SOLUTIONS FOR ADMISSIONS TEST IN
MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE
WEDNESDAY 4 NOVEMBER 2009

Mark Scheme:

Each part of Question 1 is worth four marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A. We have

$$\begin{aligned} I(a) &= \int_0^1 \left(x^2 - a\right)^2 \, \mathrm{d}x \\ &= \int_0^1 \left(x^4 - 2ax^2 + a^2\right) \, \mathrm{d}x \\ &= \frac{1}{5} - \frac{2a}{3} + a^2 \\ &= \left(a - \frac{1}{3}\right)^2 + \left(\frac{1}{5} - \frac{1}{9}\right) \\ &= \left(a - \frac{1}{3}\right)^2 + \frac{4}{45}. \end{aligned}$$

So the smallest value of $I(a)$ is $4/45$, achieved when $a = 1/3$. **The answer is (b).**

B. Completing the square we rewrite $x^2 + y^2 + 6x + 8y = 75$ as

$$(x+3)^2 + (y+4)^2 = 100.$$

Hence the circle has centre $(-3, -4)$ and radius 10. The radius from the centre, back through the origin, meets the circle at $(3, 4)$ which is at a distance 5 from the origin. **The answer is (c).**

C. Taking square roots of both sides of $x^4 = (x - c)^2$, we see

$$x^2 = x - c, \qquad \text{or} \qquad x^2 = c - x.$$

The first quadratic has discriminant $1 - 4c$ and the second quadratic has discriminant $1 + 4c$. In order that the original equation has four real solutions both of these quadratic must have two real solutions and so nonnegative discriminants. Hence $c \leqslant 1/4$ *and* $c \geqslant -1/4$. **The answer is (b).**

D. Summing the first few terms of the series

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n,$$

we see that we get the sequence

$$1, -1, 2, -2, 3, -3, 4, -4, \ldots$$

The first time that this equals or exceeds 100 is when $n = 199$. **The answer is (c).**

E. As $0 \leqslant \sin^2 x \leqslant 1$ and $0 \leqslant \cos^2 x \leqslant 1$ then

$$1 \leqslant 2^{\sin^2 x} \leqslant 2, \qquad 1 \leqslant 2^{\cos^2 x} \leqslant 2.$$

So

$$2^{\sin^2 x} + 2^{\cos^2 x} = 2$$

is only possible when $\sin^2 x = 0 = \cos^2 x$, but this is true for no value of x as $\cos^2 x + \sin^2 x = 1$ for all x. **The solution is (a).**

F. Setting

$$y = 3x^4 - 16x^3 + 18x^2 + k$$

we see

$$y' = 12x^3 - 48x^2 + 36x = 12x\,(x-1)\,(x-3)$$

and so y has turning points at $x = 0, 1, 3$ which are respectively minimum, maximum, minimum because of the shape of a quartic. There will be four real solutions when both minima are below the x-axis and the maximum is above the x-axis. Now

$$y\,(0) = k, \qquad y\,(1) = 5 + k, \qquad y\,(3) = 3^2\,(27 - 48 + 18) + k = k - 27.$$

Hence we need

$$k < 0, \qquad k > -5, \qquad k < 27$$

each to be true. **The answer is (d).**

G. Thinking about the graph/periodicity of sine, we know that

$$\sin\,(x + 2\pi) = \sin x$$

and so all the lines $y = x + 2n\pi$ where n is an integer will be part of the graph of $\sin y = \sin x$. But also we have

$$\sin\,(\pi - x) = \sin x$$

and so all the lines $y = (2n+1)\,\pi - x$ where n is an integer will also be part of $\sin y = \sin x$. **The answer is (c).**

H. Using the trapezium rule with N subintervals to calculate $\int_0^1 2^x\,\mathrm{d}x$ we get

$$\begin{aligned}
& \frac{h}{2}\left[y_0 + 2y_1 + \cdots + 2y_{N-1} + y_N\right] \\
= & \frac{1}{2N}\left[1 + 2 \times 2^{1/N} + \cdots + 2^{(N-1)/N} + 2\right] \\
= & \frac{1}{2N}\left[2\left(1 + 2^{1/N} + \cdots + 2^{(N-1)/N}\right) + 1\right] \\
= & \frac{1}{2N}\left[\frac{2\left(\left(2^{1/N}\right)^N - 1\right)}{2^{1/N} - 1} + 1\right] \\
= & \frac{1}{2N}\left\{1 + \frac{2}{2^{1/N} - 1}\right\}.
\end{aligned}$$

The answer is (b).

