THE COLLEGES OF OXFORD UNIVERSITY

MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

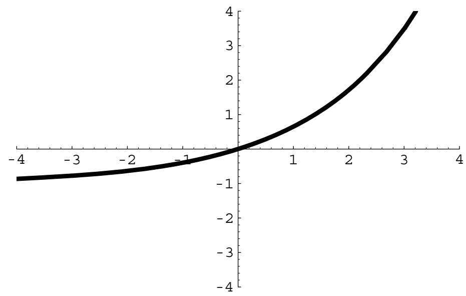
Sample Solutions for *Specimen Test 2*

1. A. The vector from $P(2,3)$ to $Q(8,-3)$ is $\overrightarrow{PQ} = (6,-6)$. If the point R lies on PQ such that $PR : RQ = 1 : 2$ then R is one third of the way along PQ. Hence

$$R = P + \frac{1}{3}\overrightarrow{PQ} = (2,3) + \frac{1}{3}(6,-6) = (4,1)$$

and **answer is (d).**

B. The graph of $y = f(x)$ is given below.



The graph of $y = f(x+1)$ is a translation one to the left of the above graph. This is then followed by a reflection in the x-axis to get the graph of $y = -f(x+1)$. So the **answer is (a).**

C. As tan has period π then

$$\tan\left(\frac{5\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1.$$

Also

$$\sin^2\left(\frac{5\pi}{4}\right) = \left(\frac{-1}{\sqrt{2}}\right)^2 = \frac{1}{2}.$$

Now $3 < \pi < 4$ and so $3.75 < 5\pi/4 < 5$. This means

$$\log_{10}\left(\frac{5\pi}{4}\right) < 1,$$

but as $2 < 3.75$ then

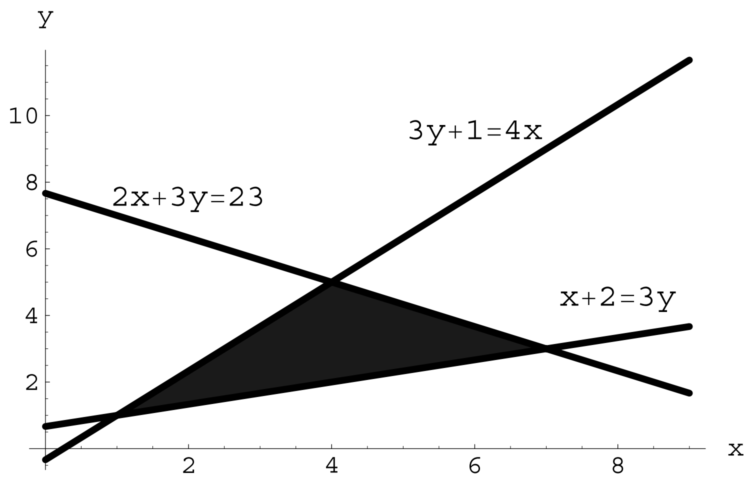
$$\log_2\left(\frac{5\pi}{4}\right) > 1.$$

Hence the **answer is (d).**

D. Sketching the three lines

$$2x+3y=23, \quad x+2=3y, \quad 3y+1=4x,$$

we see (as in the diagram below)



that the intersection of the three given regions is the interior of the triangle with vertices $(1,1)\,,(4,5)\,,(7,3)$. Hence the **answer is (b).**

E. The equation $\cos\theta = 1/2$ has solutions

$$\theta = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \pm\frac{7\pi}{3}, \pm\frac{11\pi}{3}, \cdots$$

So if $\cos(\sin x) = 1/2$ then

$$\sin x = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \pm\frac{7\pi}{3}, \pm\frac{11\pi}{3}, \cdots$$

Note that the smallest numbers on this list are $\pm\pi/3$; as $\pi > 3$ both of these lie outside sine's range which is $-1 \leqslant \sin x \leqslant 1$. So $\sin x$ can never equal any of the numbers on the above list and there are no solutions. The **answer is (a).**

