SOLUTIONS FOR ADMISSIONS TEST IN
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
WEDNESDAY 6 NOVEMBER 2013

Mark Scheme:

Each part of Question 1 is worth four marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A. The quadratic $x^2 + ax + a = 1$ has discriminant ("$b^2 - 4ac$") equal to

$$a^2 - 4 \times 1 \times (a-1) = a^2 - 4a + 4 = (a-2)^2 .$$

For the quadratic to have distinct real roots this needs to be positive and so we need $a \neq 2$. **The answer is (a).**

B. Either by using sketches of the relevant graphs, or arguing algebraically as below, we see that:

- on reflection in the line $x = \pi$ the graph of $y = \sin x$ becomes

$$y = \sin(2\pi - x) = \sin(-x) = -\sin x.$$

- on reflection in the line $y = 2$ the graph $y = -\sin x$ becomes

$$y = 4 - (-\sin x) = 4 + \sin x.$$

The answer is (c).

C. If we take the two given equations

$$f'(x) = g(x+1), \qquad g'(x) = h(x-1),$$

and differentiate the first with respect to x we have

$$f''(x) = g'(x+1), \qquad g'(x) = h(x-1).$$

Hence

$$f''(x) = g'(x+1) = h((x+1)-1) = h(x).$$

Hence $f'' = h$ (as functions) and in particular $f''(2x) = h(2x)$. **The answer is (c).**

D. The equation $x^4 - y^2 = 2y + 1$ rearranges to

$$x^4 = (y+1)^2 \Longleftrightarrow x^2 = y + 1 \quad \text{or} \quad -x^2 = y + 1.$$

So the graph consists of two parabolae $y = x^2 - 1$ and $y = -1 - x^2$ which form **the graph in (b).**

E.

- The leading term in $(2x-1)^4(1-x)^5$ is $(2x)^4(-x)^5 = -16x^9$ which differentiates twice to

$$-9 \times 8 \times 16x^7.$$

- The leading term $(2x+1)^4(3x^2-2)^2$ is $(2x)^4(3x^2)^2 = 16 \times 9x^8$ which differentiates to

$$16 \times 9 \times 8x^7.$$

Hence the leading terms cancel and so the polynomial actually has degree less than 7. **The answer is (d).**

F. If we take exponents of the three given equations we get

$$a = b^2, \qquad c - 3 = b^3, \qquad c + 5 = a^2.$$

Hence, eliminating a and c we get

$$b^3 + 3 = b^4 - 5 \Longrightarrow b^3(b-1) = 8.$$

We are only interested in positive solutions b to this equation. Note that $b^3(b-1)$ is negative for $0 < b < 1$ and then both b^3 and $b-1$ and positive and increasing (without bound) for $b > 1$. So there is only one positive solution to the equation – it is not hard to note in fact $b = 2$ (so that $a = 4$ and $c = 11$). **The answer is (a).**

G. By summing the defining arithmetic progression we see

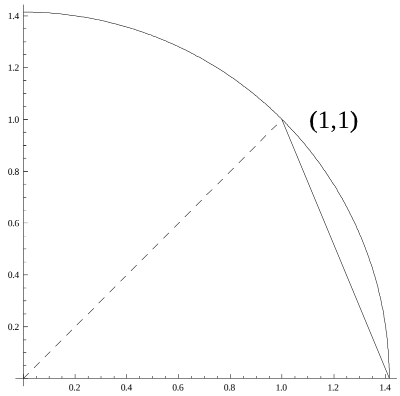
$$p_n(x) = nx - \frac{n(n+1)}{2}, \qquad p_{n-1}(x) = (n-1)x - \frac{n(n-1)}{2}.$$

One can then either do the long division, noting $p_{n-1}(x)$ divides $\frac{n}{n-1}$ times into $p_n(x)$ leaving remainder

$$\begin{aligned} p_n(x) - \frac{n}{n-1}p_{n-1}(x) &= \left(nx - \frac{n(n+1)}{2}\right) - \frac{n}{n-1}\left((n-1)x - \frac{n(n-1)}{2}\right) \\ &= -\frac{n(n+1)}{2} + \frac{n}{n-1}\frac{n(n-1)}{2} \\ &= \frac{-n^2 - n + n^2}{2} = \frac{-n}{2}, \end{aligned}$$

or one could apply the Remainder Theorem and evaluate $p_n(n/2)$ to find the same result. **The answer is (d).**

H. The graph of $y = \sqrt{2 - x^2}$ is a semicircle above the x-axis with centre $(0, 0)$ and radius $\sqrt{2}$. The line $x + \left(\sqrt{2} - 1\right) y = \sqrt{2}$ crosses this semicircle at $\left(\sqrt{2}, 0\right)$ and $(1, 1)$. So the region whose area needs to be determined is as in the sketch below:



