson’s rule; but how would you evaluate numerically the integral \( \int_0^1 \frac{e^x}{x^{1/2}} \, dx \) or \( \int_0^\infty \frac{\sin x^2}{x^{1/5}} \, dx \)? There are special techniques for the numerical approximation of such integrals. What are they and what can be said about their accuracy?


6.3 Newton’s method for nonlinear systems

Part B extended essay

Newton’s method is a standard technique for solving a nonlinear equation of the form \( f(x) = 0 \) where \( f \) is a continuously differentiable function. Consider the generalisation of Newton’s method for solving the nonlinear system \( f(x, y) = 0, \ g(x, y) = 0 \). What can be said about the convergence of Newton’s method? What is a good choice of starting value and how to locate it? How does the speed of convergence of Newton’s method compare with that of a simple fixed-point iteration?


6.4 Finite element methods for singularly perturbed problems

Part C dissertation

Conventional finite element methods are known to provide poor approximations when applied to singularly perturbed two-point boundary value problems of the form $-\epsilon u'' + u' = 0, u(0) = 0, u(1) = 1$: while the analytical solution is a smooth function, its numerical approximation exhibits unacceptable oscillations. Why does this happen? Can you improve matters by using a suitable non-uniform mesh? How should such a mesh be designed? Would generalising the concept of Galerkin finite element method by allowing a trial space that is different from the test space cure the problem?


6.5 A posteriori error analysis of finite element methods

Part C dissertation
Conventional \textit{a priori} error estimates for finite element approximations of boundary value problems are of limited practical use since they bound the computational error in terms of powers of the mesh size and norms of derivatives of the \textit{unknown} analytical solution. How would you derive an \textit{a posteriori} bound on the error by exploiting a computed solution? Now suppose that you have estimated the size of the error by means of an \textit{a posteriori} bound; how would you adapt the computational mesh to ensure that the error in a given norm does not exceed a given tolerance?


### 6.6 Fast iterative methods for systems of linear equations

\textbf{Part C dissertation}

Gaussian elimination is a standard technique for solving a system of linear equations of the form $Ax = b$ where $A$ is a non-singular matrix. Suppose that $A$ is a sparse matrix (i.e. it contains a very large number of zero entries). Is it a good idea to solve such a system by Gaussian elimination, or would it perhaps be better to use an iterative method instead? Consider the performance of various iterative methods (such as the Conjugate Gradient Method and its relatives) for the solution of sparse systems. How would you accelerate the convergence of an iterative method by \textit{preconditioning} $A$, i.e. by pre-multiplying the system by a non-singular matrix $P$ such that $P \approx A^{-1}$, and solving $PAx = Pb$ instead? How would you choose $P$?


6.7 Numerical simulation of multi-species interaction within porous tissues

Part C dissertation

Reaction-diffusion systems can explain many phenomena taking place in diverse disciplines such as industrial and environmental processes, biomedical applications, population dynamics, etc. These models allow to reproduce chaos, spatio-temporal patterns, rhythmic and oscillatory scenarios, and so on. Nevertheless, in most of the applications mentioned above, the reactions do not occur in complete isolation. The species are rather immersed in a fluid, or they move within (and interact with) a fluid-solid continuum.

The task consists in employing a primal-mixed finite element method to actually solve the system, extending the existing results to some of the cases listed below. By primal-mixed we mean that, at both continuous and discrete levels, the elasticity equations are set in a mixed form (that is, the associated formulation possesses a saddle-point structure involving additional unknowns, as for instance, the strain, or the rotations), whereas the formulation of the reaction-diffusion system is written exclusively in terms of the species concentrations. Such a structure of the governing equations is motivated by the need of recovering strains without postprocessing them from a (typically low-order) discrete displacement (which usually leads to insufficiently reliable approximations).

Examples: Electromechanics of perfused living tissues, biofilm characterization, limb morphogenesis.

References.