F. As

$$y = x^2 - 2ax + 1 = (x-a)^2 + \left(1-a^2\right)$$

then the parabola's turning point is at

$$\left(a, 1-a^2\right).$$

The distance D from the origin to this point satisfies

$$\begin{aligned} D^2 &= a^2 + \left(1-a^2\right)^2 \\ &= a^4 - a^2 + 1 \\ &= \left(a^2 - \frac{1}{2}\right)^2 + \frac{3}{4}. \end{aligned}$$

So D^2, and hence D, are at a minimum when $a^2 = 1/2$. The **answer is (d).**

G. The two-digit multiples of 13 are

$$13, 26, 39, 52, 65, 78, 91.$$

So if N begins with a 9 its first two digits must be 91, its second and third must be 13, its third and fourth 39, its fourth and fifth 91, etc. We can see that N must begin

$$913913913\ldots$$

Going through this repeating pattern of 913 for 100 digits we will have 33 lots of 913 making up the first 99 digits, and the 100th digit will be a 9. The **answer is (d).**

H. Completing the square on the RHS of the equation gives

$$\begin{aligned}\left(x^2+1\right)^{10} &= 2x - x^2 - 2 \\ &= -1-(x-1)^2\end{aligned}$$

For real x, the LHS is always positive and the RHS is always negative and so the equation has no real solutions. The **answer is (b).**

I. We are told that

$$2^3 < 3^2 < 3^3 < 2^5.$$

So, taking logarithms to base 2 of these inequalities (remembering $\log_2 x$ is an increasing function) then

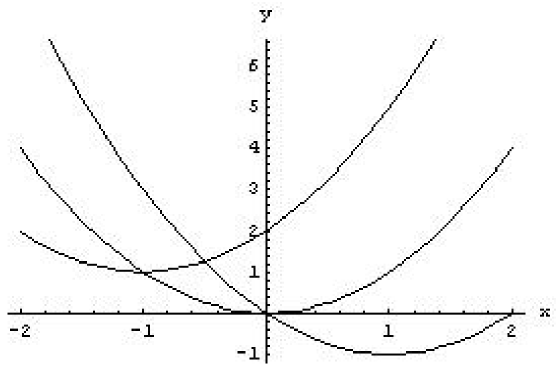
$$3 < 2\log_2 3 < 3\log_2 3 < 5.$$

Hence

$$\frac{3}{2} < \log_2 3 < \frac{5}{3},$$

and the **answer is (b).**

J. From the sketch below we can see that there are 7 regions.



Alternatively notice that solving the equations pairwise

$$\begin{aligned} y &= x^2, \quad y = x^2-2x \implies x = 0, \\ y &= x^2, \quad y = x^2+2x+2 \implies x=-1, \\ y &= x^2-2x, \quad y = x^2+2x+2 \implies x = -\frac{1}{2},\end{aligned}$$

we see that each of the three parabolas intersects with each of the other two.

The parabola $y = x^2$ splits the plane into two. Adding the second parabola $y = x^2 - 2x$, this crosses the first parabola once and so splits each of the other two regions into two — making four regions now. As we add the third parabola, $y = x^2+2x+2$, this cuts the two curves in two different points, passing through three of the four regions and dividing them into two, and so adding three more regions. In all then there are 7 regions and the **answer is (d).**