I. In order for the polynomial

$$p(x) = n^2 x^{2n+3} - 25nx^{n+1} + 150x^7$$

to have $x^2 - 1 = (x-1)(x+1)$ as a factor, the polynomial must be zero at both $x = 1$ and $x = -1$. Now

$$p(1) = n^2 - 25n + 150 = (n-10)(n-15)$$

is zero when $n = 10$ or $n = 15$. And

$$p(-1) = -n^2 + (-1)^n 25n - 150 = \begin{cases} -(n-10)(n-15) & n \text{ is even} \\ -(n+10)(n+15) & n \text{ is odd} \end{cases}$$

is zero when $n = 10$ or $n = -15$. Only $n = 10$ meets both requirements and so **the answer is (b)**

J. Factorising the given equation we see it now reads as

$$(x + 2y)^3 = 2^{30}$$

and so

$$x + 2y = 2^{10},$$

where x and y are both positive integers. The solutions (x, y) are then of the form

$$\left(2^{10} - 2, 1\right), \qquad \left(2^{10} - 4, 2\right), \ldots\ldots \left(4, 2^9 - 2\right), \qquad \left(2, 2^9 - 1\right)$$

and so **the answer is (c).**

2. (i) [2 marks] Using the given recurrence relation

$$\begin{aligned} x_4 &= 2x_3 - x_2 + 1 = 2 \times 6 - 3 + 1 = 10; \\ x_5 &= 2x_4 - x_3 + 1 = 2 \times 10 - 6 + 1 = 15. \end{aligned}$$

(ii) [4 marks] Setting $n = 1$, $n = 2$, $n = 3$ into $A+Bn+Cn^2$ and using the known values of x_1, x_2, x_3, we see that

We have

$$A + B + C = 1, \qquad A + 2B + 4C = 3, \qquad A + 3B + 9C = 6.$$

So, subtracting

$$B + 3C = 2, \qquad B + 5C = 3,$$

giving

$$C = \frac{1}{2}, \qquad B = \frac{1}{2}, \qquad A = 0.$$

(iii) [3 marks] If $n(n+1)/2 \geqslant 800$ then

$$n(n+1) \geqslant 1600 = 40^2.$$

Now $n(n+1)$ is an increasing function, and by inspection $39 \times 40 < 40^2 < 40 \times 41$, so $n = 40$ is the least possible n.

(iv) [6 marks] For for $n \geqslant 2$,

$$\begin{aligned} y_n &= 1 + 4 + 6 + \cdots + 2n \\ &= 1 + \left(\frac{n-1}{2}\right)(4 + 2n) \qquad \text{[AP formula]} \\ &= 1 + (n-1)(2+n) \\ &= n^2 + n - 1. \end{aligned}$$

As

$$y_n = n^2 + n - 1 = 2x_n - 1$$

then x_n/y_n is approximately $1/2$ for large values of n because x_n and y_n both become large.

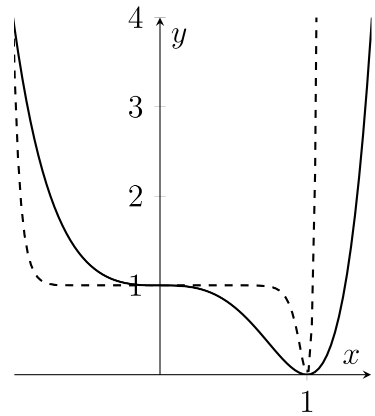
3. (i) [4 marks] We have $f_2(x) = (x^3 - 1)^2$. Note that $f_2(x) = 0$ only when $x = 1$ and $f_2(0) = 1$; so the graph crosses the axes only at $(1, 0)$ and $(0, 1)$.

Now differentiating $f_2(x) = x^6 - 2x^3 + 1$ gives

$$f_2'(x) = 6x^5 - 6x^2 = 6x^2\left(x^3 - 1\right)$$

is zero only at $x = 0$ and $x = 1$. As $f_2(x)$ is large for large positive and negative values, and as $f_2(x) \geqslant 0 = f_2(1)$ then $f_2(x)$ has a minimum at $x = 1$ and an inflexion point at $x = 0$.