This is best calculated as $\frac{1}{8}$ of the circle whose total area is $\pi(\sqrt{2})^2 = 2\pi$ subtracting a triangle with base $\sqrt{2}$ and height 1. So the area is

$$\frac{2\pi}{8} - \frac{1}{2} \times \sqrt{2} \times 1 = \frac{\pi}{4} - \frac{1}{\sqrt{2}},$$

and **the answer is (b).**

I. The recursions $F(2k) = F(k), F(2k+1) = F(k)$ mean that the kth value achieved gets repeated consecutively at $2k$ and $2k+1$. With the given starting values this means that the values of $F(k)$ when listed are

$$1, 1, -1, \underbrace{1, 1,}_{\times 2}\underbrace{-1, -1,}_{\times 2}\underbrace{1, 1, 1, 1,}_{\times 4}\underbrace{-1, -1, -1, -1,}_{\times 4}\underbrace{1, 1, 1, 1, 1, 1, 1, 1,}_{\times 8} -1, -1, \ldots$$

The lists of -1s end at $3, 7, 15, 31, \ldots$ so if one adds the first 3 or first 7 or first 15 terms etc. then the sum is 1 (from the very first term, all other later terms cancelling). So if one adds the first 127 terms then one gets 1 again. The previous 32 terms, and in particular $F(101), \ldots, F(127)$ are all -1s. So

$$F(1) + F(2) + F(3) + \cdots + F(100) + \underbrace{F(101) + \cdots + F(127)}_{\text{all } -1\text{s}} = 1$$

giving $F(1) + F(2) + F(3) + \cdots + F(100) = 1 - (-27) = 28$. **The answer is (b).**

J. Note that for $0 \leqslant x < n$ that $[2^x]$ takes all integer values from 1 to $2^n - 1$. Further for $1 \leqslant k < 2^n$ we see

$$[2^x] = k \quad \text{for} \quad \log_2 k \leqslant x < \log_2(k+1).$$

Hence we see the desired integral equals

$$\begin{aligned}
& (\log_2(2) - \log_2 1) + 2(\log_2(3) - \log_2 2) + 3(\log_2(4) - \log_2 3) + \cdots + (2^n - 1)(\log_2 2^n - \log_2(2^n - 1)) \\
= \; & -\log_2 2 - \log_2 3 - \cdots - \log_2(2^n - 1) + (2^n - 1)\log_2 2^n \\
= \; & 2^n \log_2 2^n - (\log_2 2 + \log_2 3 + \cdots + \log_2 2^n) \\
= \; & n2^n - \log_2((2^n)!).
\end{aligned}$$

The answer is (b).

2. (i) [4 marks] By replacing t with $1-t$ we now have two identities

$$\begin{aligned} f(t) - kf(1-t) &= t, &\text{(A)} \\ f(1-t) - kf(t) &= 1-t. &\text{(B)} \end{aligned}$$

We can solve these simultaneous equations by adding (A) to $k\times$ (B) to get

$$(1-k^2)f(t) = t + k(1-t)$$

and hence

$$f(t) = \frac{k + (1-k)\,t}{1-k^2}.$$

(ii) (a) [2 marks] With $g(t) = t$ then $(*)$ reads

$$f(t) - f(1-t) = t.$$

Consider the identity arrived at from the above by replacing t with $1-t$, namely

$$f(1-t) - f(t) = 1-t.$$

Adding these two identities we obtain $0 = 1$, a contradiction.

(ii) (b) [3 marks] If there is a solution to $(*)$, then adding that identity to $(+)$ we get

$$0 = g(t) + g(1-t) \qquad \text{(C)}$$

and hence $g(t)$ must necessarily satisfy (C) for there to be a solution $f(t)$ to $(*)$.

(ii) (c) [6 marks] We have $(2t-1)^3 = 8t^3 - 12t^2 + 6t - 1$. If we set $f(t) = t^3$ then we see

$$f(t) - f(1-t) = t^3 - (1-t)^3 = 2t^3 - 3t^2 + 3t - 1.$$

If we set $f(t) = t$ then we see $f(t) - f(1-t) = 2t - 1$. As

$$(2t-1)^3 = 4(2t^3 - 3t^2 + 3t - 1) - 3(2t-1)$$

then

$$f(t) = 4t^3 - 3t$$

is one solution. There are, in fact, infinitely many solutions; for example, the general cubic solution is

$$f(t) = 4t^3 + A(2t-1)^2 - 3t + B.$$

An alternative more general method: assuming that $g(t)$ satisfies the condition (C) found in (iii) then one can see that $f(t) = tg(t)$ is generally a solution. This follows as

$$f(t) - f(1-t) = tg(t) - (1-t)\,g(1-t) = tg(t) + (1-t)g(t) = g(t).$$

Another solution that works equally well and is even easier to verify, is $f(t) = g(t)/2$.