2. (i)

$$\begin{aligned}\left(x^2+ax+b\right)\left(x^2-ax+b\right) &= x^4+(a-a)\,x^3+\left(b+b-a^2\right)x^2+(-ab-ab)\,x+b^2\\ &= x^4+\left(2b-a^2\right)x^2+b^2\end{aligned}$$

which equals x^4+Ax^2+B when

$$A=2b-a^2, \qquad B=b^2.$$

(ii) We can write

$$\left(x^2+ax+b\right)\left(x^2-ax+b\right)=x^4-20x^2+16$$

if we can solve

$$2b-a^2=-20 \text{ and } b^2=16.$$

These equations are solved by $b=4$ and $a=\sqrt{28}=2\sqrt{7}$. So

$$x^4-20x^2+16=\left(x^2-2\sqrt{7}x+4\right)\left(x^2+2\sqrt{7}x+4\right).$$

(iii) Finding the roots of these two quadratics we get

$$\begin{aligned}x^2-2\sqrt{7}x+4 &= 0\\ \Longrightarrow \left(x-\sqrt{7}\right)^2=\left(\sqrt{7}\right)^2-4 &= 3,\\ \Longrightarrow x &= \sqrt{7}\pm\sqrt{3}.\end{aligned}$$

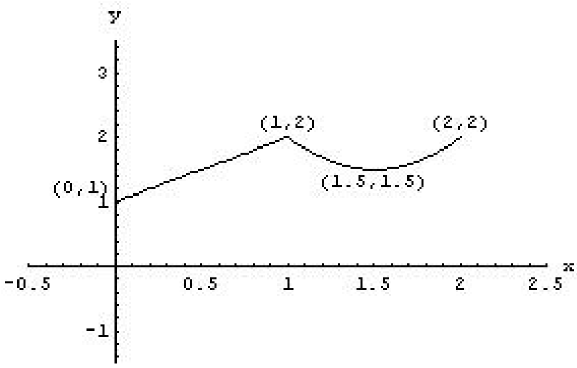
and

$$\begin{aligned}x^2+2\sqrt{7}x+4 &= 0\\ \Longrightarrow \left(x+\sqrt{7}\right)^2=\left(\sqrt{7}\right)^2-4 &= 3,\\ \Longrightarrow x &= -\sqrt{7}\pm\sqrt{3}.\end{aligned}$$

Hence the four roots of x^4-20x^2+16 are

$$\pm\sqrt{7}\pm\sqrt{3}$$

3. (i)



To find the turning point in the range $1 \leqslant x \leqslant 2$ we note $y'(x) = 4x - 6 = 0$ when $x = 3/2$. Then

$$y\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 6 = \frac{3}{2}.$$

(ii) Splitting the integral of $f(x)$ about $x = 1$ we see

$$\begin{aligned} g(t) &= \int_{t-1}^{1} (x+1)\,\mathrm{d}x + \int_{1}^{t} 2x^2 - 6x + 6\,\mathrm{d}x \\ &= \left[\frac{(x+1)^2}{2}\right]_{t-1}^{1} + \left[\frac{2x^3}{3} - 3x^2 + 6x\right]_{1}^{t} \\ &= \left\{2 - \frac{t^2}{2}\right\} + \left\{\frac{2t^3}{3} - 3t^2 + 6t - \frac{2}{3} + 3 - 6\right\} \\ &= \frac{2t^3}{3} - \frac{7t^2}{2} + 6t - \frac{5}{3}. \end{aligned}$$

(iii)

$$g'(t) = 2t^2 - 7t + 6 = (2t-3)(t-2).$$

(iv) The minimum/maximum values will occur at the ends ($t = 1$ or $t = 2$) or as a local extremum ($t = 3/2$). Note that

$$\begin{aligned} g(1) &= \frac{2}{3} - \frac{7}{2} + 6 - \frac{5}{3} = \frac{3}{2}; \\ g\left(\frac{3}{2}\right) &= \frac{9}{4} - \frac{63}{8} + 9 - \frac{5}{3} = \frac{9}{4} + \frac{9}{8} - \frac{5}{3} = \frac{54 + 27 - 40}{24} = \frac{41}{24}; \\ g(2) &= \frac{16}{3} - 14 + 12 - \frac{5}{3} = \frac{16 - 6 - 5}{3} = \frac{5}{3}. \end{aligned}$$

These answers are respectively 36/24, 41/24 and 40/24. Hence the minimum is at $t = 1$ and the maximum at $t = 3/2$.

