**SOLUTIONS FOR ADMISSIONS TEST IN**
# MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
# WEDNESDAY 4 NOVEMBER 2015

**Mark Scheme:**

Each part of Question 1 is worth 4 marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

## QUESTION 1:

**A.** Let $n$ be our starting whole number. Then we get $x = 4(n+1)^2 - 3$. $x$ will always be odd, since $4(n+1)^2$ will always be even. Consider $n = 1$, which gives $x = 13$ so neither **III** nor **IV** is true. $n = 2$ gives $x = 33$, which is precisely a multiple of 3 so **II** is not true. **The answer is (e).**

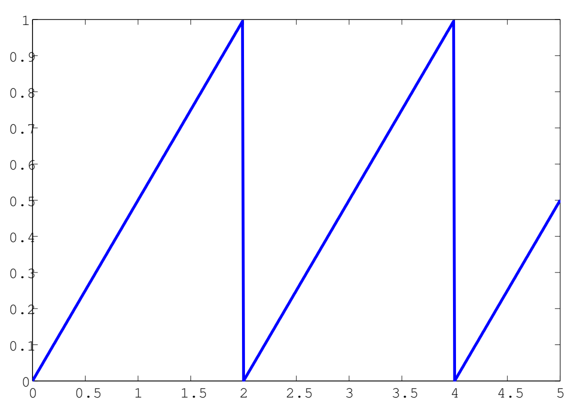
**B.** If $f(x)$ is an even function then $f'(x)$ will be an odd function, and vice versa. $f(x)$ and $f'(x)$ will cross the $x$-axis at $(-a, 0)$ and will have one further intersection when $y$ and $x$ are greater than 0. **The answer is (b).**

**C.** **I** is false when $x = 0$. **II** can be rearranged as $(\sin(x) + 1)^2 \geqslant 0$, using $\cos^2(x) + \sin^2(x) = 1$, and hence is true. **III** translates the $\sin(x)$ graph by $\frac{3\pi}{2}$ along the $x$-axis, giving $\cos(x)$, and $\cos(\pi - x)$ translates the $\cos(x)$ graph by $\pi$ and reflects in the $x$-axis, giving $\cos(x)$, and hence is true. **The answer is (c).**

**D.** Evaluating the integrals $f(x) = \frac{x^2}{3}$ and $g(x) = \frac{x^3}{3}$. Hence $g(f(A)) = \frac{A^6}{3^4}$ and $f(g(A)) = \frac{A^6}{3^3}$. Hence **the answer is (b).**

**E.** $\sin(x) = 0$ when $x = 0, \pi$, or $2\pi$ in the interval $0 \leqslant x \leqslant 2\pi$. Hence $2\cos(2x) + 2 = 0, \pi$, or $2\pi$. Rearranging gives $\cos(2x) = -1, \frac{\pi-2}{2}$, or $\pi - 1$. $\cos(2x)$ achieves the first of these values twice for two different values of $x$, achieves the second of these values four times, and never achieves the value of $\pi - 1$ and so **the answer is (d).**

**F.** Sketching f(x) produces a saw-tooth curve (a series of right-angled triangles).




The integral is equal to $\frac{2\times1}{2} + \frac{2\times1}{2} + \frac{0.5\times1}{2} = 2.25$ by inspection. 2 strips produces an underestimate (either by inspection or by calculation of the area of the resulting triangle $\frac{5\times0.5}{2} = 1.25$). 3 strips produces an overestimate - either by inspection or by algebra. Either using the trapezium rule, or by breaking the area down into the first triangle (area $= \frac{5}{3} \times \frac{5}{6} \times \frac{1}{2} = \frac{25}{36}$) and the following part triangle (area $= \left(\frac{25}{3} \times \frac{5}{6} \times \frac{1}{2}\right) - \left(\frac{15}{3} \times \frac{1}{2} \times \frac{1}{2}\right) = \frac{80}{36}$). Since $\frac{105}{36} > \frac{81}{36}$ **the answer is (b)**.

**G.** Consider integer multiples of $\pi$. When $y = n\pi$, $\cos^2(n\pi) = 1$. Hence this is satisfied when $x = m\pi$, so all the points $(n\pi, m\pi)$ must exist on the graph. **The answer is (c).**

**H.** Recall that $\log_a(b) = c$ describes the exponential relationship between $a, b$, and $c$ - namely that $a^c = b$. Hence, the equation becomes

$$4 - 5x^2 - 6x^3 = (x^2 + 2)^2.$$

Rearranging and cancelling terms gives

$$x^4 + 6x^3 + 9x^2 = 0.$$

This factorises to

$$x^2(x + 3)^2 = 0.$$

This has two roots and so **the answer is (c)**.