3. (i) [2 marks] Clearly

$$A(k) = \int_0^k x(x-k)(x-2)\,\mathrm{d}x - \int_k^2 x(x-k)(x-2)\,\mathrm{d}x$$

(ii) [4 marks] These integrals can be written as

$$A(k) = \int_0^k [p_3(x) + kp_2(x)]\ \mathrm{d}x - \int_k^2 [p_3(x) + kp_2(x)]\ \mathrm{d}x$$

where p_2 and p_3 are polynomials of degrees 2 and 3. If these integrate to q_3 and q_4 respectively we can see that

$$A(k) = [q_4(x) + kq_3(x)]_0^k - [q_4(x) + kq_3(x)]_k^2$$

is a polynomial in k of degree at most 4. (Fr those who show this by direct calculation

$$A(k) = \frac{4}{3} - \frac{4k}{3} + \frac{2k^3}{3} - \frac{k^4}{6}.$$

(iii) [2 marks] We have

$$\begin{aligned} -f_{2-k}(1-t) &= -(1-t)\,(1-t-2+k)\,(1-t-2) \\ &= -(1-t)\,(k-t-1)\,(-t-1) \\ &= (t+1)\,(t+1-k)\,((t+1)-2) \\ &= f_k(1+t). \end{aligned}$$

(iv) [3 marks] What the algebra in (iii) means geometrically is that the graph of $y = f_{2-k}(x)$ is a half-turn rotation of $y = f_k(x)$ about the point $(1,0)$. Equivalently this can be thought of as a reflection in $x = 1$ then a reflection in the x-axis.

As a rotation is area-preserving then the area between the graph and the x-axis remains unchanged – i.e. $A(k) = A(2-k)$.

(v) [4 marks] We can write

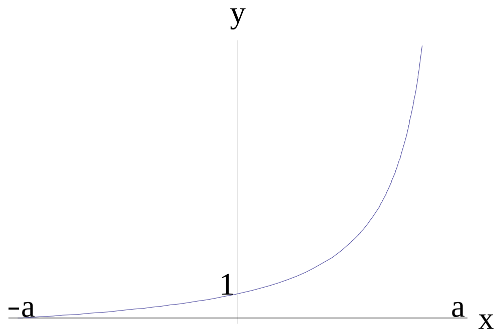
$$A(k) = \alpha\,(k-1)^4 + \beta(k-1)^3 + \cdots + \varepsilon$$

for some $\alpha, \beta, \gamma, \delta, \varepsilon$ and then

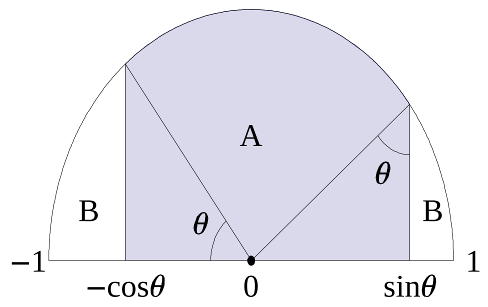
$$A(2-k) = \alpha(1-k)^4 + \beta(1-k)^3 + \cdots + \varepsilon.$$

Comparing coefficents we see that $\beta = \delta = 0$. Equivalently one could note that $A(k) = A(2-k)$ means that $A(k)$ is an even function about $k = 1$ (and a polynomial) so has no odd powers. For those that had an expression for $A(k)$ in (ii) they may show directly by algebraic manipulation that $a = \alpha = -1/6$, $b = \gamma = 1$, $c = \varepsilon = 1/2$.

4. (i) [4 marks]



(ii) [3 marks] Drawing in lines as in the diagram below



we can see that the area of A equals

$$\text{area}\,(A) = \text{area of quarter-disc} + \text{area of two triangles} = \frac{\pi}{4} + 2 \times \frac{\sin\theta\cos\theta}{2} = \pi/4 + \sin\theta\cos\theta.$$

Alternatively one can calculate this area by integration, though is more complicated:

$$\begin{aligned} A &= \int_{-\cos\theta}^{\sin\theta} \sqrt{1-x^2}\,\mathrm{d}x = \int_{\theta-\pi/2}^{\theta} \sqrt{1-\sin^2 u}\,\cos u\,\mathrm{d}u = \frac{1}{2}\int_{\theta-\pi/2}^{\theta} (1+\cos 2u)\,\mathrm{d}u \\ &= \frac{1}{2}\left[u + \frac{\sin 2u}{2}\right]_{\theta-\pi/2}^{\theta} = \frac{\pi}{4} + \frac{1}{4}\sin(2\theta) - \frac{1}{4}\sin(2\theta-\pi) = \frac{\pi}{4} + \frac{1}{2}\sin 2\theta. \end{aligned}$$