4. The points P and Q have co-ordinates $(7,1)$ and $(11,2)$ respectively.

(i) R has co-ordinates $(7,-1)$ and P,Q,R are all marked on the grids below.

(ii) Consider all possible paths PXQ where X is a point on the x-axis. Such a path is drawn in the first grid below.

Each such path PXQ is of the same length as the path RXQ as R is the mirror image of P in the x-axis. So the shortest possible path PXQ will be of the same length as the shortest possible path RXQ. But it's clear that RXQ is shortest when X lies on the line segment RQ. In which case, Pythagoras' Theorem tells us that the shortest distance is

$$|RQ| = \sqrt{(11-7)^2 + (2-(-1))^2} = \sqrt{4^2+3^2} = 5.$$

(iii) The mirror image of $Q\,(11,2)$ in the line ℓ with equation $y=x$ is $S\,(2,11)$.

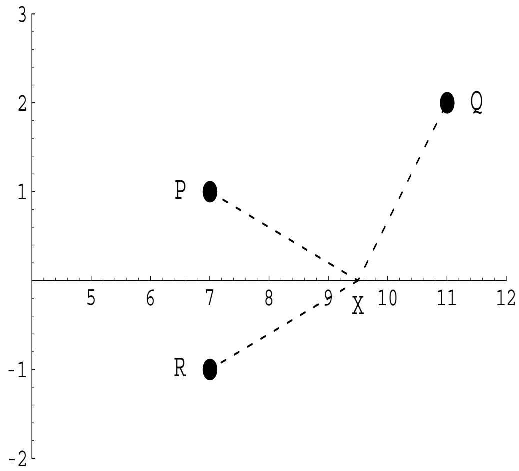
(iv) Consider paths of the form $PYZQ$ where Y lies on the x-axis and Z lies on the line ℓ.

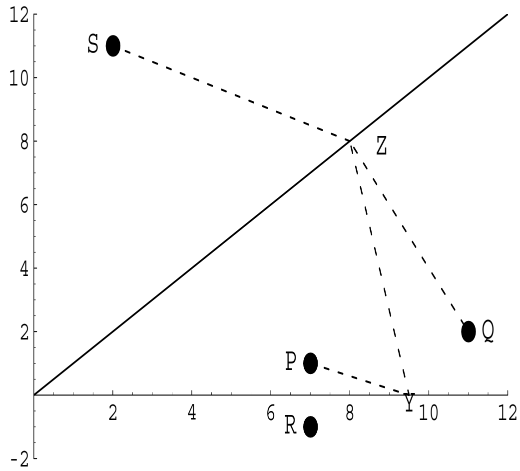
As we've already noted the part of the path PYZ is of the same length as RYZ.

Similarly the part of the path YZQ is of the same length as the path YZS.

So the path $PYZQ$ is of the same length as $RYZS$ and the shortest such path is the straight line RS which is of length

$$|RS| = \sqrt{(7-2)^2 + (-1-11)^2} = \sqrt{5^2+12^2} = 13.$$





5. (a) (i) C produces 1 from a 1×2 grid if both entries are equal. So a 2×2 grid produces a 1, after R then C, if the entries of each column agree (the top row below) or if the entries of each column disagree (the bottom row). So the eight 2×2 grids which produce a 1 (after R, then C) are

$$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$$

$$\begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}$$

(a) (ii) The first, fourth, sixth, seventh grids are symmetric in the top-left-bottom-right diagonal and the second/fifth and third/eighth are reflections of one another about this diagonal. Alternatively, we can say that the grids listed above are those that contain an even number of 1s, in which case their reflections in the leading diagonal will also have an even number of 1s.

The effect of doing R then C on $\begin{matrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{matrix}$ is the same as doing C then R on $\begin{matrix} \mathrm{a} & \mathrm{c} \\ \mathrm{b} & \mathrm{d} \end{matrix}$.