**I.** Sketching the three equations suggests the answer 7, however it is necessary to note the behaviour of the three equations when $0 \leqslant x \leqslant 1$. All three equations pass through 0, and $x^5 > x^4 > x^3$ when $x > 1$. When $0 \leqslant x \leqslant 1$ it should be noted that $x^3 \geqslant x^4 \geqslant x^5$ and that all three equations pass through $(1, 1)$. This behaviour produces two further regions (in addition to the 7 already discussed), and as such **the answer is (d).**

**J.** Squaring all answers results in $\frac{7}{4}$ which is larger than (b) $\frac{25}{16}$. After squaring (c) simplifies to $\frac{10\times9\times8\times7}{9\times6!}$, which further simplifies to $\frac{7}{9}$ which is smaller than (a). $\log_2(30) \approx 5$ and $\log_3(85) \approx 4$, hence (d) is smaller than (a) (after squaring). Comparing (a) with (e) after squaring results in a comparison of $\frac{7}{4}$ and $\frac{7+2\sqrt{6}}{9}$. As $2 < \sqrt{6} < 3$, (e) squared must be less than $\frac{13}{9}$ and hence less than $\frac{7}{4}$. **The answer is (a).**

**2.** (i) [2 marksl] Expanding to $a^{n+1}+a^nb+a^{n-1}b^2+\ldots+a^2b^{n-1}+ab^n-a^nb-a^{n-1}b^2-\ldots-ab^n-b^{n+1}$. Cancelling terms correctly to give $a^{n+1}-b^{n+1}$.

(ii) [2 marks] Want $n$ such that $n^2-1$ is prime, but $(n^2-1)=(n-1)(n+1)$.
In order for this to be prime we require $n-1=1$ (as $n-1<n+1$).
So $n=2$ and this leads to $2^2-1=3$ which is indeed prime.
So 3 is the only prime of this form.

(iii) [3 marks] Want $n$ such that $n^3+1$ is prime, but $n^3+1=(n+1)(n^2-n+1)$.
So in order for this to be prime we need $n+1=1$ or $n^2-n+1=1$.
$n+1=1\to n=0$, but $0^3+1$ is not prime.
$n^2-n+1=1\to n(n-1)=0\to n=0$ or $n=1$; as above 0 is no good, but $1^3+1=2$ is prime.
So 2 is the only prime of this form.

(iv) [3 marks] Using GP from (i), note that $3^{2015}-2^{2015}=(3^5-2^5)(3^{2010}+3^{2005}.2^5+3^{2000}.2^{10}+\ldots+3^5.2^{2005}+2^{2010})$.
But neither factor is 1, so $3^{2015}-2^{2015}$ is not prime.

(v) [5 marks] Note that $(k)^3<k^3+2k^2+2k$.
Note that $(k+1)^3=k^3+3k^2+3k+1$.
So for $k>0$ we have $k^3<k^3+2k^2+2k+1<(k+1)^3$.
So $k^3+2k^2+2k+1$ lies between two consecutive cubes, so is not a cube.

**Alternative approach:**
Assume that $k^3+2k^2+2k+1=n^3$, and using GP from (i), note that $n^3-k^3=2k^2+2k+1=(n-k)(n^2+nk+k^2)$.
From assumption, note that $n>k$.
So $n\geqslant k+1$, so $n^2+nk+k^2\leqslant 3k^2+3k+1$.
For minimum value $n=k+1$, $2k^2+2k+1=3k^2+3k+1$, for which RHS larger than LHS. So no positive integer $k$ satisfies, and so there is no cube number which can be expressed in that form.

**3. 3. Solution:**
(i) [1 mark] Many things work, for example $f(x) \equiv 0$ and $g(x) \equiv 1/100$. Then

$$\frac{1}{320} < |f(x) - g(x)| = \frac{1}{100} \leqslant \frac{1}{100}$$

and so $f$ is a good approximation to $g$ but not an excellent approximation to $g$.

(ii) [2 marks] Note that

$$|f(x) - g(x)| = \left| x - \left( x + \frac{\sin(4x^2)}{400} \right) \right| = \left| \frac{\sin(4x^2)}{400} \right| \leqslant \frac{1}{400} \leqslant \frac{1}{320}$$

and hence $f$ is an excellent approximation to $g$.

(iii) [3 marks] Start by noting that

$$\begin{aligned} g(x) = 1 + \int_0^x f(t)dt &= 1 + \int_0^x \left( 1 + t + \frac{t^2}{3} + \frac{t^3}{6} \right) dt \\ &= 1 + \left[ t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} \right]_0^x \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \end{aligned}$$

Hence

$$|f(x) - g(x)| = \frac{x^4}{24} \leqslant \frac{1}{384} \leqslant \frac{1}{320}$$

whenever $0 \leqslant x \leqslant 1/2$. It follows that $f$ is an excellent approximation to $g$.

