SOLUTIONS FOR ADMISSIONS TEST IN
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
WEDNESDAY 5 NOVEMBER 2014

Mark Scheme:

Each part of Question 1 is worth four marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A. We can rewrite the inequality as

$$x^4 - 8x^2 - 9 < 0$$

which rearranges to

$$(x^2 - 9)(x^2 + 1) < 0.$$

As $x^2 + 1 > 0$ for all values of x then we can divide by this term and see that the inequality is equivalent to $x^2 < 9$. **The answer is (a).**

B. We can complete the square in the quadratic to get $(x-1)^2+1$. So substituting $x = 1$, we know that the graph meets the x-axis at (1,0), since $\log_{10}(1) = 0$. This eliminates graphs (a), (b), (c), and (d). **The answer is (e).**

C. The first derivative is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3kx^2 - (2k+2)x + (2-k)$$

which at $x = 1$ takes the value $3k - 2k - 2 + 2 - k = 0$. So the gradient is zero for all values of k. However for this to be a minimum we also need that the second derivative is positive. The second derivative is equal to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6kx - 2(k+1)$$

which equals $4k - 2$ when $x = 1$. This is positive when $k > \frac{1}{2}$ and so **the answer is (c).**

D. One might note that (d) is the only vector that is a unit vector for all values of m; as a reflection is distance-preserving then (d) is the only possible answer. Alternatively we might consider the case when m becomes large (the line $y = mx$ begins to verge upon the y-axis) and (d) is the only answer that agrees with $(-1,0)$ in this extreme. Or one might calculate the image using geometry.

The line through $(1,0)$ and perpendicular to $y = mx$ has gradient $-1/m$ and so has equation

$$y = (1-x)/m.$$

This normal intersects $y = mx$ at $(1/(m^2+1), m/(m^2+1))$ which is a displacement of

$$\left(\frac{-m^2}{m^2+1}, \frac{m}{m^2+1}\right)$$

away from the original point $(1,0)$. So, using vectors, the reflected image is at

$$(1,0) + 2\left(\frac{-m^2}{m^2+1}, \frac{m}{m^2+1}\right) = \left(\frac{1-m^2}{m^2+1}, \frac{2m}{m^2+1}\right).$$

Hence **the answer is (d).**

E. Consider first the extremes of the expression in the bracket. If we use the identity $\sin^2 x = 1 - \cos^2 x$ then we see

$$4\sin^2 x + 4\cos x + 1 = 5 + 4\cos x - 4\cos^2 x = 6 - (1 - 2\cos x)^2.$$

Thus the expression in the bracket can take values from -3, when $\cos x = -1$, to 6, when $\cos x = 1/2$. Hence the greatest the square of the expression can be is $6^2 = 36$ and **the answer is (b).**

F. Whilst $t = 1$ and $s = 8$ (in that order) do lead to the function $8 - x$ there are clearly other ways of achieving this function – e.g. $t = 3$ and $s = 8$ in that order. So we cannot deduce (a). As S is translation a unit to the right and T is reflection in the origin then after some selection of the two functions we will have translated the real line to the right or left by an integer (wherever the origin has moved to) and it will be "pointing" in the same direction or in the reverse direction; more precisely we will have yielded a function $n \pm x$ where n is an integer. If t is even we will have a $+x$ and if t is odd we will have a $-x$. In our example we know that the origin has moved to the point 8. The order in which the various S and T are performed affects the position of the origin as

$$ST(x) = 1 - x \qquad \text{and} \qquad TS(x) = -1 - x$$

but as these two possibilities differ in how they move the origin by 2, whilst we cannot conclude that $s = 8$ we can conclude that s is even. So **the answer is (c)**.

G. Recall from the Binomial Theorem for $(a + b)^n$ that the sum of the exponents of a and b in each term always equals n, so an expansion of $(xy + y^2)^k$ would lead solely to terms of total exponent $2k$ (in x and y together). As x^3y^5 has total exponent 8 then we need to focus on the $k = 4$ term. The correct term in the binomial expansion of $(1 + xy + y^2)^n$ is

$$\binom{n}{4}(xy + y^2)^4.$$

and as $x^3y^5 = (xy)^3y^2$ we need to choose 3 xy-terms and 1 y^2 term, so and there are $\binom{4}{1} = 4$ ways of doing this. **The answer is (d).**

H. Between 1 and 6000 there are 3000 values of n which are divisible by 2 and 2000 values of n which are divisible by 3. Even multiples of 3 are divisible by 2, and there are 1000 such values of n. Thus

$$f(6000) = 2 * (3000 - 1000) + 3 * (2000 - 1000) + 4 * (1000) = 4000 + 3000 + 4000 = 11000.$$

The answer is (c).