Contact: Dr Ricardo Ruiz Baier (ricardo.ruizbaier@maths.ox.ac.uk)
6.8 High order finite volume element methods

Part C dissertation

Despite numerous advances in handling the complexity of flow equations, most numerical techniques used by practitioners still lack essential features to reliably couple flow and transport processes. In addition to efficiency, it is crucial that the schemes are accurate and robust under various ranges of model parameters and geometry configurations. Moreover, discrete mass conservation is key to avoid artificial sinks or sources. Unfortunately, up to date there is no ultimate numerical tool able to resolve all these issues at once: Some methods are easy to implement and can be readily parallelized while others are more suitable for unstructured meshes and complicated geometries, or mass conservative by construction, or allow the natural derivation of error estimates, etc. Consequently, to resolve multiphysics problems, one must resort to schemes that combine, at least some of, these properties.

This project deals with the analysis of finite finite volume element (FVE) methods, which feature local mass conservativity, flexibility for choosing accurate numerical fluxes, and suitability for error analysis. These schemes exploit intrinsic joint properties of finite element (FE) and finite volume (FV)-based formulations. The envisaged schemes will show robustness and applicability in a wide range of problems, whilst being suitable for establishing well-posedness and deriving error estimates.

References.


Contact: Dr Ricardo Ruiz Baier (ricardo.ruizbaier@maths.ox.ac.uk)
7 Mathematical Physics

7.1 Extremum principles in theoretical physics

For example: (a) Fermat’s principle; (b) Hamilton’s principle; ...; (f) variational methods in quantum mechanics; ...; (r) geodesics in general relativity for a (given) Schwarzschild metric; ...; (z) path integral formulations of quantum mechanics [rather advanced].

7.2 Hamilton–Jacobi theory

Action-angle variables and the early development of quantum theory.

7.3 Concepts of quantum mechanics

For example: Schrödinger’s cat; Bell’s inequality. Another possibility is a critical summary of the Bohr–Einstein dialogues.

7.4 Symmetries in quantum mechanics

Various groups which occur naturally in quantum mechanics, particle-field systems can be studied in a simple manner.

7.5 Quantum mechanics of scattering

There are many calculational examples which are within reach of the advanced undergraduate. Examples: the Born approximation; simple atoms and molecules (in electric and/or magnetic fields); one- or two-electron systems.

7.6 Waveguides

The undergraduate can start with the excellent exposition on waveguides in the Feynman Lectures, and go a bit further from there.
7.7 The microwave background

A systematic list of the consequences of the microwave background would make a worthwhile and approachable project.

7.8 Tests of general relativity

For example, simple calculations involved in gravitational wave experiments; gravitational lensing effect (quite a lot of interesting geometry there).

7.9 Hot big bang versus steady state in cosmology

This is a topic that can be expanded in various interesting ways, expository and historical.

7.10 Appearance of moving objects in special relativity

In spite of Lorentz contraction, a spherical object does look spherical to an observer. Why?

7.11 Study of a special metric

Even apart from the Schwarzschild metric, general relativity is full of special metrics (“exact solutions”) which repay even simplistic studies. In particular, the NUT metrics have intriguing topological properties.

7.12 SO(3), SU(2), Euler angles and angular momentum

Work out the exact 2-to-1 map of SU(2) to SO(3); generalize Euler angles to higher dimensions; angular momentum and spin.
7.13 Momentum space in quantum mechanics

States as functions of momentum. Fourier transform. Plancherel theorem. Time evolution.

7.14 Translation of some well known theorem in euclidean space to Minkowski space

Many interesting problems relating to Euclidean and hyperbolic geometry can be tackled.

7.15 Quantum Turbulence and Chaos

Part C Dissertation

When modeling turbulent dynamics of quantized vortex filaments in superfluid Helium, one encounters a variety of apparently chaotic behaviors (see the review article [1]). These dynamics were modeled as stochastic nonlinear Kelvin waves by [2,3,4] and the results were related back to superfluid turbulence. When simulating stochastic or even deterministic dynamics for these problems, one must also consider that local models (such as the quantum form of the LIA which includes mutual friction and normal fluid effects [5]) will work well enough only sometimes, while in other regimes models involving non-local Biot-Savart dynamics may be needed. This was shown recently to be true for tightly coiled Kelvin waves under parametric amplification leading to turbulence [6]. It remains to be seen whether one can generate deterministic chaos from these types of models, as many of the deterministic dynamics end up being fairly regular [7]. The interested student will pick a problem within this area.