(iv) [2 marks] Note that

$$\begin{aligned} h(x) - f(x) &= g(x) - f(x) + h(x) - g(x) \\ &= g(x) - f(x) + \left( 1 + \int_0^x h(t)dt \right) - \left( 1 + \int_0^x f(t)dt \right) \\ &= g(x) - f(x) + \int_0^x (h(t) - f(t))dt. \end{aligned}$$

(v) [2 marks] The integral is at most the length of the range of integration times the largest value of the integrand, and so

$$\int_0^x (h(t) - f(t))dt \leqslant x(h(x_0) - f(x_0)) \leqslant \frac{1}{2}(h(x_0) - f(x_0)).$$

(vi) [5 marks] By (iv) and then (v) we have

$$\begin{aligned} h(x_0) - f(x_0) &= g(x_0) - f(x_0) + \int_0^{x_0} (h(t) - f(t))dt \\ &\leqslant g(x_0) - f(x_0) + \frac{1}{2}(h(x_0) - f(x_0)). \end{aligned}$$

It follows that

$$h(x_0) - f(x_0) \leqslant 2(g(x_0) - f(x_0)) \leqslant \frac{2}{320} \leqslant \frac{1}{100}.$$

By definition of $x_0$ we conclude that for all $0 \leqslant x \leqslant 1/2$ we have

$$h(x) - f(x) \leqslant h(x_0) - f(x_0) \leqslant \frac{1}{100}$$

by assumption $h(x) - f(x) \geq 0 \geq -\frac{1}{100}$ and so $f$ is a good approximation to $h$.

**4.** (i) [3 marks]

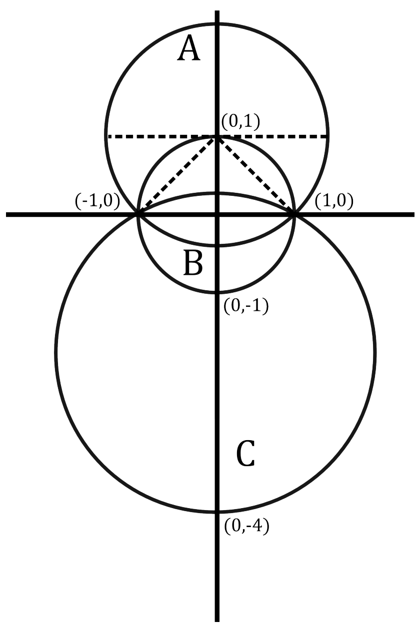
$$
\begin{aligned}
(x-m)^2+(y-h)^2&=r^2\\
(1-m)^2+h^2&=r^2\\
(-1-m)^2+h^2&=r^2.
\end{aligned}
$$

Rearrange to get $4m=0$ so $m=0$. Hence $r=\sqrt{1+h^2}$.

(ii) [3 marks] Let $(x_0,y_0)$ be our third point. Then $(x_0-m)^2+(y_0-h)^2=r^2$.
From part (i) know that $m=0$. Then sensible algebra to get $x_0^2+y_0^2=1+2hy_0$.
Substituting in $h=\sqrt{r^2-1}$, gives $x_0^2+y_0^2=1+2y_0\sqrt{r^2-1}$, which is uniquely dependent on $x_0$ and $y_0$.

(iii) [5 marks] A diagram is essential. The calculation necessary is [area of circle A $=2\pi$] - ([sector formed by points (-1,0) and (1,0) of circle A $=\frac{\pi}{2}$] - [triangle formed by centre of circle A and points (-1,0) and (1,0) $=1$]) - [semicircle of circle B $=\frac{\pi}{2}$]
Hence the lopsidedness $L(A)$ equals

$$
\begin{aligned}
L(A)&=\frac{2\pi-(\frac{\pi}{2}-1)-\frac{\pi}{2}}{2\pi}\\
&=\frac{\pi+1}{2\pi}.
\end{aligned}
$$




(iv) [4 marks] A diagram is essential. Observe that the centre of circle $A$ is at $(0,p)$ and the centre of circle $C$ is at $(0,-p)$.
The lopsidedness of $B$ is minimised when $B$ is split into three equal regions (of area $\frac{\pi}{3}$).
The area of the middle region of $B$ is the area of the two sectors formed by $A$ and $C$ minus the area of the two triangles formed by the centres of the circles $A$ and $C$ and the points $(-1,0)$ and $(1,0)$.
The line $y=0$ splits this middle region precisely in two, so the difference between one sector and one triangle should be $\pi/6$.
Each sector has area $(p^2+1)\tan^1(\frac{1}{p})$ and each triangle has area $p$.
Hence the lopsidedness is minimised when the equation is satisfied.