The graph of $y = f_2(x)$ is shown with a solid line below.



(ii) [2 marks] Consider $f_n(x) = (x^{2n-1} - 1)^2$ when n is large and positive. Then $f_n(x)$ still crosses the axes at $(1, 0)$ and $(0, 1)$ only, is large for large negative and positive values and takes only nonnegative values. For $-1 < x < 1$ we have that $x^{2n-1} - 1$ is very close to -1 and so $f_n(x)$ is close to 1 in the range $-1 < x < 1$. For large values of x it is now the case that $f_n(x)$ increases even more rapidly than $f_2(x)$ does. The graph of $y = f_9(x)$ is shown with a dashed line above.

(iii) [2 marks] Note

$$\int_0^1 \left(x^{2n-1} - 1\right)^2 \mathrm{d}x = \int_0^1 \left(x^{4n-2} - 2x^{2n-1} + 1\right) \mathrm{d}x = \frac{1}{4n-1} - \frac{1}{n} + 1.$$

(iv) [4 marks] So

$$\int_0^1 f_n(x)\,\mathrm{d}x \leqslant \frac{1}{4n-1} - \frac{1}{n} + 1 \leqslant 1 - \frac{A}{n+B} \iff \frac{3n-1}{(4n-1)\,n} \geqslant \frac{A}{n+B}$$

Rearranging this gives

$$\begin{aligned}(3n-1)(n+B) &\geqslant A(4n-1)\,n. \\ 3n^2 - n - B + 3Bn &\geqslant 4An^2 - An\end{aligned}$$

Comparing coefficients of n^2 (the dominant term) we need $3 \geqslant 4A$ so that $A \leqslant 3/4$. Or one might expand the brackets to get

$$(3 - 4A)\,n^2 + (3B + A - 1)\,n - B \geqslant 0 \qquad \text{for } n \geqslant 1;$$

if $4A > 3$ the LHS is an "upside-down" parabola and so negative for large n which is a contradiction. Hence $A \leqslant 3/4$.

(v) [3 marks] When $A = 3/4$, we need

$$
\begin{aligned}
4\,(3n-1)\,(n+B) &\geqslant 3\,(4n-1)\,n \qquad \text{for } n \geqslant 1; \\
12n^2 + 12nB - 4n - 4B &\geqslant 12n^2 - 3n \qquad \text{for } n \geqslant 1; \\
(12B-1)\,n - 4B &\geqslant 0 \qquad \text{for } n \geqslant 1.
\end{aligned}
$$

The graph of the LHS against n is a line with gradient $12B - 1$; so we need $12B - 1 \geqslant 0$ for a non-decreasing line and also $(12B - 1) - 4B \geqslant 0$ for the first $n = 1$ case to hold. So $B \geqslant 1/8$ and $1/8$ is the smallest value.

4. (i) [4 marks] By differentiating, we know that the gradient at Q is $2a$ and so the normal has gradient $-1/(2a)$ when $a \neq 0$. Hence the line L has equation

$$y - a^2 = \frac{-1}{2a}(x - a).$$

If $a = 0$ the equation is $x = 0$. The equation can be better written as

$$x + 2ay = 2a^3 + a$$

which also includes the $a = 0$ case.

(ii) [3 marks] So L passes through $(0, 1)$ if

$$0 + 2a = 2a^3 + a \implies a\left(2a^2 - 1\right) = 0 \implies a = \frac{\pm 1}{\sqrt{2}}, \text{ or } a = 0.$$

(iii) [2 marks] As $Q = (a, a^2)$ and $P = (0, 1)$ then the distance $|PQ|^2$ is determined as

$$|PQ|^2 = a^2 + \left(a^2 - 1\right)^2 = a^4 - a^2 + 1.$$

(iv) [3 marks] This can be differentiated, and set to zero, so that

$$4a^3 - 2a = 0 \implies a = 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$$

though 0 is a local maximum. [Note these were the three values determined in part (ii).] Or by completing the square we have

$$a^4 - a^2 + 1 = \left(a^2 - \frac{1}{2}\right)^2 + \frac{3}{4}$$

which we can see is minimal when $a^2 = 1/2$.

(v) [3 marks] We need to find a point R, in the xy-plane, not on C, such that $|RQ|$ is smallest for a unique value of a. There are many points where this is the case but we are only asked to find one such point..Now $R = (0, -1)$ is such a point and it is relatively easy to justify this: the point $(0, 0)$ on C is distance 1 away; no other point in the upper half-plane, and hence no other point on C, is as close as $(0, 0)$. [The same argument can be used for any point on the negative y-axis.]