Prerequisites: Some experience with solving PDEs: asymptotic, perturbation, numerical approaches. Particular courses like B5.6: Nonlinear Systems, B5.2: Applied Partial Differential Equations, B5.4: Waves and Compressible Flow, C4.3: Functional Analytic Methods for PDEs, C4.4: Hyperbolic Equations, and C5.5: Perturbation Methods are useful, but no one course is essential.

References:


Contact: Dr Robert A Van Gorder (Robert.VanGorder@maths.ox.ac.uk)

**Relativistic Superfluidity and Vorticity**

**Part C Dissertation**

Quantized vortex filament dynamics (including reconnection events) in superfluids are often studied under the local induction approximation (LIA) [1], which can be put into correspondence with cubic NLS (see [2] for classical case, [3] for the quantum case). In the relativistic regime, superfluidity and vorticity can be studied under the framework of a nonlinear Klein-Gordon model in curved spacetime whose phase dynamics give rise to superfluidity [4]. The interested student will explore this latter situation.

Prerequisites: Some experience with PDEs: asymptotic, perturbation, numerical approaches. Particular courses like B5.6: Nonlinear Systems, B5.2: Applied Partial Differential Equations, B5.4: Waves and Compressible Flow,
C4.3: Functional Analytic Methods for PDEs, C4.4: Hyperbolic Equations, and C5.5: Perturbation Methods are useful, but no one course is essential.

References:


Contact: Dr Robert A Van Gorder (Robert.VanGorder@maths.ox.ac.uk)

8 Stochastics, Discrete Mathematics and Information

8.1 Combinatorics — Graphs of large chromatic number

BE Extended Essay.

It is easy to see that the chromatic number of a graph is at least as large as its clique number (the number of vertices in a largest complete subgraph). A graph is perfect if its chromatic number equals its clique number, and the same holds for all its induced subgraphs. An essay on this topic could investigate the theory of perfect graphs, including the theorem of Lovász that a graph is perfect if and only if its complement is perfect. An essay should certainly discuss the Strong Perfect Graph Theorem of Chudnovsky, Robertson, Seymour and Thomas, which gives a structural characterization of perfect graphs, although there would not be space to include a proof.
Alternatively, the essay might look at what subgraphs must appear in graphs with large chromatic number. Relevant topics would include graphs with large girth and large chromatic number, and $\chi$-bounded classes and the Gyárfás-Sumner Conjecture.

8.2 Mathematical models in finance

Possible topics might include a rigorous discussion of Itô’s lemma and its relation to random walks; the rôle of martingales in models of markets; stochastic volatility; time series methods. The lecturer may be able to suggest others.

8.3 Mathematical models in evolution

Modern methods for understanding genetic data, and using this to learn about the processes of evolution, rely heavily on mathematical models. These models usually involve probability, to capture the various sources of randomness in genetics processes. Study these models and their uses to untangle evolutionary questions, such as how different species are related, or what we can learn about early human evolution from genetics data.

8.4 Duality and Random Walks

Duality, the relation between a set of paths and the reversed set is a classical tool in the study of random walks. One project would be to review some problems in which this technique has been successfully applied, notably in the queuing literature.

For the very able student there is the possibility of novel work on the study and classification of dual times for walks in $\mathbb{R}^n$ or in the understanding of duality relations that have arisen from recent work on stopping rules for Markov chains.

8.5 The Coupling Method

Loosely, coupling refers to the study of one or two marginal probability distributions by way of the construction of an appropriate joint distribution. The success of this method lies in its ability to convert difficult analytic
problems into ones of probabilistic construction. There are many areas in applied probability in which to study how this most elegant method is used, including Poisson Approximation, Random Walks and Markov Chains.