**5.** (i) [2 marks] Given expression $= s(s(1, -1, -2), \ldots, \ldots) = s(-2, s(1, -1, 2), \ldots) = 2$.

(ii) [3 marks] $f(5, 2) = s(2, p(5), p(f(5, 1)) = p(f(5, 1)) = p(s(1, p(5), p(f(5, 0)))) = p(p(f(5, 0))) = p(p(s(0, p(5), f(5, -1)))) = p(p(p(5))) = 8$.

(iii) [6 marks] For all positive integers $a$ and $b$ the value of $f(a, b)$ is given by $a + 1 + b$.

This is because, when $b$ is positive the value is given by the second argument of the $s$ function, which is one more than the value of the recursive call with its second argument reduced by one. I.e.

$$f(a, b) = f(a, b - 1) + 1, \qquad \text{for } b > 0.$$

After $b$ recursive calls, the second argument reaches zero, so the value is given by the first argument of the $s$ function, which is $a + 1$. Hence the overall value is $b$ increments greater than $a + 1$, which gives $a + 1 + b$.

(iv) [4 marks] One solution is

$$g(a, b) = s(p(b),\ m(g(a, p(b))),\ a).$$

The base case is when $p(b) = b + 1 > 0$, which will be reached when $b = 0$; at this point, $a$ is the correct result. The recurion mirrors the $f$-recursion, incrementing $b$ but decrementing the result:

$$g(a, b) = g(a, b + 1) - 1, \qquad \text{for } b \leq 0.$$

An alternative recursion is

$$g(a, b) = s(p(b),\ g(m(a), p(b))),\ a).$$

An alternative base case (for use with either recursion) is

$$g(a, b) = s(b,\ \ldots, p(a)).$$

Here the base case will be reached when $b$ reaches 1, at which point $a + 1$ is returned.

**6.** (i) [2 marks] If Beela is a werewolf, then the statement is true, so Azrael must be a werewolf too. If Beela is a vampire, then the statement is false, so Azrael again must be a werewolf. Beela could be either a werewolf or a vampire.

(ii) [3 marks] If the statement about Dita is true, then (by analogy with part (i)), Dita is a vampire, so Cesare is a vampire, and we can tell nothing about Elith.

If the statement about Dita is false, then Cesare and Dita are different species. We can tell nothing about Elith: for example, maybe Dita says nothing about Elith.

(iii) [4 marks] If the first two creatures are of the same species, then the statement is true, so the third creature is of the other species. Alternatively, if the first two of different species, then the statement is false, so the third creature is the same species as the second. In either case, the first and third creatures are different species.

Hence the order of species must be VVWWVVWW..., with an arbitrary offset; and $N$ must be divisible by 4.

(iv) [6 marks] If the first creature is the same species as the third, then the fourth must be different from both. If the first creature is different from the third, then the fourth must be the same as the third. In either case, the fourth is different from the first.

Thus either

- the order goes VVVWWWVVVWWW..., with an arbitrary offset, and $N$ is divisible by 6;
- or the order goes VWVW..., with an arbitrary offset, and $N$ is divisible by 2.

**7.** (i) [2 marks] Each such word can be produced in the following way:

$$S \to AB \to ... \to A^nB \to ... \to a^nB \to a^nbb$$

(i.e. repeating the second rule $n-1$ times, and the third rule $n-1$ times).

Every production must start with $S \to AB$. $A$ can be reduced only to strings of the form $a^n$ with $n \geq 1$. And $B$ can be reduced only to $bb$.

(ii) [2 marks] All words of the form $a^nb^n$ for $n \geq 1$.

(iii) [3 marks]

$$S \to aSa, \qquad S \to bSb, \qquad S \to a, \qquad S \to b, \qquad S \to aa, \qquad S \to bb$$

(iv) [4 marks] Either of the following will do

$$S \to aSb, \qquad S \to bSa, \qquad S \to SS, \qquad S \to ab, \qquad S \to ba.$$

or

$$S \to aSbS, \qquad S \to bSaS, \qquad S \to abS, \qquad S \to baS, \qquad S \to ab, \qquad S \to ba.$$

(v) [4 marks] Rename the start variables to new variables $S_1$ and $S_2$. If necessary, replace the variables in one set of rules by new variables so that the sets of rules have distinct variables. Combine the two sets of rules and add the rule

$$S \to S_1, \qquad S \to S_2.$$