I. Completing the square rearranges the exponent to $(x-2)^2-1$. This means that

$$2^{x^2-4x+3} = 2^{(x-2)^2-1} = \frac{1}{2}2^{(x-2)^2}.$$

We are therefore translating the graph parallel to the x-axis (from the $(x-2)$ term), and then performing a stretch parallel to the y-axis (from the $\frac{1}{2}$ multiplier), and so **the answer is (b).**

J. We are interested in the integral between -1 and 1 of $f(x)$. If we integrate the original identity between -1 and 1, we find

$$12 + \int_{-1}^{1} f(x)\,\mathrm{d}u = 2\int_{-1}^{1} f(-x)\,\mathrm{d}u + \left[x^3\right]_{-1}^{1} \int_{-1}^{1} f(x)\,\mathrm{d}u.$$

Note that $\int_{-1}^{1} f(-x)\,\mathrm{d}u = \int_{-1}^{1} f(x)\mathrm{d}u$ as the graph of $y=f(-x)$ is just a reflection in the y-axis of the graph of $y=f(x)$. Substituting $A = \int_{-1}^{1} f(x)\,\mathrm{d}u$ means we are left with

$$12 + A = 2A + 2A,$$

so that $A=4$ and **the answer is (a).**

2. (i) [5 marks] As $x = 1$ is a root then

$$1 + 2b - a^2 - b^2 = 0$$

which rearranges (on completing the square) to

$$a^2 + (b-1)^2 = 2.$$

As $a^2 \geqslant 0$ then

$$(b-1)^2 \leqslant 2 \implies 1 - \sqrt{2} \leqslant b \leqslant 1 + \sqrt{2}.$$

(ii) [5 marks] Substituting $a^2 = 1 + 2b - b^2$ into $(*)$ and factorizing it we get

$$0 = x^3 + 2bx^2 + (b^2 - 2b - 1)x - b^2 = (x-1)(x^2 + (2b+1)x + b^2).$$

If $(*)$ has $x = 1$ as a repeated root, then $x = 1$ is also a root of the quadratic factor and so

$$1 + 2b + 1 + b^2 = 1 + (b+1)^2 = 0$$

which is impossible for real b.

Alternative approach: Write the cubic as $(x-1)^2(x-\gamma)$ to gain equations in b, γ (and a); by comparing coefficients we arrive at a contradiction.

(iii) [5 marks] As $x = 1$ can never be a repeated root, then $(*)$ can only have a repeated root by the quadratic factor having a repeated root. This happens when the discriminant is zero – that is, when

$$(2b+1)^2 = 4 \times 1 \times b^2 \implies 1 + 4b = 0 \implies b = -1/4.$$

When $b = -1/4$ then

$$x^2 + (2b+1)x + b^2 = x^2 + x/2 + 1/16 = (x + 1/4)^2$$

and so the repeated root is also $-1/4$. As the repeated root is less than the single root, then (from knowledge of a cubic's shape) we know there is a local maximum at the repeated root.

Alternative approach: Write the cubic as $(x-1)(x-\gamma)^2$ and compare coefficients to to gain equations in b and γ. We see $2b = -2\gamma - 1$ and $-b^2 = -\gamma^2$, so that $b = -1/4$. Then follow as above to find x, and either by observation or differentiation know that there is a local maximum at the repeated root.

3. (i) [2 marks] Setting $x = y = 0$ in (A) we get

$$f(0) = f(0)f(0).$$

As $f(0) > 0$ by (C) then $f(0) = 1$.

(ii) [3 marks] By property (B) we have

$$I = \int_0^1 f(x)\,\mathrm{d}x = \int_0^1 \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = [f(x)]_0^1 = f(1) - f(0) = a - 1.$$

(iii) [6 marks] The trapezium rule estimates the area as

$$I_n = \frac{1/n}{2}\left[f(0) + 2f\left(\frac{1}{n}\right) + 2f\left(\frac{2}{n}\right) + \cdots 2f\left(\frac{n-1}{n}\right) + f(1)\right].$$