8.6 Applied Probability

Queues are often used to model communication and manufacturing systems: a topic with manufacturing applications would be to discuss the stability and behaviour of networks of queues when the arrivals at a network almost saturate its service capacity; a topic with applications in the telecommunications context would be to discuss rare events in large systems.

8.7 Operational Research

Gather appropriate data from a filling station or a supermarket, and use it, with the help of an appropriate simulation software package, to investigate the characteristics of the queues which would form under different possible service regimes.

9 History of Mathematics

It is difficult to offer specific projects in the history of mathematics because the possibilities are so varied and the choice will depend very much on each student’s personal inclination and skills. Those who have taken O1 as a third-year option will already have a good grounding in the history behind the present day mathematics curriculum and may choose to go more deeply into a particular topic, person, or debate. Others may wish to work on a more general theme. There is also plenty of untranslated source material and those with some Latin, French, or German might like to undertake a translation and commentary; there is no better way to enter into the mind of a first rank mathematician, and Euler, Lagrange, and Cauchy, for example, all offer material that is both accessible and engaging.

To give some idea of the range and style of projects that are possible, here are examples of some topics that have been the subject of some recent OE essays or OD dissertations:

Mathematics and World War II
Mathematics in the Early Years of the St Petersburg Academy of Sciences
Robert Recorde’s presentation of Euclidean geometry
Hilbert’s seventh and eighth problems
The lives and work of Emilie du Chatelet and Sophie Germain
The life and work of Edmund Halley
Arthur Cayley’s contribution to group theory
A translation (from Latin) on summation of series by Euler
A comparison of the contributions to analysis of Cauchy and Bolzano
A translation (from French) of Galois’ work on finite fields
A Historical Study of the Church-Turing Thesis
The Historical Development of the Theory of Matrices

Anyone interested in working on a historical subject is encouraged to come and discuss ideas with Dr Chris Hollings (christopher.hollings@maths.ox.ac.uk) before the end of Trinity Term.

10 General

A host of tractable and interesting problems are to be found in journals such as

- *The American Mathematical Monthly,*
- *The Mathematical Gazette,*
- *The Mathematical Intelligencer,*
- *SIAM Review.*

Many of these have a past, and ramifications, the tracing of which would provide a project—including, perhaps, the solution of the problem, though this would not be essential.
11 Titles of Previous Projects

Listed below are the titles of some projects undertaken by students in recent years.

11.1 BE Extended Essays

Alhazen’s Problem and Galois Groups
A Study on Waring’s Problem
Investigating the Community Structure of Federal Election Commission Data on Congressional Donations
The Applications of Quantum Phase Estimation
Graphs of Large Chromatic Number
Mathematical Models of Dermal Wound Healing
Probabilistic Models Behind Spread Betting in Football
A Study of Distribution Theory in One Dimension

11.2 CD Dissertations

Extended Series Solutions to Flow Through Curved Pipes (single unit)
Thin Viscous Flow over a Flexible Beam (single unit)
Frobenius Algebras, Topological Quantum Field Theories and the Cobordism Hypothesis (single unit)
Of Galois Groups in Polynomial Time (single unit)
A Comparison of Intuitionistic and Classical Mathematics and Logic
The Poincare Conjecture for Dimensions 5 and Above
Numerical Solution of Elliptic PDEs with Stochastic Coefficients
Trees and Excursions
Hilbert’s Irreducibility Theorem and Applications to the Inverse Galois Problem
Mean-Variance Portfolio Selection Under Constraints
Contact Line Dynamics of an Evaporating Droplet
Irreducible Representations of Some Finite Groups Over Finite Fields
The Hole in the Wall: a look at the singular barrier found in a model of cell invasion
Stochastic Models of Chemical Processes
An Analysis of Changes in Mathematical Subfields via Time-dependent Co-authorship Network
Baroclinic Instability in The Earth’s Atmosphere
Homotopy Types
Catalysis in Quantum State Transformations
Transcendental Numbers and Algebraic Independence
Measure Theory, Riez Spaces and the Radon-Nikodym Theorem