5. (i) [3 marks] A tour of an even by even grid can be produced by generalizing the sketch below.By moving right along the first row, dropping a row, left to column two, dropping a row, and so, and then finally coming back all the way along the bottom row and coming up the leftmost column, we complete a tour. As n is even then this general E-shapes has $n/2$ "arms" and so it is the case that having descended all the rows the tour returns left along the bottom row.

(ii) [2 marks] Given it is possible to draw a tour which goes through all the squares, our new starting point is somewhere on the original tour and so a new tour can begin from there and proceed along the original tour to the top-left corner and then catch up on the part of the original tour that it had missed.

(iii) [3 marks] We have $f = n^2$ as the robot has to move through each of the squares of the grid without repetition and moves into a new square with each F command. As it starts in a corner going right it must return to that corner by moving up and so to go from travelling right to up means that robot has turned 270 degrees over all, or one right angle short of a number of whole turns.Equivalently $r + 1$ clockwise quarter turns have led to a whole number of turns in all and so $r + 1$ must be a multiple of 4 (there being four quarter turns in a whole turn).

(iv) [3 marks] As the robot still travels through all the squares we still have $f = n^2$. But if, for example, the robot set off going right along one of the flat parts of the E-like tour above then it would overall turn through 360 degrees and, instead, r would be a multiple of 4

(v) [4 marks] Suppose now that n is odd. Then $f = n^2$ also is odd. But as any tour begins and ends in the same place, for every move to the right there must be one to the left, and for every up there must be a down i.e. f must be even. Hence no tour is possible for odd n.

6. (i) [3 marks] Clearly Charlie is telling the truth. Consequently Bob is lying which makes Alice's statement true.

(ii) [6 marks] It is impossible for all three to be lying because of Charlie's statement. It is impossible likewise for them all to be telling the truth. Alice and Bob lying and Charlie being honest is consistent. Alice and Bob both honest and Charlie lying is also consistent. So there may be one or two truth-tellers.

(iii) [6 marks] If Alice is telling the truth then Bob's statement is honest contradicting Alice's, so Alice is lying. This makes Charlie's statement a lie also. Bob's statement (because of the second clause) is then true.

7. (i) [2 marks] There are 7 valid sequences of length 2: MM, MX, XM, XX, XW, WX, and WW.

(ii) [9 marks] Every valid string starting with an M, can be turned into a sequence starting with a W by changing all the Ms to Ws and vice versa. Likewise all W-sequences will be turned into M-sequences. So there are equal numbers of each of length n.

Starting with a M, then the next letter must be an M or an X; so any sequence of length n which starts with an M is an M followed by a valid sequence of length $n-1$ starting with an M or an X. That is

$$m(n) = m(n-1) + x(n-1).$$

Starting with a X, then the next letter can be an M, an X or a W; that is

$$x(n) = m(n-1) + x(n-1) + w(n-1) = 2m(n-1) + x(n-1)$$

as $w(k) = m(k)$ for all k.

Because an sequence of length n must start with one (and only one) of M,X,W then

$$g(n) = m(n) + x(n) + w(n) = 2m(n) + x(n).$$

We can fill in the $n = 2$ column of the table below using the numbers from part (i). Later columns we can fill in using the relations above

n	2	3	4
$m(n) = m(n-1) + x(n-1)$	2	5	12
$x(n) = 2m(n-1) + x(n-1)$	3	7	17
$g(n) = 2m(n) + x(n)$	7	17	41

So we have $g(3) = 17$ and verify $g(4) = 2 \times 12 + 17 = 41$.

(iii) [4 marks] As the sequence is reflexive when it is reversed the middle letter stays in the same position and so must remain unchanged by the M-W swap. That is, it has to be an X. To make an even length reflexive sequence we can take a sequence that starts with an X, reverse it swapping Ms for Ws and vice versa, and put the new word in front of the original. (All even length reflexive sequences can be made this way because there can't be adjacent M,W in the middle.) An odd length reflexive sequence can be made in the same way but dropping one of the middle Xs. Put mathematically this gives the formulas as in the question

$$r(n) = \begin{cases} x\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd,} \\ x\left(\frac{n}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$