Now $b = f(1/n)$ and by (B) we have $f(2/n) = f(1/n)f(1/n) = b^2$, $f(3/n) = b^3$, etc. Note in particular that

$$b^n = f(n/n) = f(1) = a.$$

Hence

$$\begin{aligned}
I_n &= \frac{1}{2n}\left[1 + 2b + 2b^2 + 2b^3 + \cdots 2b^{n-1} + b^n\right] \\
&= \frac{1}{2n}\left[1 + \frac{2b(b^{n-1}-1)}{b-1} + b^n\right] \\
&= \frac{1}{2n}\left[\frac{2b^n - 2b + b^{n+1} - b^n + b - 1}{b-1}\right] \\
&= \frac{1}{2n}\left[\frac{b^{n+1} + b^n - b - 1}{b-1}\right] \\
&= \frac{1}{2n}\left[\frac{(b+1)(b^n-1)}{b-1}\right] \\
&= \frac{1}{2n}\left(\frac{b+1}{b-1}\right)(a-1).
\end{aligned}$$

(iv) [4 marks] We are given that $I_n \geqslant I$ for large n and hence

$$\frac{1}{2n}\left(\frac{b+1}{b-1}\right)(a-1) \geqslant a - 1. \qquad (+)$$

Then

$$\begin{aligned}
\frac{b+1}{b-1} &\geqslant 2n \\
1 + \frac{2}{b-1} &\geqslant 2n
\end{aligned}$$

Then

$$a^{\frac{1}{n}} = b \leqslant 1 + \frac{2}{2n-1}.$$

4. (i) [2 marks] As ABC is isosceles then $\angle ABC = \pi - 2\alpha$. So

$$\begin{aligned} \text{area of triangle } ABC &= \frac{1}{2}(AB)(BC)\sin(\angle ABC) \\ &= \frac{1}{2} \times 1 \times 1 \times \sin(\pi - 2\alpha) \\ &= \frac{1}{2}\sin 2\alpha. \end{aligned}$$

(ii) [3 marks] When $\beta = \alpha$ then $F = 1$ and when $\beta = \pi/2$ then $F = 0$. Further F is a decreasing function of β and hence each value of $0 \leqslant k \leqslant 1$ is uniquely attained by F.

(iii) [3 marks] AXB will have half the area of ABC when $\angle AXB$ is a right angle. Hence $\angle ABX = \pi/2 - \alpha$. But we also have that $\angle ABX = \pi - 2\beta$ then it follows that

$$\pi - 2\beta = \pi/2 - \alpha \qquad \Longrightarrow \qquad \beta = \pi/4 + \alpha/2.$$

(iv) [4 marks] As $\angle ABX = \pi - 2\beta$ and $\angle AXB = 2\beta - \alpha$ then, by the sine rule, we have

$$\frac{AX}{\sin(\pi - 2\beta)} = \frac{1}{\sin(2\beta - \alpha)}.$$

Hence

$$\text{area of } ABX = \frac{1}{2}(AX)(AB)\sin\alpha = \frac{\sin(\pi - 2\beta)\sin\alpha}{2\sin(2\beta - \alpha)} = \frac{\sin 2\beta \sin\alpha}{2\sin(2\beta - \alpha)}.$$

Thus

$$F = \frac{\sin 2\beta \sin\alpha}{2\sin(2\beta - \alpha)} \Big/ \frac{\sin 2\alpha}{2} = \frac{\sin 2\beta \sin\alpha}{\sin(2\beta - \alpha.)\sin 2\alpha}.$$

(v) [3 marks] When $0 < \beta < \alpha \leqslant \pi/2$, we can swap the roles of α and β and hence

$$\text{area of triangle } ABX = \frac{\sin 2\alpha \sin\beta}{2\sin(2\alpha - \beta)}.$$

Hence

$$F = \frac{\sin 2\alpha \sin\beta}{2\sin(2\alpha - \beta)} \Big/ \frac{\sin 2\alpha}{2} = \frac{\sin\beta}{\sin(2\alpha - \beta)}.$$

5. (i) [1 mark] AAA, AAB, ABA, ABB, ABC.

(ii) [2 marks] $c_{n,1} = 1$: the rhyming scheme must contain just n As.
$c_{n,n} = 1$: the only rhyming scheme is ABC. . . K, containing the first k letters of the alphabet.

(iii) [5 marks] If the last symbol matches an earlier symbol then the first $n-1$ characters have k different symbols, so there are $c_{n-1,k}$ ways of choosing the first $n-1$ symbols of the rhyming scheme; and there are k ways of choosing the last symbol – total $kc_{n-1,k}$.

If the last symbol does not match an earlier one, then the first $n-1$ characters have $k-1$ different symbols, so there are $c_{n-1,k-1}$ ways of choosing the first $n-1$ symbols; the final symbol must be the kth symbol of the alphabet – total $c_{n-1,k-1}$.