This means that if the first effect is a 1 then so will is the second. Similarly if the first effect is a 0, and so it not amongst the above eight grids, then neither will its reflection be; hence doing C then R on the reflection and will also produce a 0.

(b) (i) If we consider the right $n-2$ columns of the $n \times n$ grid then C has no effect on them whatsoever, whether done first or second. The effect of R is to compare the top two rows, but this is the same effect whether done first or second.

(b) (ii) From the previous part the effect on the bottom $n-2$ rows, and right $n-2$ columns, is the same irrespective of order. From part (a) of the question the effects on the top-left 2×2 entries are also the same and so the order in which R and C are performed does not matter.

6. (i) If Bob is telling the truth, then so is Alice, contradicting the given fact that only one person is telling the truth.

So Bob is lying which means Alice also is.

Similarly Charlie is also lying. Hence Dianne is the one telling the truth.

(ii) Clearly Dianne is telling the truth.

If Alice is telling the truth, then Bob is lying and Charlie is telling the truth – but that leaves three telling the truth, a contradiction.

So Alice is lying, which means Bob is telling the truth and Charlie is lying.

(iii) Now three are telling the truth.

Either Dianne and Egbert are both telling the truth, or both are lying.

If they are both lying, the other three must be telling the truth – but Charlie telling the truth means Alice is lying, a contradiction.

So Dianne and Egbert are telling the truth and one of the others.

Either Alice and Bob are each telling the truth and Charlie is lying, or vice versa. As only one of these three is truthful it must be Charlie.

So Charlie, Dianne, Egbert are telling the truth.

7. (i) The eight ways three Bs or Ws can be put in a row are:

$$BBB,\ BBW,\ BWB,\ BWW,\ WBB,\ WBW,\ WWB,\ WWW.$$

(ii) More generally there are 2^N ways of placing N pebbles in a row, as at each place in the row a pebble can be independently B or W.

(iii) In the list of eight rows of length three above, the five rows without Bs side-by-side are

$$BWB,\ BWW,\ WBW,\ WWB,\ WWW.$$

(iv) Consider the r_N possible rows of B and W of length N which don't contain adjacent Bs. These r_N rows can either end in B or W (and clearly not both).

- If such a row ends in a W then the previous $N-1$ characters make up a row of length $N-1$ which can't have adjacent Bs.
- If such a row ends in a B then the previous character has to be W. But then the previous $N-2$ characters are a row of Bs and Ws without adjacent Bs.

What we've shown is that each of the r_N rows of length N, without adjacent Bs, can be written as precisely one of the following

$$\begin{gathered}(\text{a row of length } N-1 \text{ without adjacent } B\text{s})\, W,\\ (\text{a row of length } N-2 \text{ without adjacent } B\text{s})\, WB.\end{gathered}$$

There are r_{N-1} rows of the former type and r_{N-2} of the latter. Hence

$$r_N = r_{N-1} + r_{N-2}.$$

(v) We now require that the first and last pebble cannot both be black.

- If such a row of length N begins in a W then the only restriction on next $N-1$ characters is that they have no adjacent Bs.
- If such a row of length N begins in a B then it must be followed by a W and the last character must also be a W, because of the new restriction. The intervening $N-3$ characters make up a row without adjacent Bs.

What this shows is that the w_N rows of length N each come in precisely one of the following forms:

$$\begin{gathered}W\,(\text{row of length } N-1 \text{ without adjacent } B\text{s})\,,\\ BW\,(\text{row of length } N-3 \text{ without adjacent } B\text{s})\, W.\end{gathered}$$

There are r_{N-1} of the first type and r_{N-3} of the second form. Hence

$$w_N = r_{N-1} + r_{N-3}.$$

(These sample solutions have been produced by Dr. Richard Earl, who is the Organising Secretary for admissions in Mathematics, Statistics and Computer Science in Oxford. Any questions regarding the test, applying to Oxford to read for these subjects, or comments on these solutions, would be welcome at earl@maths.ox.ac.uk)