(iii) [3 marks] As $\sin^2\theta + \cos^2\theta = 1$ then

$$\sin\theta\cos\theta = \frac{\left(\sin^2\theta + \cos^2\theta\right) - (\sin\theta - \cos\theta)^2}{2} \leqslant \frac{1}{2}.$$

(iv) [5 marks] The area of B is $\pi/4 - \sin\theta\cos\theta$. So

$$\frac{\text{area of } A}{\text{area of } B} = \frac{\pi/4 + \sin\theta\cos\theta}{\pi/4 - \sin\theta\cos\theta}.$$

If we write $x = \sin\theta\cos\theta$ then we know from (i) that this ratio is an increasing function of x. We know from (iii) that $x \leqslant 1/2$ and that $x = 1/2$ when $\theta = \pi/4$. Hence the largest the ratio of the areas can be is

$$\frac{\pi/4 + \sin\theta\cos\theta}{\pi/4 - \sin\theta\cos\theta} = \frac{\pi/4 + 1/2}{\pi/4 - 1/2} = \frac{\pi+2}{\pi-2}.$$

5. Below, we'll include leading 0s, for clarity.

(i) [2 marks] 9: 08, 17, 26, ..., 71, 80.

(ii) [1 mark] $n + 1$, by analogy with the previous part.

(iii) [4 marks] If the hundreds digit is h, the other two digits sum to $i = n - h$, and there are $i + 1$ possibilities. i can be anywhere in the range 0 to n, hence the total is

$$\sum_{i=0}^{n} i + 1 = \frac{1}{2}(n+1)(n+2)$$

(iv) [2 marks] The last two digits sum to either 0, 1, 2 or 3, giving $1 + 2 + 3 + 4 = 10$ possibilities.

(v) [3 marks] By the previous part, there are 10 such where the first digit is at least 5. By symmetry, there are another 10 where the second digit is at least 5, and 10 where the last digit is at least 5. These three sets are disjoint. Total: 30.

(vi) [3 marks] Each digit from 0 to 9 appears 100 times in each position, so 300 times in total. So the total of the digit sums is

$$300 \times (0 + 1 + 2 + \cdots + 9) = 300 \times 45 = 13500.$$

6. Write A for Alice's number, and B for Bob's number.

(i) [4 marks] Bob's number is 1. Alice then realises her number is 2, because it can't be 0.

If Bob had any other number n, then Alice's number could be either $n+1$ or $n-1$, so Alice wouldn't know her number.

(ii) [6 marks] After Charlie's initial statement, the possibilities are:

B	A
1 or 3	2
2	3
4	3 or 5
6	5 or 7
8	7

If Alice could see 1, 2, 3 or 8, she would deduce her number, so those possibilities are ruled out.

If Bob could then see 3 or 7, he would deduce his number was 4 or 6, respectively, so those possibilities are ruled out. Hence Alice's number is 5.

(iii) [5 marks] Alice's statement implies Bob's number is not 1 or 10. Charlie's second statement implies that precisely one out of $B-1$ and $B+1$ is a square. The possibilities are:

B	A
2	1 or 3
3	2 or 4
5	4 or 6
8	7 or 9

If Bob could see 1, 3, 2, 6, 7 or 9, he would know his number. Hence Alice's number is 4.

7. (i) [2 marks.] Use rule (1) three times, then rule (2) twice.

(ii) [3 marks] All words of the form $\mathbf{A}^i\mathbf{B}^j$ with $i \geqslant j \geqslant 0$. Clearly, the rules preserve this form. To produce an arbitrary word of this form, apply rule (1) i times, then rule (2) $i-j$ times.

(iii) [4 marks] All words of the form $\mathbf{A}^i\mathbf{B}^j$ with $i \geqslant 0$ and $j \geqslant 0$. Clearly, the rules preserve this form. To produce an arbitrary word of this form with $i \geqslant j$, proceed as in the previous part. For $j > i$, proceed as in the previous part to produce $\mathbf{A}^j\mathbf{B}^i$, then apply rule (3).

(iv) [6 marks] (a) Given w, we may apply rule (1) then rule (2) to produce $\mathbf{A}w$.

(b) Write $\overline{x}$ for the word produced from x by replacing all $\mathbf{A}$ s with $\mathbf{B}$ s, and vice versa. Given w, use rules (3) and (4) (or the opposite order) to produce $\overline{w}$; then use rules (1) and (2) to produce $\mathbf{A}\overline{w}$; then use rules (3) and (4) to produce $\overline{\mathbf{A}\overline{w}} = \mathbf{B}w$.

Clearly we can repeat these steps to build up an arbitrary word.