Adding the above gives the total.

(iv) [4 marks]

$$r_n = \sum_{k=1}^{n} c_{n,k}$$

Here's a table of $c_{n,k}$ values.

$n\backslash k$	1	2	3	4
1	1	–	–	–
2	1	1	–	–
3	1	3	1	–
4	1	7	6	1

The first column and main diagonal come from the base cases. The other entries are calculated from the recursive equation as follows

$$\begin{aligned} c_{3,2} &= 2.c_{2,2} + c_{2,1} = 2.1 + 1 = 3 \\ c_{4,2} &= 2.c_{3,2} + c_{3,1} = 2.3 + 1 = 7 \\ c_{4,3} &= 3.c_{3,3} + c_{3,2} = 3.1 + 3 = 6 \end{aligned}$$

Hence $r_4 = 1 + 7 + 6 + 1 = 15$.

(v) [3 marks] $c_{n,2} = 2^{n-1} - 1$. The first symbol is A; the next $n-1$ are either A or B (2^{n-1} possibilities), but not all A's (so subtract 1).

Alternative approach: Take $k = 2$ in the formula of part (iii):

$$c_{n,2} = 2c_{n-1,2} + c_{n-1,1} = 2c_{n-1,2} + 1.$$

6. (i) [2 marks] We must have that the product has a unique factorisation, so $x = 1$ and y is a prime or 1. If the product is not a prime it would have more than one factorisation, so Pam would not know x and y.

(ii) [2 marks] If we had $x = 1,\ y = 3$, Pam would know x and y, as in the previous part. Thus $x = 2$ and $y = 2$.

(iii) [3 marks] If we had $x = 2$ and $y = 2$ then Sam would know x and y, as in the previous part. Thus $x = 1$ and $y = 4$.

(iv) [4 marks] Sam knows that the sum is 9, and so knows that Pam didn't originally know x and y (since 9 is not $1 +$ a prime).

If we had $x = 2$ and $y = 4$ then Sam would know the sum is 6, and so would not initially know that Pam does not know x and y: from Sam's point of view, it could be that $x = 1$ and $y = 5$, in which case Pam would know x and y. Hence $x = 1$ and $y = 8$.

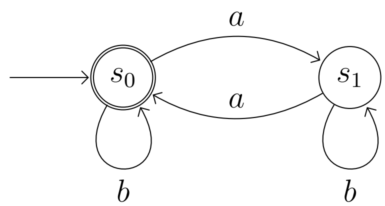
(v) [4 marks] Pam knows the product is 6, so says no at her first statement. Sam knows the sum if 5, so at her first statement, she considers it possible that $x = 1$ and $y = 4$. At Pam's second statement, she still considers it possible that $x = 1$ and $y = 6$ (in this case, Sam would know the sum is 7, and would still consider $x = 2$, $y = 5$ or $x = 3$, $y = 4$ possible).

If we had $x = 1$ and $y = 4$ then Pam would know the product is 4. In this case, Sam's first statement would tell Pam that we don't have $x = y = 2$, by part (ii). Hence Pam would know x and y at the point of her second statement — contradiction. Hence $x = 2$ and $y = 3$.

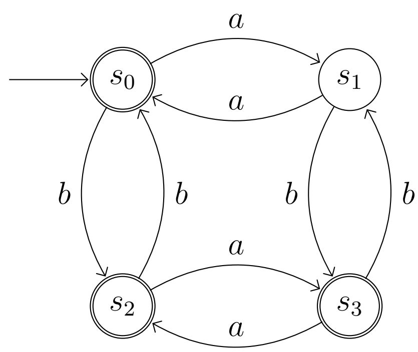
7. (i) [2 marks] a, aba, $ababa$, ... All words composed of alternating a's and b's, starting and ending with an a.

(ii) [2 marks] All words except those in part (i).

(iii) [2 marks] Automaton such as the one illustrated below.



(iv) [3 marks] Automaton such as the one illustrated below.



(v) [6 marks] From the state reached after a^i, following the path labelled b^i leads to an accepting state, since $a^i b^i$ is in L.

However, from the state reached after a^j, following the path labelled b^i does not lead to an accepting state, since $a^j b^i$ is not in L.

Hence these are different states.

Hence each word a^i (for $i = 0, 1, 2, \ldots$) leads to a different state, so there must be an infinite number of states. But this contradicts the fact that there are a finite number of states.